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GUARANTEED ESTIMATION OF SPEECH FUNDAMENTAL FREQUENCY WITH BOUNDED COMPLEXITY ALGORITHM

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Abstract
The first stage of the Pitch estimation for a short time speech signal analysis consists of an exhaustive search of all integer Pitch periods in the harmonic model. For each Pitch period the cost function is calculated and then optimized. The general complexity is proportional to N^2 where N is the length of the window in samples. It is shown in this paper that the cost function can be approximated with the geometric rate of accuracy while the complexity of the algorithm remains around $N \log_2 N$. The trade-off between accuracy and complexity is discussed.

Key words
frequency estimation, fast algorithms, harmonic model

1 Introduction

Speech is the main instrument of communication between people. In everyday understanding human speech is the sound wave emitted by the human mouth and audible to the human ear. Such wave is called a speech signal.

There is complex physical process behind human speech production (Fry, 1979; Erath, Zañartu, Stewart, Plesniak, Sommer and Peterson, 2013; Erath et al., 2013; Encina, Yuz, Zanartu and Galindo, 2015).

One of the basic physical parameters of speech is the fundamental frequency (or pitch). This frequency f_0 defined as the inverse of the fundamental period. Fundamental period for periodic signal defined as:

$$T_0 = \min[T > 0 | \forall t : x(t) = x(t + T)]. \tag{1}$$

and $f_0 = 1/T_0$. Also, fundamental frequency may be defined by other definition:

$$\begin{aligned} f_0 = \max[f > 0 | \exists \alpha_k, \exists \phi_k : \\ x(t) = \textstyle\sum_{k=0}^{\infty} \alpha_k \sin(2\pi k f t + \phi_k)]. \end{aligned} \tag{2}$$

These two definitions are mathematically equivalent but carry a significantly different meanings. The first property describes the signal in the time domain, while the second describes the signal in the frequency domain. The basic principle for the determination of (1) is the periodicity of the signal $x(t) = x(t + T)$. And to determine the (2) — existence of the multiple frequency decomposition.

Precise estimation of the Fundamental frequency is necessary for correct calculation of harmonic amplitudes especially for the high frequency formants. An estimation error causes a multiple error for the high frequency harmonics. Pitch estimation error of 1 sample can completely reject harmonics in the estimated model at the frequency band near 2 kHz.

The Least Squares approach is successfully implemented for estimation of the complex amplitudes of the harmonic polynomial model of a voiced signal (Stylianou, 1996; De Cheveigné and Kawahara, 2002). But the Pitch estimation problem remains highly nonlinear with several local minima that can cause a standard multiple frequency error.

A general complexity of the estimation algorithms is proportional to N^2 where N is the window length. Such exhaustive search of admissible Pitch values is too expensive.

The "unbiased criterion" for Pitch estimation was proposed in (Griffin and Lim, 1988). Its complexity is proportional to $N \log_2 N$ where N is the frame window length. This criterion is also independent of the additive white noise.

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Accurate speech signal parameters estimation can be useful for physical models synthesis of the human vocal system. Also it may be useful for vocal tract physical parameters estimation (Kobayashi, Nisimura, Irino and Kawahara, 2013).

In our previous work (Barabanov, Melnikov, Magerkin and Vikulov, 2015) the unbiased criterion from (Griffin and Lim, 1988) for Pitch estimation was generalized to short time intervals. We proposed fast algorithm called Noise Variance Minimization (NVM) for precise estimation of the Fundamental frequency on a short time interval. The new algorithm gives an approximate solution with a complexity of $N \log_2 N$ operations.

In this work detailed proofs of our results is presented.

2 Problem Statement

Let $s = (s_t)_{t=-N/2}^{N/2-1}$ be a voiced signal of the length N. The stationary model of the signal is

$$\widehat{s}_t = \sum_{k=-M}^{M} a_k e^{\frac{2\pi i}{N} Fkt} + v_t, \quad -N/2 \le t \le N/2-1,$$

where P is the Pitch period of the model, $M = [(P-1)/2]$ is the number of harmonics, a_k are the complex amplitudes and $a_k = \bar{a}_{-k}$ for all k. v_t — white noise with σ^2 variance. The Pitch period P corresponds to the Fundamental frequency $F = N/P$ calculated in periods per frame.

The full set of the model parameter contains the value of P and the vector $A = (a_k)_{k=-M}^{M}$. All these values can be arbitrary. Accuracy of the model can be measured by the squared norm of the windowed estimation error

$$J(A,P) = \frac{1}{N} \sum_{t=-N/2}^{N/2-1} |w_t(s_t - \widehat{s}_t)|^2,$$

where $w_t = [1 + \cos(2\pi t/N)]/2$ is the Hanning window. The estimation problem is then reduced to minimization of the function J by all variables.

It is numerically effective to make a successive minimization:

$$J_{\min}(P) = \min_A J(A,P), \qquad J_{\min}(P) \to \min_P .$$

The first minimization problem is to be solved for all P, and the last minimization in one variable P is made by a constrained search.

As shown in (Barabanov et al., 2015) the function $J_{\min}(F)$ can be presented in explicit form (i.e. for the class of stationary models):

$$J_{\min}(F) = \frac{1}{N} \left(\sum_{t=-N/2}^{N/2-1} w_t^2 s_t^2 - \frac{1}{F} \sum_{m=0}^{P-1} \frac{|y_m|^2}{C_m} \right),$$

where

$$y_m(P) = \sum_{n=[\frac{-N/2-m}{P}]_+}^{[\frac{N/2-1-m}{P}]_-} \widetilde{s}_{m+nP}, \quad 0 \le m \le P-1,$$

$$C_m(P) = \frac{8}{3F} \sum_{n=[\frac{-N/2-m}{P}]_+}^{[\frac{N/2-1-m}{P}]_-} w_{m+nP}^2, \; 0 \le m \le P-1,$$

$\widetilde{s}_t = w_t^2 s_t$, $[\cdot]_+$ means round up and $[\cdot]_-$ — round down.

The minimal admissible value of P in this case corresponds to $F = 1.6$. If the number of signal periods in the frame window is less than 1.6 then a signal cannot be distinguished from the white noise.

The expectation of the minimal cost function is equal to

$$\mathsf{E} J_{\min}(P) = \frac{3}{8}\sigma^2 \left(1 - \frac{h_\infty(F)}{F}\right)$$

On interval $F \in [1.6, 3.0]$ function $h_\infty(F)$ can be approximate by

$$h_\infty^s(F) \approx -1.2635 + 3.0399 \cdot F - 0.9621 \cdot F^2 + 0.1018 \cdot F^3, \quad 1.6 \le F \le 3.$$

If $F \ge 3$ a good approximation is $h_\infty^s(F) = 1.9444$.

The function $J_{\min}(P)$ cannot be taken for the final decision of the Pitch estimate. The unbiased criterion $\mathcal{E}_{UB}(P)$ derived in (Barabanov et al., 2015) corrects the two standard errors: the multiple frequency error and influence of the white noise.

The unbiased criterion in this problem coincides with the Maximum Likelihood criterion: to minimize the unbiased estimate of the noise variance σ^2:

$$\mathcal{E}_{UB}(P) = \frac{J_{\min}(P)}{1 - \frac{h_\infty(F)}{F}}.$$

Due to high nonlinearity of $\mathcal{E}_{UB}(P)$ with many local minima, it is necessary to perform exhaustive search of global minimum. Search process is divided into two stages. In the first stage, integer pitch candidates are obtained from $\mathcal{E}_{UB}(P)$ for all pitch candidates from desired range. Next, integer pitch candidate is refined over all continuous values.

Thus, at the first stage, it is required to calculate the function (right part for the $J_{\min}(P)$ formulation).

$$\phi(P) = \sum_{m=0}^{P-1} \frac{|y_m(P)|^2}{C_m(P)}$$

for all integer P from the admissible interval $[P_{\min}, P_{\max}]$ where $P_{\min}$ is a fixed small integer and $P_{\max} \approx 5N/8$.

Values of $C_m(P)$ can be considered as tabulated.

To calculate the value of $\phi(P)$ by definition it's required to fold N/P values for each $m = 0, \ldots, P-1$. This step requires about N operations. Also P squaring, divisions and additions are needed. The complexity of the calculation is proportional to N. The number of different values of P, for which it is required to calculate $\phi(P)$ is also proportional to N. Therefore, the complexity of $\phi(P)$ calculation for all admissible P is proportional to N^2. It is required to reduce this complexity.

3 Multipliers Factorization

Associate rate $F = N/P$ with the length of the window N and period P, which expresses the number of periods in the window. Subsequently, the variables P and F are always related by the equation $PF = N$. Quality score $\phi(P)$ is to be calculated only for $F \geq 1.6$.

