STABILITY PRESEVVATION PROBLEM IN THE METHODS THAT FIND RATIONAL APPROXIMATION OF FRACTIONAL ORDER SYSTEMS

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Abstract
This paper deals with stability preservation issue in methods which present an integer order model as an approximation of a fractional order system. First, the stability region of the approximating model obtained in these methods is determined and compared with the stability region of the original system. Then, one of the popular methods in this category (Charef's method) is studied from the stability preservation point of view.

Key words
Fractional order system, Approximation, Stability, Charef's method.

1 Introduction

Fractional calculus as an extension of ordinary calculus is a mathematical topic with more than 300 years old history. Even though fractional calculus has a long history, its application to physics and engineering has been attracted lots of attention only in the last few decades. It has been found that many real world physical systems can be described by fractional differential equations. Some examples from fractional order dynamics can be found in [Podlubny, 1999b; Hilfer, 2001] and references therein. Also, in recent years fractional order dynamic systems have been widely studied in the design and practice of control systems (for example [Podlubny, 1999a; Feliu-Batlle, Rivas Perez, and Sanchez Rodriguez, 2007; Tavazoei and Haeri, 2008; Calderon, Vinagre and Feliu, 2006]).

Although the integer order models can be considered as a special form of the more general fractional order models, there are basic differences between fractional order and integer order models. The main difference between them arises from inherent attribute of fractional derivatives. In fact, in contrary to the integer derivatives, the fractional derivatives are not local operators [Podlubny, 1999b]. In other words, the fractional derivative of a function depends on its whole past values. This property makes a fractional order model to behave like a system with an "infinite memory" or "long memory". Due to this property, the fractional order models are not easy to simulate or implement [Oustaloup, 1995]. Use of the rational approximations of the fractional order models is a way to unravel this difficulty. For instance, to implement a fractional order controller, it is common to replace the controller by its integer order approximation [Vinagre, Podlubny, Hernandez, Feliu, 2000; Petras, Podlubny, O'Leary, Dorcak and Vinagre, 2002; Charef, 2006]. These rational approximations have also been widely used in simulation of the fractional order systems [Aoun, Malti, Levron, and Oustaloup, 2004; Hartley, Lorenzo and Qammer, 1995; Li and Chen, 2004].

In this paper, we give a brief discussion about stability preservation in the methods that approximate a fractional order system with an integer order model. The paper is organized as follows. In Section 2, we find the stability region for the approximating models obtained using the rational approximation of the fractional operators in the frequency domain. The obtained stability region is compared with the stability region of the original system. The critical regions in which the stability does not preserve are determined. A more rigorous stability analysis on one of the popular methods of finding the rational approximation of a fractional operator i.e. the Charef's method is done in Section 3. Finally, conclusions in Section 4 close the paper.

2 Stability Analysis for Integer Order Approximation of Fractional Order Systems

In this section, we discuss about the stability of integer order approximation of a fractional order system and compare it with the stability of the original system. Suppose that the original system is described by a fractional order model such as

$$T(s^\alpha) = \frac{B(s^\alpha)}{A(s^\alpha)} = \frac{b_m (s^\alpha)^m + b_{m-1}(s^\alpha)^{m-1} + \cdots + b_0}{a_n (s^\alpha)^n + a_{n-1}(s^\alpha)^{n-1} + \cdots + a_0} \quad (1)$$

where $0 < \alpha < 1$, and $A(s)$ and $B(s)$ are coprime polynomials. We know that $T(s^\alpha)$ is BIBO stable if and only if $|\arg(s)| > \alpha\pi/2$ for every $s \in \mathbb{C}$ such that $A(s) = 0$ [Matignon, 1996]. The stable and unstable regions for fractional order system (1) have been shown in Fig. 1. Now, assume that we approximate system (1) using rational approximation of the fractional operator s^α. Let the approximating filter for the operator s^α has the transfer function $G(s) = p(s)/q(s)$, where $p(s)$ and $q(s)$ are two coprime polynomials with real coefficients and $q(s)$ has no root with nonnegative real part. Using this approximating filter, the original system is approximated by

$$T\left(\frac{p(s)}{q(s)}\right) = \frac{B\left(\frac{p(s)}{q(s)}\right)}{A\left(\frac{p(s)}{q(s)}\right)} = \frac{b_m \left(\frac{p(s)}{q(s)}\right)^m + b_{m-1}\left(\frac{p(s)}{q(s)}\right)^{m-1} + \cdots + b_0}{a_n \left(\frac{p(s)}{q(s)}\right)^n + a_{n-1}\left(\frac{p(s)}{q(s)}\right)^{n-1} + \cdots + a_0} \quad (2)$$