Introduce the auxiliary function $\eta(t) = \cos^4(t/2)$ on the interval $|t| \leq \pi$, extended by zero outside the interval $[-\pi, \pi]$ as well as its inverse Fourier transform

$$\widehat{\eta}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \eta(t)\, e^{itx}\, dt, \qquad x \in \mathrm{R},$$

and the normalized function $\widehat{\eta}_0(t) = \frac{8}{3}\widehat{\eta}(t)$.

Theorem 1. *1. For any $F > 0$*

$$C_m(P) = d_P(z_P^m), \qquad 0 \leq m \leq P-1,$$

where $z_P = e^{-\frac{2\pi i}{P}}$ and d_P is the Laurent series

$$d_P(z) = \sum_{n=-\infty}^{\infty} \widehat{\eta}_0(nF) z^n, \qquad |z| = 1.$$

2. The function $\widehat{\eta}_0(x)$ can be represented as

$$\widehat{\eta}_0(x) = \frac{4\sin(\pi x)}{\pi x(x^2-1)(x^2-4)},$$

for all x except singular points where

$$\widehat{\eta}_0(0) = 1, \qquad \widehat{\eta}_0(\pm 1) = \frac{2}{3}, \qquad \widehat{\eta}_0(\pm 2) = \frac{1}{6}.$$

This is a continuous function and $\widehat{\eta}_0(x) = \mathcal{O}(x^{-5})$ with $x \to \infty$.

3. For all $F \geq 1.6$ the function $d_P(z)$ can be approximated as follows:

$$\left| d_P(z) - |1 - \alpha_F z|^2 \right| \leq 0.01, \qquad |z| = 1,$$

where

$$K_F = \frac{1}{2}\left(\sqrt{1 + 2\widehat{\eta}_0(F)} + \sqrt{1 - 2\widehat{\eta}_0(F)}\right)$$
$$\alpha_F = -\frac{\widehat{\eta}_0(F)}{K_F^2}.$$

Proof. 1. By definition of function η square of Hanning function on interval $[N/2, N/2-1]$ is equal to

$$w_t^2 = \eta\left(\frac{2\pi}{F}\frac{t}{P}\right), \qquad -\frac{N}{2} \leq t \leq \frac{N}{2} - 1,$$

and coefficients $C_m(P)$ can be represented by their definition in form

$$C_m(P) = \frac{8}{3F} \sum_{k=-\infty}^{\infty} \eta\left(\frac{2\pi}{N}m + \frac{2\pi}{F}k\right).$$

Express these coefficients through an inverse Fourier transform of $\eta(t)$. To do this, apply the equality of generalized functions

$$\sum_{k=-\infty}^{\infty} \delta(t - 2\pi k) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{int}.$$

to function $\eta(\tau + t/F)$ for any τ. After the change of variable in the integral we get that

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} \eta\left(\tau + \frac{2\pi}{F}k\right) = \\
&= \frac{F}{2\pi} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} e^{inF(s-\tau)}\, \eta(s)\, ds \\
&= \frac{F}{2\pi} \sum_{n=-\infty}^{\infty} \widehat{\eta}(nF) e^{-inF\tau} \\
&= \frac{3F}{8} d_P(e^{-iF\tau}).
\end{aligned}$$

By definition the function d_P takes only real positive values, since any number on the unit circle can be represented in the form $z{=}e^{iF\tau}$ with some τ. By taking $\tau = 2\pi m/N$ we get

$$C_m(P) = d(z_P^m), \quad 0 \leq m \leq P-1,$$

that proves assertion 1 of the theorem.

The function $d_P(z)$ depends only on the number F of signal periods in window, and does not depend on the sampling frequency. The set of points z_P^m on the unit circle is completely determined by the value of P which is proportional to the sampling frequency and does not depend on the window length N.

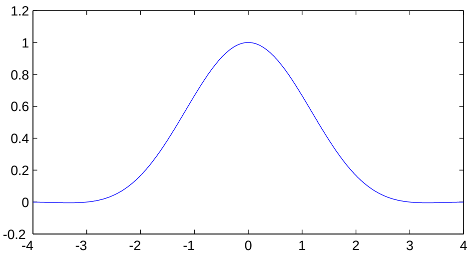

Figure 1. Normalized function $\widehat{\eta}_0(x) = \frac{8}{3}\widehat{\eta}(x)$.

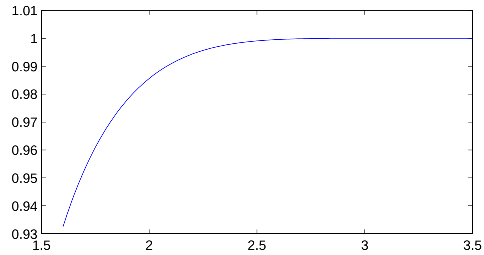

Figure 2. Multiplier K_F.

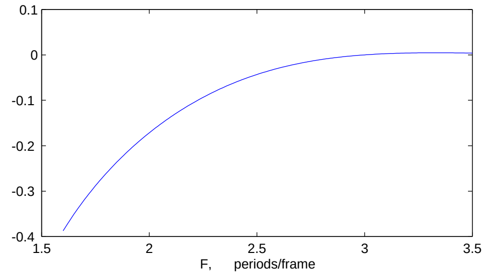

Figure 3. First order filter coefficient α_F.

2. Assertion 2 is proved by direct summation of geometric progressions.

3. The function $\widehat{\eta}_0(x)$ is even and equals to product of $\sin(\pi x)$ with the monotonically decreasing function proportional to x^{-5}. When $|x| > 3$ function is less than 0.005. For $|x| > 4$ — less than 0.001.

Sequence $(\widehat{\eta}_0(nF))_{n=-\infty}^{\infty}$ is the coefficients of the Laurent series for $d(z)$. By assumption, $F \geq 1.6$. Thus $|nF| \geq 3.2$ for $n \geq 2$ and hence

$$|\widehat{\eta}_0(nF)| \leq 0.005, \qquad |n| \geq 2, \quad F \geq 1.6.$$

The first-order approximation admits a factorization

$$\sum_{n=-1}^{1} \widehat{\eta}_0(nF)z^n = h_F(z)h_F(z^{-1}),$$
$$h_F(z) = K_F(1 - \alpha_F z),$$

where the coefficients are determined from the factorization equation:

$$K_F = \frac{1}{2}\left(\sqrt{1 + 2\widehat{\eta}_0(F)} + \sqrt{1 - 2\widehat{\eta}_0(F)}\right),$$
$$\alpha_F = -\frac{\widehat{\eta}_0(F)}{K_F^2}.$$

This proves assertion 3 and completes the proof of the theorem. □

Plot of the function $\widehat{\eta}_0$ is shown at fig. 1.

Plot of the functions K_F and α_F with $F \geq 1.6$ are shown at fig. 2 and 3. As expected, when $F \geq 3$ admissible approximation are $K_F \approx 1$ and $\alpha_F \approx 0$.

According to statements 1 and 3 of the theorem 1 the relative approximation error of values $C_m(P)$ on the ray $F \geq 1.6$ is less than 0.01. Thus,

$$\frac{1}{C_m(P)} \approx \frac{1}{|K_F|^2}\,|g_P(z_P^m)|^2,$$
$$g_P(z) = \frac{1}{1 - \alpha_F z} = \sum_{k=0}^{\infty} \alpha_F^k\, z^k.$$

Replace the original function $\phi(P)$ with appropriate approximation:

$$\phi(P) \approx \frac{1}{|K_F|^2}\phi_0(P),$$
$$\phi_0(P) = \sum_{m=0}^{P-1} |g_P(z_P^m)y_m|^2.$$

4 Approximation Of $\phi_0(P)$ By Smoothing Function

This section highlights the intermediate statements, which then take final form in the section 11.

Introduce the notation

$$v_P(t) = s_t w_t^2 g_P(z_P^t), \qquad -\frac{N}{2} \leq t \leq \frac{N}{2} - 1.$$

Lemma 1. *Let P — integer, $1 \leq P < N/2$. Then 1. Function ϕ_0 can be calculated by the formula*

$$\phi_0(P) = r_P(0) + 2\sum_{k=1}^{[N/P]} r_P(kP),$$

where $r_P(t)$ — correlation function of the signal $v_P(t)$, filled with zeros on $|t| \geq N/2$.

2. Vector $(r_P(k))_{k=-N}^{N}$ is the inverse DFT of the vector $(|Q_P(n)|^2)_{n=-N}^{N-1}$, where

$$Q_P(n) = \sum_{t=-N/2}^{N/2-1} v_P(t)e^{-\frac{2\pi i}{2N}nt}, \quad -N \leq n \leq N-1.$$

Proof. Define values $\widetilde{s}_t = s_t w_t^2$ with zeros on $|t| \geq N/2$. By definition of the values y_m and $v_P(t)$:

$$g_P(z_P^m)y_m = \sum_{n=-\infty}^{\infty} g_P(e^{-\frac{2\pi i}{P}m})\widetilde{s}_{m+nP} = \sum_{n=-\infty}^{\infty} g_P(e^{-\frac{2\pi i}{P}(m+nP)})\widetilde{s}_{m+nP} = \sum_{n=-\infty}^{\infty} v_P(m+nP).$$

Thus,

$$\phi_0(P) = \sum_{m=0}^{P-1}\left|\sum_{n=-\infty}^{\infty} v_P(m+nP)\right|^2$$
$$= \sum_{m=0}^{P-1}\sum_{n=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} v_P(m+nP)v_P^*(m+\ell P)$$
$$= \sum_{m=0}^{P-1}\sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} v_P(m+nP)v_P^*(m+(n+k)P)$$
$$= \sum_{k=-\infty}^{\infty}\sum_{t=-\infty}^{\infty} v_P(t)v_P^*(t+kP)$$
$$= \sum_{k=-\infty}^{\infty} r_P(kP) = r_P(0) + 2\sum_{k=1}^{[N/P]} r_P(kP),$$

since the function r_P is even and $r_P(t) = 0$ for $t \geq N$. This proves assertion 1 of the lemma.

The 2 statement of the lemma is a standard expression of the correlation function in terms of the DFT of the vector with length equal to $2N$. □

Thus the problem of fast calculation $\phi(P)$ is reduced to a fast calculation Q_P for all desired P.

5 Smoothing Function

Introduce the notation for the spectrum of the speech signal, multiplied by the Hanning window,

$$S(n) = \sum_{t=-N/2}^{N/2-1} w_t s_t e^{-\frac{2\pi i}{2N}tn}, \qquad -N \leq n \leq N-1,$$

and for the normalized DFT of the product of the frequency of the harmonic f on the Hanning window

$$B_N(f) = \frac{2}{N}\sum_{t=-N/2}^{N/2-1} w_t e^{-\frac{2\pi i}{2N}f\,t}, \qquad f \in \mathsf{R},$$

hereinafter referred to as «bell».

Lemma 2. *Let $1 < P < N/2$ and $F = N/P$ — harmonic frequency with a period P. Then*

$$Q_P(n) = \tfrac{1}{4}\sum_{k=0}^{\infty}\alpha_F^k \sum_{m=-N}^{N-1} S_m B_N(n-m+2kF), \quad -N \leq n \leq N-1,$$

where values of α_F is defined in theorem 1.

Proof. Define values

$$G_P(n) = \sum_{t=-N/2}^{N/2-1} w_t g_P(z_P^t)e^{-\frac{2\pi i}{2N}tn}, \quad -N \leq n \leq N-1.$$

Let $a_t = w_t s_t$ and $b_t = w_t g(z_P^t)$ for $-N/2 \leq t \leq N/2-1$. Define the function as zero for $-N \leq t < -N/2$ and for $N/2 \leq t < N$. Extend the functions a_t and b_t periodically on the set of all integers with period $2N$.

By the property of periodic functions, DFT of the product on the period is equal to the circular convolution of the DFT:

$$Q_P(n) = \sum_{t=-N}^{N/2-1} a_t b_t e^{-\frac{2\pi i}{2N}tn} = \frac{1}{2N}\sum_{m=-N}^{N-1}\left(\sum_{t=-N}^{N-1} a_t e^{-\frac{2\pi i}{2N}tm}\right)\cdot \left(\sum_{t=-N}^{N-1} b_t e^{-\frac{2\pi i}{2N}t(n-m)}\right)$$
$$= \frac{1}{2N}\sum_{m=-N}^{N-1} S(m)G_P(n-m)$$

by definition of the S and G_P.

On the other hand, by the definition of g_P and from the condition $|\alpha_F|<1$ is follows that

$$g_P(z_P^t) = \frac{1}{1-\alpha_F z_P^t} = \sum_{k=0}^{\infty}\alpha_F^k e^{-\frac{2\pi i}{N}Fkt}.$$

Thus,

$$G_P(n-m) = \sum_{k=0}^{\infty}\alpha_F^k \sum_{t=-N/2}^{N/2-1} w_t e^{-\frac{2\pi i}{2N}t(n-m)}e^{-\frac{2\pi i}{N}Fkt}$$
$$= \sum_{k=0}^{\infty}\alpha_F^k \sum_{t=-N/2}^{N/2-1} w_t e^{-\frac{2\pi i}{2N}t(n-m+2kF)}$$
$$= \frac{N}{2}\sum_{k=0}^{\infty}\alpha_F^k B_N(n-m+2kF).$$

Substitution of this equation into the formula for $Q_P(n)$ leads to the assertion of the lemma. □

The first series in the lemma 2 converges as a geometric progression because $|\alpha_F| \leq 0.4$ for $F \geq 1.6$. Fast algorithm for approximate calculation $Q_P(n)$ for all P and n is based on the method of calculating the convolution

$$\sum_{m=-N}^{N-1} S_m B_N(n-m-2kF),$$

where (S_n) — input signal spectrum multiplied by a window, and B_N — bell. The properties of bells are studied in the next section.

6 Bell Calculation

Function $B_N(f)$ is called «bell». This function is real, even, with period $2N$. Therefore, it is completely determined by its values on the interval $0 \leq f \leq N$.

Lemma 3.

$$B_N(f) = -\frac{1}{N}\frac{\sin\left(\frac{\pi}{2}f\right)\cos\left(\frac{\pi}{2N}f\right)\sin^2\left(\frac{\pi}{N}\right)}{\sin\left(\frac{\pi}{2N}f\right)\sin\left(\frac{\pi}{2N}(f-2)\right)\sin\left(\frac{\pi}{2N}(f+2)\right)}$$

with the definition at the singular points:

$$B_N(0) = 1, \quad B_N(\pm 2) = \frac{1}{2}.$$

Limit value of the function B_N by N is

$$B_\infty(f) = \lim_{N\to\infty} B_N(f) = -\frac{8\sin\frac{\pi f}{2}}{\pi f(f^2-4)}.$$

Proof. Because of

$$w_t = \frac{1}{2} + \frac{1}{4}e^{\frac{2\pi i}{N}t} + \frac{1}{4}e^{-\frac{2\pi i}{N}t},$$

then

$$B_N(f) = \frac{1}{N}\left[\frac{e^{-\frac{\pi i}{2}f} - e^{\frac{\pi i}{2}f}}{e^{-\frac{2\pi i}{2N}f} - 1} + \frac{1}{2}\frac{e^{-\frac{\pi i}{2}(f-2)} - e^{\frac{\pi i}{2}(f-2)}}{e^{-\frac{2\pi i}{2N}(f-2)} - 1} + \frac{1}{2}\frac{e^{-\frac{\pi i}{2}(f+2)} - e^{\frac{\pi i}{2}(f+2)}}{e^{-\frac{2\pi i}{2N}(f+2)} - 1}\right].$$

Numerators of the three fractions in brackets are equal, respectively, $-2i\sin(\pi f/2)$, $2i\sin(\pi f/2)$ and $2i\sin(\pi f/2)$. Reduce this amount to a common denominator. The common denominator is

$$D = (e^{-\frac{2\pi i}{2N}f} - 1)(e^{-\frac{2\pi i}{2N}(f-2)} - 1)(e^{-\frac{2\pi i}{2N}(f+2)} - 1)$$
$$= e^{-\frac{3\pi i}{2N}f} i \sin\frac{\pi f}{2N}\sin\frac{\pi(f-2)}{2N}\sin\frac{\pi(f+2)}{2N}.$$

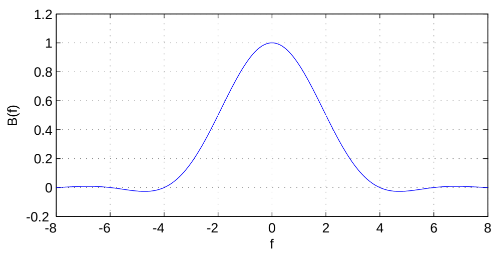

Figure 4. Limiting bell B_∞.

For the calculation of the numerator introduce the notation $x = -\pi f/N$. Numerator without the factor $i\sin(\pi f/2)/N$ is equal to

$$\begin{aligned} M &= -2(e^{ix+\frac{2\pi i}{N}} - 1)(e^{ix-\frac{2\pi i}{N}} - 1) + \\ &(e^{ix} - 1)(e^{ix-\frac{2\pi i}{N}} - 1) + (e^{ix} - 1)(e^{ix+\frac{2\pi i}{N}} - 1) = \\ &e^{2ix}\left[-2 + e^{\frac{2\pi i}{N}} + e^{-\frac{2\pi i}{N}}\right] + e^{ix}\left[-2 + e^{\frac{2\pi i}{N}} + e^{-\frac{2\pi i}{N}}\right] \\ &= e^{\frac{3}{2}ix} 2\cos\frac{x}{2}\left(2\cos\frac{2\pi}{N} - 2\right) \\ &= -8e^{\frac{-3\pi i}{2N}f}\cos\frac{\pi f}{2N}\sin^2\frac{\pi}{N}. \end{aligned}$$

After substitution we get that

$$\begin{aligned} B_N(f) &= \frac{1}{N}i\sin\left(\frac{\pi f}{2}\right)\frac{M}{D} \\ &= -\frac{1}{N}\frac{\sin\frac{\pi f}{2}\cos\frac{\pi f}{2N}\sin^2\frac{\pi}{N}}{\sin\frac{\pi f}{2N}\sin\frac{\pi(f-2)}{2N}\sin\frac{\pi(f+2)}{2N}}, \end{aligned}$$

which coincides with the conclusion of the lemma.