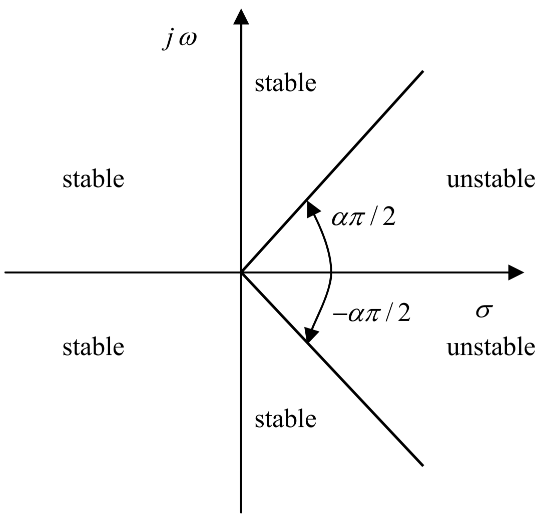

Figure 1. Stability region of system (1).

Since the approximating filter $G(s) = p(s)/q(s)$ is stable, $B(p(s)/q(s))$ has no pole with nonnegative real part. Also, $A(p(s)/q(s))$ and $B(p(s)/q(s))$ have no common root. Thus, $T(p(s)/q(s))$ is stable if and only if $A(p(s)/q(s))$ has no zero with nonnegative real part. For an arbitrary set $M \subseteq \mathbb{C}$, we define

$$f(M) = \{f(s) \mid s \in M\} \quad (3)$$

It can be easily verified that $T(p(s)/q(s))$ is stable if and only if equation $A(s) = 0$ has no root in the area $G(D)$, where

$$D = \{s \in \mathbb{C} \mid \mathrm{Re}\{s\} \geq 0\} \quad (4)$$

Since $G(s) = p(s)/q(s)$ is an analytic function over D, for obtaining the area $G(D)$ it suffices to map the boundary of D under function $G(s) = p(s)/q(s)$. Let γ be the boundary of D. We may partition γ to two curves γ_1 and γ_2 which are defined as follows,

$$\gamma_1 = \{s \in \mathbb{C} \mid s = j\omega, -\infty < \omega < \infty\} \quad (5)$$

$$\gamma_2 = \{s \in \mathbb{C} \mid s = re^{j\theta}, -\pi/2 \leq \theta \leq \pi/2, r \to \infty\} \quad (6)$$

When $p(s)$ and $q(s)$ have the same degree, $G(\gamma_2)$ is a constant number. Also since $p(s)$ and $q(s)$ have real coefficients, $G(\gamma_1)$ is a symmetrical curve with respect to the real axis of the s-plane. Therefore, the curve $G(\gamma_1)$ is the boundary between stable and unstable regions of the approximated model according to location of denominator roots of the original system. Fig. 2 schematically shows the instability region for the approximated model. If the original system is unstable but its unstable poles are not in the region $G(D)$, its approximated model will be stable. Also, when the boundary of $G(D)$ crosses the lines $|\arg(s)| = \alpha\pi/2$, the approximated model could be unstable, while the original system is stable. These two forms of inconsistencies may occur when we use the frequency based approximation methods.

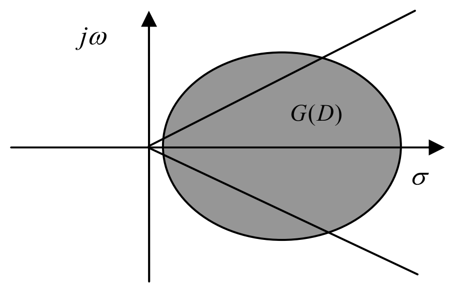

Figure 2. Instability region of approximation model.

3 Stability Preservation Problem in Charef Method

In this section, we investigate the stability preservation problem for the Charef's method [Charef, 2006]. In this method, the approximation for operator s^α with maximum discrepancy y dB in the frequency band (ω_L, ω_H) is given by

$$G(s) = K_D \prod_{k=0}^{N} \frac{(1 + s/z_k)}{(1 + s/p_k)} \quad (7)$$

where the poles and zeros of (7) are found to be in a geometric progression. The first zero, z_0, is chosen using the following equation

$$z_0 = \omega_c 10^{(y/20\alpha)} \quad (8)$$

where $\omega_c = \omega_L \sqrt{10^{(\varepsilon/10\alpha)} - 1}$ and ε is the maximum permitted error between the slopes of s^α and its fractional power zero in the frequency band (ω_L, ω_H). The gain K_D is selected as $K_D = \omega_c^\alpha$. By fixing the first zero, z_0, the other zeros and poles are determined as

$$z_k = (ab)^k z_0, \quad k = 1, 2, ..., N \quad (9)$$

$$p_k = a(ab)^k z_0, \quad k = 0, 1, ..., N \quad (10)$$

where $a = 10^{(y/10(1-\alpha))}$ and $b = 10^{(y/10\alpha)}$. Also, the number of poles and zeros is ($N+1$), where

$$N = \left[\frac{\log(\omega_{\max}/z_0)}{\log(ab)} \right] + 1 \quad (11)$$

and $\omega_{\max} = 100\omega_H$.