Limiting transition for $N \to \infty$ and for a fixed f is satisfied trivially by the replacement of the sine by argument. □

Plot of the function B_∞ is show on fig. 4. The function decreases rapidly: $|B_\infty(f)| < 0.001$ for $|f| > 5.6$.

Approximation error $|B_N(f) - B_\infty(f)|$ decreases rapidly with increasing N. For $N = 128$ the maximum of this error on the interval $f \in [-6, 6]$ is less than $2 \cdot 10^{-8}$.

7 Bells Approximation

Suppose that for each integer m on the interval $[m - 1/2, m + 1/2]$ function B_N expanded in a rapidly converging series of the same functions system on the interval $x \in [-0.5, 0.5]$. Due to the smoothness of A_N a function system can be polynomial:

$$B_N(x+m) = \sum_{j=0}^{\infty} C_{m,j}x^j,\ x \in [-0.5, 0.5], \quad m \in \mathrm{Z}.$$

For each j sequence $C_{m,j}$ has period of $2N$. In practice, we can assume that $C_{m,j} \approx 0$, starting from small $|m| < N/2$.

Expand the limiting bell in a series. Represent $\sin\frac{\pi x}{2}$ and $\cos\frac{\pi x}{2}$ as series:

$$\sin\frac{\pi x}{2} = \sum_{n=0}^{\infty} C_n^{\sin} x^n,$$

where

$$C_{2n+1}^{\sin} = \left(\frac{\pi}{2}\right)^{2n+1} \frac{(-1)^n}{(2n+1)!}, \quad C_{2n}^{\sin} = 0.$$

and

$$\cos\frac{\pi x}{2} = \sum_{n=0}^{\infty} C_n^{\cos} x^n,$$

where

$$C_{2n+1}^{\cos} = 0, \quad C_{2n}^{\cos} = \left(\frac{\pi}{2}\right)^{2n} \frac{(-1)^n}{(2n)!}.$$

Lemma 4. *Let*

$$B_\infty(f) = -\frac{8\sin\frac{\pi f}{2}}{\pi f(f^2-4)}$$

and $f = m + x$, where m — integer, $|x| \le 0.5$, then:

$$B_\infty(x+m) = \sum_{n=0}^{\infty} C_{m,n} x^n,$$

where

$$C_{2m,k} = \sum_{l=0}^{k} C_l^{\sin} \cdot \widehat{C}_{2m,k-l},$$

$$C_{2m+1,k} = \sum_{l=0}^{k} C_l^{\cos} \cdot \widehat{C}_{2m+1,k-l},$$

$$\widehat{C}_{2m,n} = \frac{(-1)^{n+m+1}}{\pi} \cdot \left[-\frac{2}{2m^{n+1}} + \frac{1}{(2m-2)^{n+1}} + \frac{1}{(2m+2)^{n+1}}\right],$$

$$\widehat{C}_{2m+1,n} = \frac{(-1)^{n+m+1}}{\pi} \cdot \Big[-\frac{2}{(2m+1)^{n+1}} + \frac{1}{(2m+1-2)^{n+1}} + \frac{1}{(2m+1+2)^{n+1}}\Big].$$

In singular points

$$C_{0,k} = \frac{2}{\pi} C_{k+1}^{\sin} + \sum_{l=0}^{k} C_l^{\sin} \cdot \widehat{C}_{0,k-l},$$

$$C_{2,k} = \frac{1}{\pi} C_{k+1}^{\sin} + \sum_{l=0}^{k} C_l^{\sin} \cdot \widehat{C}_{2,k-l},$$

$$C_{-2,k} = \frac{1}{\pi} C_{k+1}^{\sin} + \sum_{l=0}^{k} C_l^{\sin} \cdot \widehat{C}_{-2,k-l}.$$

Proof. Decompose $B_\infty(f)$ into partial fractions:

$$B_\infty(f) = -\frac{1}{\pi}\sin\frac{\pi f}{2}\left(-\frac{2}{f} + \frac{1}{f-2} + \frac{1}{f+2}\right)$$

Let $f = m + x$, where m — integer, $|x| \le 0.5$, then

$$B_\infty(x+m) = -\frac{1}{\pi}\sin\frac{\pi(x+m)}{2} \cdot \left(-\frac{2}{x+m} + \frac{1}{x+m-2} + \frac{1}{x+m+2}\right)$$

First, consider the case where $m \ne [-2, 0, 2]$. Because $|x| \le 0.5$, then

$$\frac{1}{x+m-2} = \sum_{n=0}^{\infty}(-1)^n (m-2)^{-n-1} x^n$$

$$\frac{1}{x+m+2} = \sum_{n=0}^{\infty}(-1)^n (m+2)^{-n-1} x^n$$

$$\frac{1}{x+m} = \sum_{n=0}^{\infty}(-1)^n (m)^{-n-1} x^n$$

Thus,

$$\begin{aligned} &B_\infty(x+m) = \\ &-\frac{1}{\pi}\sin\frac{\pi(x+m)}{2}\sum_{n=0}^{\infty}(-1)^n \cdot \\ &\left[-\frac{2}{m^{n+1}} + \frac{1}{(m-2)^{n+1}} + \frac{1}{(m+2)^{n+1}}\right] x^n \\ &= -\frac{1}{\pi}\left(\sin\frac{\pi x}{2}\cos\frac{\pi m}{2} + \cos\frac{\pi x}{2}\sin\frac{\pi m}{2}\right) \cdot \\ &\sum_{n=0}^{\infty}(-1)^n \cdot \\ &\left[-\frac{2}{m^{n+1}} + \frac{1}{(m-2)^{n+1}} + \frac{1}{(m+2)^{n+1}}\right] x^n \end{aligned}$$

or

$$B_\infty(x+2m) = \sin\frac{\pi x}{2}\sum_{n=0}^{\infty}\widehat{C}_{2m,n}x^n$$

$$B_\infty(x+2m+1) = \cos\frac{\pi x}{2}\sum_{n=0}^{\infty}\widehat{C}_{2m+1,n}x^n$$

where

$$\widehat{C}_{2m,n} = \frac{(-1)^{n+m+1}}{\pi}\cdot \left[-\frac{2}{2m^{n+1}}+\frac{1}{(2m-2)^{n+1}}+\frac{1}{(2m+2)^{n+1}}\right],$$

$$\widehat{C}_{2m+1,n} = \frac{(-1)^{n+m+1}}{\pi}\cdot \left[-\frac{2}{(2m+1)^{n+1}}+\frac{1}{(2m-1)^{n+1}}+\frac{1}{(2m+3)^{n+1}}\right].$$

According to the rule of the Cauchy for product of power series:

$$B_\infty(x+2m) = \sum_{n=0}^{\infty}C_n^{\sin}x^n\cdot\sum_{n=0}^{\infty}\widehat{C}_{2m,n}x^n = \\ = \sum_{k=0}^{\infty}C_{2m,k}x^k$$

where

$$C_{2m,k} = \sum_{l=0}^{k}C_l^{\sin}\cdot\widehat{C}_{2m,k-l}.$$

Similarly,

$$B_\infty(x+2m+1) = \sum_{n=0}^{\infty}C_n^{\cos}x^n\cdot\sum_{n=0}^{\infty}\widehat{C}_{2m+1,n}x^n = \\ = \sum_{k=0}^{\infty}C_{2m+1,k}x^k$$

where

$$C_{2m+1,k} = \sum_{l=0}^{k}C_l^{\cos}\cdot\widehat{C}_{2m+1,k-l}.$$

Now consider the case $m = [-2, 0, 2]$.
Let $m = 0$:

$$B_\infty(x) = -\frac{1}{\pi}\sin\frac{\pi x}{2}\left(-\frac{2}{x}+\frac{1}{x-2}+\frac{1}{x+2}\right) = \\ \sin\frac{\pi x}{2}\cdot \\ \left(\frac{2}{\pi x}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{\pi}\left[\frac{1}{-2^{n+1}}+\frac{1}{2^{n+1}}\right]x^n\right) = \\ = \frac{2}{\pi}x^{-1}\sum_{n=0}^{\infty}C_n^{\sin}x^n+\sum_{n=0}^{\infty}C_n^{\sin}x^n\cdot\sum_{n=0}^{\infty}\widehat{C}_{0,n}x^n = \\ = \sum_{n=0}^{\infty}\frac{2}{\pi}C_{n+1}^{\sin}x^n+\sum_{n=0}^{\infty}C_n^{\sin}x^n\cdot\sum_{n=0}^{\infty}\widehat{C}_{0,n}x^n = \\ = \sum_{k=0}^{\infty}C_{0,k}x^k$$

where

$$C_{0,k} = \frac{2}{\pi}C_{k+1}^{\sin}+\sum_{l=0}^{k}C_l^{\sin}\cdot\widehat{C}_{0,k-l}$$

Let $m = 2$:

$$B_\infty(x+2) = \\ \sin\frac{\pi x}{2}\cdot \\ \left(\frac{1}{\pi}x^{-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{\pi}\left[\frac{-2}{2^{n+1}}+\frac{1}{4^{n+1}}\right]x^n\right) = \\ \frac{1}{\pi}x^{-1}\sum_{n=0}^{\infty}C_n^{\sin}x^n+\sum_{n=0}^{\infty}C_n^{\sin}x^n\cdot\sum_{n=0}^{\infty}\widehat{C}_{2,n}x^n = \\ = \sum_{k=0}^{\infty}C_{2,k}x^k$$

where

$$C_{2,k} = \frac{1}{\pi}C_{k+1}^{\sin}+\sum_{l=0}^{k}C_l^{\sin}\cdot\widehat{C}_{2,k-l}$$

Let $m = -2$:

$$B_\infty(x-2) = \\ \sin\frac{\pi x}{2}\cdot \\ \left(\frac{1}{\pi}x^{-1}+\sum_{n=0}^{\infty}\frac{(-1)^n}{\pi}\left[\frac{-2}{-2^{n+1}}+\frac{1}{-4^{n+1}}\right]x^n\right) = \\ = \frac{1}{\pi}x^{-1}\sum_{n=0}^{\infty}C_n^{\sin}x^n+\sum_{n=0}^{\infty}C_n^{\sin}x^n\cdot\sum_{n=0}^{\infty}\widehat{C}_{-2,n}x^n = \\ = \sum_{k=0}^{\infty}C_{-2,k}x^k$$

where

$$C_{-2,k} = \frac{1}{\pi} C^{\sin}_{k+1} + \sum_{l=0}^{k} C^{\sin}_{l} \cdot \widehat{C}_{-2,k-l}.$$

□

8 $Q_P(n)$ Representation

Substitute the expansion of B_N (more precisely, to its limit function B_∞) into the expression for $Q_P(n)$:

$$Q_P(n) = \frac{1}{4} \sum_{k=0}^{\infty} \alpha_F^k \sum_{m=-N}^{N-1} S_m B_N(n-m+2kF) =$$
$$= \frac{1}{4} \sum_{k=0}^{\infty} \alpha_F^k \sum_{m=-N}^{N-1} S_m B_N([n-m+l_{kF}] + x_{kF}) =$$
$$= \frac{1}{4} \sum_{k=0}^{\infty} \alpha_F^k \sum_{m=-N}^{N-1} S_m \sum_{j=0}^{\infty} C_{n-m+l_{kF},j} x_{kF}^j.$$

Introduce notation for normalized convolution sequences S and C for fixed j:

$$D_{n,j} = \frac{1}{4} \sum_{m=-N}^{N-1} S_m C_{n-m,j}, \qquad n \in \mathsf{Z}.$$

For each j this is a periodic sequence of period $2N$. Let $k \geq 0$ and round $2kF$ for closest integer value:

$$2kF = \ell_{kF} + x_{kF}, \qquad |x_{kF}| \leq 0.5, \quad \ell_{kF} \in \mathsf{Z}.$$

Theorem 2. *The correlation function of the sequence $v_P(t)$ equal to*

$$r_P(t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \alpha_F^k \alpha_F^m \cdot$$
$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{kF}^i x_{mF}^j \, \rho_{i,j}(t, l_{mF} - l_{kF}) \, e^{-\frac{\pi i}{N} l_{kF} t}$$

where the notation for the inverse DFT is

$$\rho_{i,j}(t,\tau) = \frac{1}{2N} \sum_{n=-N}^{N-1} D_{n,j} \overline{D}_{n+\tau,i} e^{\frac{2\pi i}{2N} tn}$$

Proof. Values of $D_{n,j}$ are calculated using DFT. Let

$$c_{t,j} = \sum_{n=-N}^{N-1} C_{n,j} e^{\frac{2\pi i}{2N} tn}$$

then

$$D_{n,j} = \frac{1}{4} \sum_{t=-N/2}^{N/2-1} w_t s_t c_{t,j} e^{-\frac{2\pi i}{2N} tn}$$

In this notation,

$$Q_P(n) = \sum_{k=0}^{\infty} \alpha_F^k \sum_{j=0}^{\infty} x_{kF}^j D_{n+l_{kF},j}.$$

By definition, $r_P(k)$ is inverse DFT of $|Q_P(n)|^2$:

$$r_P(t) =$$
$$\frac{1}{2N} \sum_{n=-N}^{N-1} \left| \sum_{k=0}^{\infty} \alpha_F^k \sum_{j=0}^{\infty} x_{kF}^j D_{n+l_{kF},j} \right|^2 e^{\frac{2\pi i}{2N} in} =$$
$$\frac{1}{2N} \sum_{n=-N}^{N-1} \sum_{k=0}^{\infty} \alpha_F^k \sum_{j=0}^{\infty} x_{kF}^j D_{n+l_{kF},j} \cdot$$
$$\sum_{m=0}^{\infty} \alpha_F^m \sum_{i=0}^{\infty} x_{mF}^i \overline{D}_{n+l_{mF},i} \cdot e^{\frac{2\pi i}{2N} tn}$$

Introduce notation:

$$\rho_{i,j}(t,\tau_1,\tau_2) = \frac{1}{2N} \sum_{n=-N}^{N-1} D_{n+\tau_1,j} \cdot \overline{D}_{n+\tau_2,i} \cdot e^{\frac{2\pi i}{2N} tn}$$
$$= \frac{1}{2N} \sum_{n=-N}^{N-1} D_{(n+\tau_1),j} \cdot \overline{D}_{(n+\tau_1)+\tau_2-\tau_1,i} \cdot e^{\frac{2\pi i}{2N} tn} =$$
$$= e^{-\frac{2\pi i}{2N} t\tau_1} \cdot \frac{1}{2N} \sum_{n=-N}^{N-1} D_{n,j} \cdot \overline{D}_{n+\tau_2-\tau_1,i} \cdot e^{\frac{2\pi i}{2N} tn} =$$
$$= e^{-\frac{2\pi i}{2N} t\tau_1} \cdot \rho_{i,j}(t, \tau_2 - \tau_1)$$

where

$$\rho_{i,j}(t,\tau) = \frac{1}{2N} \sum_{n=-N}^{N-1} D_{n,j} \overline{D}_{n+\tau,i} e^{\frac{2\pi i}{2N} tn}$$

In this notation,

$$r_P(t) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \alpha_F^k \alpha_F^m \cdot$$
$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} x_{kF}^i x_{mF}^j \, \rho_{i,j}(t, l_{mF} - l_{kF}) \, e^{-\frac{\pi i}{N} l_{kF} t}$$

It leads to the theorem conclusion. □

9 Partial Sums Accuracy

Write down an estimation for ρ:

$$|\rho_{i,j}(t,\tau)| \leq \frac{1}{2N}||D_{\cdot,i}|| \cdot ||D_{\cdot,j}|| =$$
$$\frac{1}{16}||s^w \cdot c_{\cdot,i}|| \cdot ||s^w \cdot c_{\cdot,j}|| \leq$$
$$\leq \frac{1}{16}||s^w||^2 \cdot ||c_{\cdot,i}|| \cdot ||c_{\cdot,j}||$$
$$= \frac{2N}{16}||s^w||^2 \cdot ||C_{\cdot,i}|| \cdot ||C_{\cdot,j}||$$

Write down the error term and the error estimation:

$$\delta = \Big| \sum_{k=K+1}^{\infty} \sum_{m=K+1}^{\infty} \alpha_F^k \alpha_F^m \cdot$$
$$\sum_{i=J+1}^{\infty} \sum_{j=J+1}^{\infty} x_{kF}^i x_{mF}^j \, \rho_{i,j}(t, l_{mF} - l_{kF}) \, e^{-\frac{\pi i}{N} l_{kF} t}\Big|$$
$$\leq \frac{2N}{16}\|s^w\|^2 \sum_{k=K+1}^{\infty} \sum_{m=K+1}^{\infty} |\alpha_F|^k |\alpha_F|^m \cdot$$
$$\sum_{i=J+1}^{\infty} \sum_{j=J+1}^{\infty} |x_{kF}|^i |x_{mF}|^j \|C_{\cdot,i}\| \|C_{\cdot,j}\|$$
$$= \frac{2N}{16}\|s^w\|^2 \left(\sum_{k=K+1}^{\infty} |\alpha_F|^k \sum_{i=J+1}^{\infty} |x_{kF}|^i \|C_{\cdot,i}\| \right) \cdot$$
$$\left(\sum_{m=K+1}^{\infty} |\alpha_F|^m \sum_{j=J+1}^{\infty} |x_{mF}|^j \|C_{\cdot,j}\| \right)$$
$$= \frac{2N}{16}\|s^w\|^2 M_{K,J}(F)^2,$$

where

$$M_{K,J}(F) = \sum_{k=K+1}^{\infty} |\alpha_F|^k \sum_{i=J+1}^{\infty} |x_{kF}|^i \|C_{\cdot,i}\| =$$
$$\sum_{i=J+1}^{\infty} \|C_{\cdot,i}\| \left(\sum_{k=K+1}^{\infty} |\alpha_F|^k |x_{kF}|^i \right).$$