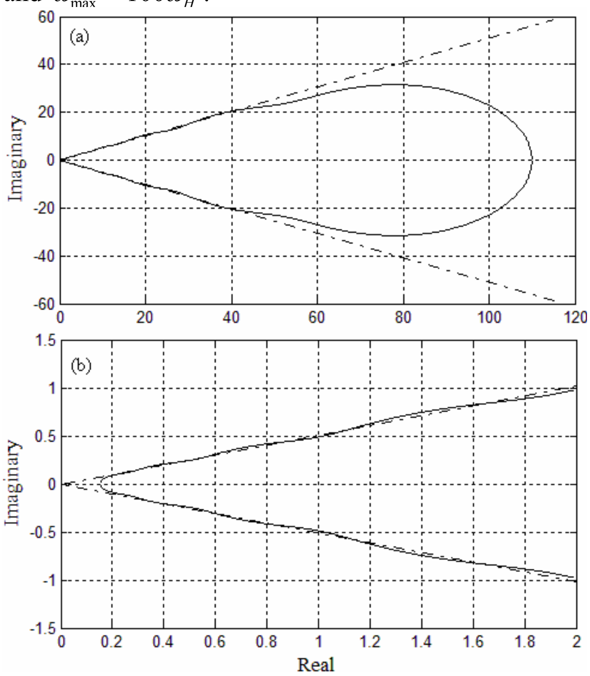

Figure 3. Stability boundary for approximated model, (a): all frequencies (b): low frequencies. The dashed lines indicate the original stability boundary.

As an example, Fig. 3 illustrates the stability boundary for approximated models constructed based on the Charef's method in a special case ($\alpha = 0.3$, $(\omega_L, \omega_H) = (10^{-3}, 10^3)$, $\varepsilon = 10^{-5}$, and $y = 2dB$). It is clear that if the denominator polynomial of original system has a root in critical regions described in the previous section, the original system and its approximated model are not the same in the sense of stability. In other words, the stable original system may have unstable approximated model and the vice versa.

Now, let us investigate the stability preservation issue in the Charef's method more rigorously. Suppose

$$\exists \delta > 0: \ |\measuredangle G(j\omega) - \alpha\pi/2| < \delta \quad (12)$$

for $\omega_L < \omega < \omega_H$. If $\delta < \alpha\pi/2$, (12) can be written as

$$\tan(\alpha\frac{\pi}{2} - \delta) < \frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))} < \tan(\alpha\frac{\pi}{2} + \delta), \quad \omega_L < \omega < \omega_H \quad (13)$$

The inequalities given in (13) guarantee that the stability boundary settles in the sector $|\measuredangle s - \alpha\pi/2| < \delta$. Therefore, the allowable phase error of the approximating filter has a very effective role in the accuracy of the stability boundary.

According to (7), phase of the approximating filter is

$$\measuredangle G(j\omega) = \sum_{k=0}^{N} \left(\tan^{-1}(\omega/z_k) - \tan^{-1}(\omega/p_k) \right) \quad (14)$$

Define

$$\measuredangle G_k(j\omega) = \tan^{-1}(\omega/z_k) - \tan^{-1}(\omega/p_k) \quad (15)$$

for $0 \le k \le N$. It can be easily verified that

$$\measuredangle G_{k+1}(j\omega) = \measuredangle G_k(j\omega/(ab)) \quad (16)$$

Now, we define $F_k(j\omega)$ by

$$F_k(j\omega) = G_k(j10^\omega) \quad (17)$$

From (16) and (17),

$$F_{k+1}(j\omega) = F_k(j(\omega - \log(ab))) \quad (18)$$

Therefore, one can write

$$\measuredangle F(j\omega) = \sum_{k=0}^{N} \measuredangle F_k(j\omega) = \sum_{k=0}^{N} \measuredangle F_0(j(\omega - k\log(ab))) \quad (19)$$

It is straightforward to prove that $\measuredangle F_k(j\omega)$ is a symmetrical function with respect to the line $\omega = \log(\sqrt{z_k p_k})$.