Thus,

Lemma 5.

$$|\widehat{r}_P(t) - r_P(t)| \leq$$
$$\leq \frac{2N}{16}\|s^w\|^2 \sum_{i \in \{0,K\}, j \in \{0,J\} \setminus \{0,0\}} M_{i,j}$$

10 Properties Of The $\rho_{i,j}(t,\tau)$

To reduce the computations one may use the following properties of the function $\rho_{i,j}(t,\tau)$:

Statement 1. *Let $\tau \neq 0$, then*

$$\rho_{i,j}(t,\tau) = \overline{\rho_{i,j}(t,-\tau)},$$

where i and j with the same parity and

$$\rho_{i,j}(t,\tau) = -\overline{\rho_{i,j}(t,-\tau)},$$

where i and j with the different parity. For $\tau = 0$ and for any $i \neq j$:

$$\rho_{i,j}(t,0) = \overline{\rho_{j,i}(-t,0)}.$$

11 $\phi_0(P)$ Approximation By Laurent Series

This section provides the final form of the approximation function $\phi_0(P)$ and their accuracy estimation.

By definition, $r_P(\tau)$ is a correlation function of sequence $v_P(t) = s_t w_t^2 g_P(z_P^t)$:

$$r_P(\tau) = \sum_{t=-N/2}^{N/2-1} v_P(t) v_P(t+\tau)^* =$$
$$\sum_{t=-N/2}^{N/2-1} s_t w_t^2 s_{t+\tau} w_{t+\tau}^2 g_P(z_P^t) g_P(z_P^{t+\tau})^*.$$

In calculation of $\phi_0(P)$ values of $r_P(t)$ is involved only for $t = qP$ (by lemma 1). If $\tau = qP$ for integer q, then $z_P^\tau = 1$, thus

$$g_P(z_P^t) g_P(z_P^{t+\tau})^* = g_P(z_P^t) g_P(z_P^t)^* =$$
$$= |g_P(z_P^t)|^2 = \frac{1}{|1 - \alpha_F z_P^t|^2}, \qquad \tau = qP,$$

and this expression is independent of τ. It can be attached to the factors, depend of the t, and further approximate the one geometric progression, instead of two.

Introduce signal notations

$$\widetilde{s}_t = s_t w_t^2, \qquad h_t = \widetilde{s}_t |g_P(t)|^2 = s_t w_t^2 |g_P(t)|^2,$$

and corresponding DFT's

$$\widetilde{S}_n = \sum_{t=-N/2}^{N/2-1} \widetilde{s}_t e^{-\frac{2\pi i}{2N} tn}, \qquad -N \leq n \leq N-1,$$

$$H_n = \sum_{t=-N/2}^{N/2-1} h_t e^{-\frac{2\pi i}{2N} tn}, \qquad -N \leq n \leq N-1.$$

Since the DFT preserves the scalar product of up to a factor $2N$, then

$$r_P(\tau) = \frac{1}{2N} \sum_{n=-N}^{N-1} \widetilde{S}_n^* H_n e^{\frac{2\pi i}{2N} \tau n},$$
$$\tau = qP, \quad |\tau| < N.$$

Generally, $\widetilde{S}_n$ do not depends of P and P enters into H_n only in $|g_P(z_P^t)|^2$. One need to approximate this function.

12 Geometric Progression Decomposition

Lemma 6.

$$r_P(\tau) = \frac{1}{1-\alpha_F^2}\Big[\frac{1}{2N}\sum_{n=-N}^{N-1}|\widetilde{S}_n|^2 e^{\frac{2\pi i}{2N}\tau n} + +2\,\mathrm{Re}\sum_{k=1}^{\infty}\alpha_F^k\frac{1}{2N}\sum_{n=-N}^{N-1}\widetilde{S}_n^* G_{n,k} e^{\frac{2\pi i}{2N}\tau n}\Big],$$

where

$$G_{n,k} = \sum_{t=-N/2}^{N/2-1}\widetilde{s}_t z_P^{kt} e^{-\frac{2\pi i}{2N}tn}.$$

Proof. On the unit circle $g_P(z)^* = g_P(z^{-1})$. Expand the analytic function $g_P(z)g_P(z^{-1})$ in Laurent series on the unit circle:

$$|g_P(z)|^2 = \frac{1}{(1-\alpha_F z)(1-\alpha_F z^{-1})} = = \frac{1}{1-\alpha_F^2}\sum_{k=-\infty}^{\infty}\alpha_F^{|k|}z^k.$$

Thus

$$H_n = \frac{1}{1-\alpha_F^2}\sum_{k=-\infty}^{\infty}\alpha_F^{|k|}G_{n,k},$$

$$G_{n,k} = \sum_{t=-N/2}^{N/2-1}\widetilde{s}_t z_P^{kt} e^{-\frac{2\pi i}{2N}tn}.$$

for $-N \le n \le N-1$. Substitution in the formula for the correlation function $r_P(\tau)$ for $\tau = qP$ gives

$$r_P(\tau) = \frac{1}{1-\alpha_F^2}\sum_{k=-\infty}^{\infty}\alpha_F^{|k|}\frac{1}{2N}\sum_{n=-N}^{N-1}S_n^* G_{n,k} e^{\frac{2\pi i}{2N}\tau n}.$$

Simplify the resulting expression. Obviously, $G_{n,0} = S_n$. For $k \ge 1$ from the definitions

$$S_{-n} = S_n^*, \qquad G_{-n,-k}^* = G_{n,k}.$$

Therefore

$$\left(\sum_{n=-N}^{N-1}\widetilde{S}_n^* G_{n,-k} e^{\frac{2\pi i}{2N}\tau n}\right)^* = \sum_{n=-N}^{N-1}\widetilde{S}_n^* G_{n,k} e^{\frac{2\pi i}{2N}\tau n}.$$

Hence, for the formula for function $r_P(\tau)$ it is possible to use only positive indices k:

$$r_P(\tau) = \frac{1}{1-\alpha_F^2}\Big[\frac{1}{2N}\sum_{n=-N}^{N-1}|\widetilde{S}_n|^2 e^{\frac{2\pi i}{2N}\tau n} + +2\,\mathrm{Re}\sum_{k=1}^{\infty}\alpha_F^k\frac{1}{2N}\sum_{n=-N}^{N-1}\widetilde{S}_n^* G_{n,k} e^{\frac{2\pi i}{2N}\tau n}\Big].$$

□

13 The Exponent Decomposition

Lemma 7. *For $\tau = qP$*

$$r_P(\tau) = \frac{1}{1-\alpha_F^2}\Big[\frac{1}{2N}\sum_{n=-N}^{N-1}|\widetilde{S}_n|^2 e^{\frac{2\pi i}{2N}\tau n} + + 2\,\mathrm{Re}\sum_{k=1}^{\infty}\alpha_F^k\sum_{j=0}^{\infty}\frac{(-\pi i x_{2Fk}/2)^j}{j!}\rho_{\ell_{2Fk},j}(\tau)\Big].$$

Proof. Let $k \ge 1$. In power of function

$$z_P^{kt} = e^{-\frac{2\pi i}{2N}2Fkt}$$

select the whole and fractional parts with respect to DFT of $2N$:

$$2Fk = \ell_{2Fk} + x_{2Fk}, \qquad |x_{2Fk}| \le 1/2.$$

The factor with a fractional part expanded in a Taylor series at zero, which gives

$$G_{n,k} = \sum_{t=-N/2}^{N/2-1}\widetilde{s}_t z_P^{kt} e^{-\frac{2\pi i}{2N}tn} = \sum_{t=-N/2}^{N/2-1}\widetilde{s}_t e^{-\frac{2\pi i}{2N}t(n+\ell_{2Fk})}\sum_{j=0}^{\infty}\frac{1}{j!}\left(-\frac{\pi i t}{N}x_{2Fk}\right)^j = \sum_{j=0}^{\infty}\frac{(-\pi i x_{2Fk}/2)^j}{j!}\sum_{t=-N/2}^{N/2-1}\widetilde{s}_t\left(\frac{2t}{N}\right)^j e^{-\frac{2\pi i}{2N}t(n+\ell_{2Fk})}.$$

Introduce the following notations:

$$F_{n,j} = \sum_{t=-N/2}^{N/2-1}\widetilde{s}_t\left(\frac{2t}{N}\right)^j e^{-\frac{2\pi i}{2N}tn}, \quad -N \le n \le N-1,\ j \ge 0.$$

$$\rho_{\ell,j}(\tau) = \frac{1}{2N}\sum_{n=-N}^{N-1}\widetilde{S}_n^* F_{n+\ell,j} e^{\frac{2\pi i}{2N}\tau n}, \quad j \ge 0, \quad \ell \ge 0,$$

moreover $F_{n,j}$ periodically continues by n, so the convolution is circular. It follows from the definition that $F_{n,0} = \widetilde{S}_n$ is DFT of $\widetilde{s}_t$.

Thus, for $\tau = qP$

$$r_P(\tau) = \frac{1}{1-\alpha_F^2}\Big[\frac{1}{2N}\sum_{n=-N}^{N-1}|\widetilde{S}_n|^2 e^{\frac{2\pi i}{2N}\tau n} +$$
$$2\,\mathrm{Re}\sum_{k=1}^{\infty}\alpha_F^k\sum_{j=0}^{\infty}\frac{(-\pi i x_{2Fk}/2)^j}{j!}\rho_{\ell_{2Fk},j}(\tau)\Big].$$

Note that $|\pi i x_{2Fk}/2| \le \pi/4 < 0.8$, so the terms are rapidly decreasing on j. □

14 Final Approximation

Approximation of the correlation function r_P is based on the selection of a set of pairs of non-negative indices $M = \{(\ell, j)\}$ for $\rho_{\ell,j}(\tau)$ calculation:

Theorem 3.

$$\widehat{r}_{P,M}(\tau) = \frac{1}{1-\alpha_F^2}\Big[\rho_{0,0}(\tau) +$$
$$+2\,\mathrm{Re}\sum_{(k,j):(\ell_{2Fk},j)\in M}\alpha_F^k\frac{(-\pi i x_{2Fk}/2)^j}{j!}\rho_{\ell_{2Fk},j}(\tau)\Big],$$

where F —frequency with relation to N: $F = N/P$,

$$2Fk = \ell_{2Fk} + x_{2Fk}, \qquad |x_{2Fk}| \le 1/2.$$

and the total number of Fourier transformations required for computation of the approximation is equal to

$$N_{\mathrm{fft}} = |M| + J_{\max} + 1,$$

Proof. Expression for $\widehat{r}_{P,M}(\tau)$ is based on previous lemma. $2N$ DFT are computed for each element of (ℓ, j) from set M to compute $\rho_{\ell,j}(\cdot)$. And for each $j \le J_{\max}$ when calculating $F_{\cdot,j}$. Here $J_{\max}$ — maximum value of j in set M. Thus, overall number of DFT's are

$$N_{\mathrm{fft}} = |M| + J_{\max} + 1,$$

where $|M|$ — number of elements in the set M. □

To improve the quality of approximation, it makes sense to include in the set M all pairs (ℓ, j) with $0 \le j \le J(\ell) - 1$ for fixed first component ℓ. We assume that this condition is satisfied.

In particular, the condition $J(\ell) = 0$ means that no pair of the form (ℓ, j) are included in set M. It's obvious that

$$|M| = \sum_{\ell=0}^{\infty} J(\ell).$$

The initial task of searching pitch period P has been reduced by Lemma 1 to calculation of the $\phi_0(P)$ for correlation function $r_P(\tau)$.

These approximation for function r_P make it possible to calculate the following approximations of the objective functions $\phi_0(P)$. Let P — integer and $F = N/P > 1$. Then

$$\widehat{\phi}_0(P) = \widehat{r}_{P,M}(0) + 2\sum_{q=1}^{\lfloor F\rfloor}\widehat{r}_P(qP)$$
$$= \frac{1}{1-\alpha_F^2}\Big\{\rho_{0,0}(0) + 2\sum_{q=1}^{\lfloor F\rfloor}\rho_{0,0}(qP) +$$
$$\sum_{(k,j):(\ell_{2Fk},j)\in M}\alpha_F^k\,2\,\mathrm{Re}\left[\frac{(-\pi i x_{2Fk}/2)^j}{j!}\rho_{\ell_{2Fk},j}(0)\right]$$
$$+\sum_{(k,j):(\ell_{2Fk},j)\in M}\alpha_F^k\,4\,\mathrm{Re}\cdot$$
$$\left[\frac{(-\pi i x_{2Fk}/2)^j}{j!}\sum_{q=1}^{\lfloor F\rfloor}\rho_{\ell_{2Fk},j}(qP)\right]\Big\}.$$

15 Accuracy Estimation

The maximum number of ℓ for which $(\ell, 0) \in M$ is denoted by L. Let P — integer and $F = N/P$. Maximum integer k, for which $2Fk \le L + 0.5$, denoted by $K(P)$. Obviously,

$$K(P) = \left\lfloor\frac{P(2L+1)}{4N}\right\rfloor.$$

The next result gives an estimation of error for approximation from theorem 3:

Theorem 4. *Approximation accuracy for ϕ_0 is:*

$$|\phi_0(P) - \widehat{\phi}_0(P)| \le$$
$$\sum_{t=-N/2}^{N/2-1}\gamma_P(t)|\widetilde{s}_t|\left(|\widetilde{s}_t| + 2\sum_{q=1}^{\lfloor \frac{t}{P}+\frac{F}{2}\rfloor}|\widetilde{s}_{t-qP}|\right),$$

where

$$\gamma_P(t) = \frac{2}{1-\alpha_F^2}\Big[\sum_{k=1}^{K(P)}|\alpha_F|^k\frac{1}{J(\ell_{2Fk})!}\left|\frac{\pi t}{N}x_{2Fk}\right|^{J(\ell_{2Fk})}$$
$$+|\alpha_F|^{K(P)+1}\frac{1+|\alpha_F|}{|1+\alpha_F z_P^t|^2}\Big],$$

Proof. Estimation error

$$r_P(\tau) - \widehat{r}_{P,M}(\tau) = \delta_\alpha(\tau) + \delta_x(\tau)$$

consists of the truncation error and geometric error of Taylor polynomial:

$$\delta_\alpha(\tau) = \frac{1}{1-\alpha_F^2} 2\,\mathrm{Re} \sum_{k=K(P)+1}^{\infty} \alpha_F^k \frac{1}{2N} \sum_{n=-N}^{N-1} \widetilde{S}_n^* G_{n,k} e^{\frac{2\pi i}{2N}\tau n},$$

$$\delta_x(\tau) = \frac{1}{1-\alpha_F^2} 2\,\mathrm{Re} \sum_{k=1}^{K(P)} \alpha_F^k \frac{1}{2N} \cdot \sum_{n=-N}^{N-1} \widetilde{S}_n^* (G_{n,k} - \widehat{G}_{n,k}) e^{\frac{2\pi i}{2N}\tau n},$$

where for $\tau = qP$

$$G_{n,k} = \sum_{t=-N/2}^{N/2-1} \widetilde{s}_t z_P^{kt} e^{-\frac{2\pi i}{2N}tn} = \sum_{t=-N/2}^{N/2-1} \widetilde{s}_t e^{-\frac{2\pi i}{2N}t(n+\ell_{2Fk})} e^{-\frac{2\pi i}{2N}x_{2Fk}t},$$

$$\widehat{G}_{n,k} = \sum_{t=-N/2}^{N/2-1} \widetilde{s}_t e^{-\frac{2\pi i}{2N}t(n+\ell_{2Fk})} \sum_{j=0}^{J(\ell_{2Fk})-1} \frac{1}{j!} \left(-\frac{\pi i t}{N} x_{2Fk}\right)^j.$$

Truncation error of geometric progression collapses as a geometric progression again:

$$\delta_\alpha(\tau) = \frac{1}{1-\alpha_F^2} 2\,\mathrm{Re} \sum_{k=K(P)+1}^{\infty} \alpha_F^k \frac{1}{2N} \cdot \sum_{n=-N}^{N-1} \widetilde{S}_n^* e^{\frac{2\pi i}{2N}\tau n} \sum_{t=-N/2}^{N/2-1} \widetilde{s}_t z_P^{kt} e^{-\frac{2\pi i}{2N}tn}$$
$$= \frac{\alpha_F^{K(P)+1}}{1-\alpha_F^2} 2\,\mathrm{Re} \frac{1}{2N} \cdot \sum_{n=-N}^{N-1} \left[\widetilde{S}_n^* \sum_{t=-N/2}^{N/2-1} \widetilde{s}_t \frac{z_P^{(K(P)+1)t}}{1-\alpha_F z_P^t} e^{-\frac{2\pi i}{2N}tn} \right] e^{\frac{2\pi i}{2N}\tau n}.$$

DFT preserves the scalar product with a constant factor and $(S_n e^{-\frac{2\pi i}{2N}\tau n})_{n=-N}^{N-1}$ is DFT on the vector $(s_{t-\tau})_{t=-N/2}^{N/2-1}$. Thus

$$\delta_\alpha(\tau) = \frac{\alpha_F^{K(P)+1}}{1-\alpha_F^2} 2\,\mathrm{Re} \sum_{t=-N/2}^{N/2-1} \widetilde{s}_{t-\tau} \widetilde{s}_t \frac{z_P^{(K(P)+1)t}}{1-\alpha_F z_P^t}.$$

For brevity $\kappa = K(P)+1$. Then

$$\delta_\alpha(\tau) = \frac{\alpha_F^\kappa}{1-\alpha_F^2} \cdot \sum_{t=-N/2}^{N/2-1} \widetilde{s}_{t-\tau} \widetilde{s}_t \left[\frac{z_P^{\kappa t}}{1-\alpha_F z_P^t} + \frac{z_P^{-\kappa t}}{1-\alpha_F z_P^{-t}} \right]$$
$$= \frac{\alpha_F^\kappa}{1-\alpha_F^2} \sum_{t=-N/2}^{N/2-1} \widetilde{s}_{t-\tau} \widetilde{s}_t \frac{2\,\mathrm{Re}(z_P^{(\kappa-1)t} - \alpha_F z_P^{\kappa t})}{|1-\alpha_F z_P^t|^2}.$$

Hence,

$$|\delta_\alpha(\tau)| \le \frac{2|\alpha_F|^{K(P)+1}}{1-|\alpha_F|} \sum_{t=-N/2}^{N/2-1} \frac{|\widetilde{s}_{t-\tau}|\,|\widetilde{s}_t|}{|1-\alpha_F z_P^t|^2}.$$

Taylor polynomial accuracy for exponent with pure imaginary index has the form

$$\left| e^{ix} - \sum_{j=0}^{n-1} \frac{(ix)^j}{j!} \right| \le \frac{|x|^n}{n!}.$$

For $1 \le k \le K(P)$ apply this inequality to function $e^{-\frac{2\pi i}{2N}x_{2Fk}t}$. This leads to the upper bound

$$|\delta_x(\tau)| \le \frac{2}{1-\alpha_F^2} \sum_{k=1}^{K(P)} |\alpha_F|^k \cdot \sum_{t=-N/2}^{N/2-1} |\widetilde{s}_{t-\tau}|\,|\widetilde{s}_t| \frac{1}{J(\ell_{2Fk})!} \left| \frac{\pi t}{N} x_{2Fk} \right|^{J(\ell_{2Fk})}.$$

As a result, when $q \ge 0$

$$|r_P(qP) - \widehat{r}_{P,M}(qP)| \le \sum_{t=-N/2}^{N/2-1} |\widetilde{s}_{t-qP}|\,|\widetilde{s}_t| \gamma_P(t),$$

where

$$\gamma_P(t) = \frac{2}{1-\alpha_F^2} \cdot \Big[\sum_{k=1}^{K(P)} |\alpha_F|^k \frac{1}{J(\ell_{2Fk})!} \left| \frac{\pi t}{N} x_{2Fk} \right|^{J(\ell_{2Fk})} + |\alpha_F|^{K(P)+1} \frac{1+|\alpha_F|}{|1+\alpha_F z_P^t|^2} \Big],$$

and this expression is independent of q.

The approximation error of a function ϕ_0 is estimated from above as follows:

$$|\phi_0(P) - \widehat{\phi}_0(P)| \le \sum_{t=-N/2}^{N/2-1} \gamma_P(t) |\widetilde{s}_t| \left(|\widetilde{s}_t| + 2 \sum_{q=1}^{\lfloor \frac{t}{P} + \frac{F}{2} \rfloor} |\widetilde{s}_{t-qP}| \right).$$

□

16 Guaranteed Estimation

Estimation accuracy of ϕ_0 is compared with the vector norm $s_w = (s_{w,t})_{t=-N/2}^{N/2-1}$. By definition, $\widetilde{s}_t = s_{w,t}w_t = s_t w_t^2$, where $(w_t)_{t=-N/2}^{N/2-1}$ — Hanning window.

Find the minimum number $\lambda > 0$, for which

$$\sum_{t=-N/2+1}^{N/2-1} \gamma_P(t)|\widetilde{s}_t| \left(|\widetilde{s}_t| + 2\sum_{q=1}^{\lfloor \frac{t}{P}+\frac{F}{2}\rfloor} |\widetilde{s}_{t-qP}|\right) \le \lambda \|s_w\|^2$$

for any input signal s. The left hand side is a quadratic form, denote it $f(s_w)$. The required number of λ is the norm of the matrix of the quadratic form.

Form $f(s_w)$ has a number of features that facilitate its study. The sum can be rearranged in such a way that in each term were the product of signals with the indices of the same equivalence class by modulo P. Perform this conversion.

For each residue $k = 0, 1, \dots, P-1$ the minimal number from $-N/2+1 \le t \le N/2-1$, comparable with k, denote by t_k^0. Amount of comparable to k numbers from this interval denote by N_k. Then

$$f(s_w) = \sum_{k=0}^{P-1}\sum_{n=0}^{N_k-1} \gamma_P(t_k^0+nP)\widetilde{s}_{t_k^0+nP} \cdot \left(\widetilde{s}_{t_k^0+nP} + 2\sum_{q=1}^{n} \widetilde{s}_{t_k^0+(n-q)P}\right).$$

Hence we see that the initial problem of guaranteed estimation falls into P independent subtasks:

$$\sum_{n=0}^{N_k-1} \gamma_P(t_k^0+nP)\widetilde{s}_{t_k^0+nP} \cdot \left(\widetilde{s}_{t_k^0+nP} + 2\sum_{q=1}^{n} \widetilde{s}_{t_k^0+(n-q)P}\right) \le \le \lambda_k \sum_{n=0}^{N_k-1} s^2_{w,t_k^0+nP}, \qquad 0 \le k \le P-1.$$

Next as a common factor λ the great of λ_k can be chosen. On practice λ can be selected as weighted average values of λ_k with taking into account the signal intensities.

Each particular problem is solved with the help of the following lemma, which is substituted $x_n = \widetilde{s}_{t_k^0+nP}$, $c_n = w_{t_k^0+nP}$, $d_n = \gamma_P(t_k^0+nP)$, $N = N_k - 1$.

Lemma 8. *Suppose positive numbers $(c_n)_{n=0}^N$ and $(d_n)_{n=0}^N$ are given. Minimum value of λ, for which*

$$\sum_{n=0}^N d_n x_n \left(x_n + 2\sum_{q=1}^n x_{n-q}\right) \le \lambda \sum_{n=0}^N \frac{x_n^2}{c_n^2}, \quad \forall\, x = (x_n)_{n=0}^N,$$

coincides with the norm of the self-adjoint matrix $A = (a_{i,j})_{i,j=0}^N$ with elements

$$a_{i,j} = c_i c_j d_{\max\{i,j\}}, \qquad 0 \le i, j \le N.$$

Proof. The left side of the inequality in the lemma is

$$\sum_{n=0}^N d_n x_n \left(x_n + 2\sum_{q=1}^n x_{n-q}\right) = = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix}^* \begin{pmatrix} d_0 & d_1 & \cdots & d_N \\ d_1 & d_1 & \cdots & d_N \\ \vdots & \vdots & \ddots & \vdots \\ d_N & d_N & \cdots & d_N \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_N \end{pmatrix}.$$

Perform the change of variable $x_n = c_n y_n$. Then the inequality in the lemma is equivalent to the:

$$y^* A y \le \lambda \|y\|^2.$$

Matrix A is self-adjoint by construction, therefore, the minimum value λ coincides with $\|A\|$. □

If $P \ge N/3$, then $N_k \le 2$ for all k and P. Thus, lemma 8 must be applied for $N = 0$, $N = 1$ and $N = 2$. In the first two cases, the problem can be solved analytically. In the last case, a cubic equation, for which one can pick up an analytical approximate solution.

Finally, it is possible to obtain an accuracy estimation depending only on signal energy:

Theorem 5. *For all $P \in [P_{\min}, P_{\max}]$*

$$|\phi_0(P) - \widehat{\phi}_0(P)| \le \lambda \|s_w\|^2,$$

where

$$\lambda = \max_{0 \le k \le P-1} \lambda_k, \quad \lambda_k = \|A\|,$$
$$a_{i,j} = c_i c_j d_{\max\{i,j\}}, \quad 0 \le i, j \le N_k - 1,$$

and

$$c_n = w_{t_k^0+nP}, \quad d_n = \gamma_P(t_k^0+nP).$$

17 Conclusion

In this work the main statements and results for NVM algorithm is presented. Previous work (Barabanov et al., 2015) only describes a part of this method, without any statements proof.

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