In [Charef, Sun, Tsao and Onaral, 1992], it has been shown that if the number of poles and zeros tends to infinity i.e. $N \to \infty$ or equivalently $ab \to 1$, the slope of amplitude of approximation (7) is the same as that of the original operator s^α i.e. $20\alpha\, dB/dec$. Now, we analyze the phase of approximation (7) when the number of poles and zeros tends to infinity. From (19), we have

$$\lim_{ab\to 1} \measuredangle F(j\omega) = \lim_{ab\to 1} \left(\sum_{k=0}^{N} \measuredangle F_0(j(\omega - k\log(ab)))\right) = \lim_{ab\to 1} \frac{\int_{\omega - N\log(ab)}^{\omega} \measuredangle F_0(j\bar{\omega})d\bar{\omega}}{\log(ab)} \quad (20)$$

According to (11),

$$\lim_{ab\to 1} N\log(ab) = \log(\omega_{\max}/z_0) \quad (21)$$

Therefore,

$$\lim_{ab\to 1} \measuredangle F(j\omega) = \lim_{ab\to 1} \frac{\int_{\omega - \log(\omega_{\max}/z_0)}^{\omega} \measuredangle F_0(j\bar{\omega})d\bar{\omega}}{\log(ab)} \quad (22)$$

Since $F_0(j\omega)$ is a symmetrical function with respect to the line $\omega = \log(\sqrt{z_0 p_0})$, the maximum of $\int_{\omega - \log(\omega_{\max}/z_0)}^{\omega} \measuredangle F_0 \quad (j\bar{\omega})d\bar{\omega}$ occurs at

$\omega = \log\sqrt{z_0 p_o} + 0.5\log(\omega_{\max}/z_0)$. Also, it is obvious that the maximum of $\measuredangle G_k(j\omega)$ in $\omega \in (z_0, \omega_{\max})$ is equal to the maximum of $\measuredangle F_k(j\omega)$ in $\omega \in (\log z_0, \log \omega_{\max})$. Thus, the maximum of $\measuredangle G_k(j\omega)$ where $N \to \infty$ (i.e. $ab \to 1$) equals to

$$\max \lim_{ab\to 1}\{\measuredangle G(j\omega)\} = \lim_{ab\to 1} \frac{\int_{\log\sqrt{z_0 p_0} - 0.5\log(\omega_{\max}/z_0)}^{\log\sqrt{z_0 p_0} + 0.5\log(\omega_{\max}/z_0)} \measuredangle F_0(j\varpi)d\varpi}{\log(ab)} = \lim_{ab\to 1} \frac{2\int_{\log z_0}^{\log\sqrt{z_0\omega_{\max}}} [\tan^{-1}(10^{\varpi}/z_0) - \tan^{-1}(10^{\varpi}/az_0)]d\varpi}{\log(ab)} \quad (23)$$

Let assume $\eta = 10^{\varpi}$ and use the equality $a = (ab)^{\alpha}$, then (23) can be written as

$$\max \lim_{ab\to 1}\{\measuredangle G(j\omega)\} = \lim_{ab\to 1} \frac{\int_{z_0}^{\sqrt{z_0\omega_{\max}}} \frac{2}{\eta \ln 10}[\tan^{-1}(\eta/z_0) - \tan^{-1}(\eta/(ab)^{\alpha} z_0)]d\eta}{\log(ab)} \quad (24)$$

Using the first L'Hopital rule, (24) equals to

$$\max \lim_{ab\to 1}\{\measuredangle G(j\omega)\} = 2\alpha\left(\tan^{-1}\left(\sqrt{\frac{\omega_{\max}}{z_0}}\right) - \frac{\pi}{4}\right) \quad (25)$$

Equation (25) gives an interesting property for the approximating filters resulted from the Charef's method when $N \to \infty$. According to equation (25), we know that

$$\max \lim_{ab\to 1}\{\measuredangle G(j\omega)\} < \alpha\frac{\pi}{2} \quad (26)$$

This means that the instability region of the original system includes the instability region of its approximated model. In other words, while the stable fractional order systems necessarily have stable approximating model if $N \to \infty$, unstable fractional order systems may have stable approximating models. Fig. 4 shows the maximum phase of the approximating filters resulted from the Charef's method for fractional operator $s^{0.3}$ with different orders.

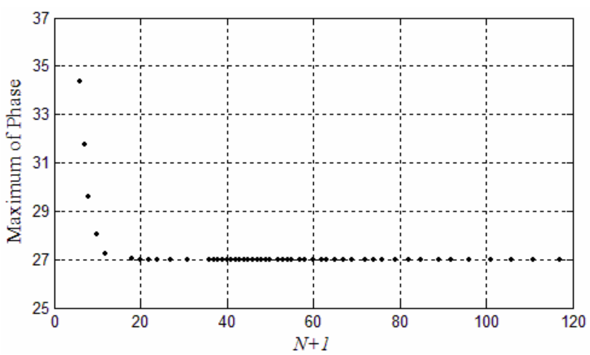

Figure 4: Maximum phase of the approximating filters for fractional operator $s^{0.3}$ for different numbers of poles and zeros.

4 Conclusion

In this paper, we investigated the stability problem for integer order approximation of commensurate fractional order transfer functions. We showed that the stability boundary of the approximating model of a fractional order system, achieved via using filter $G(s)$ to approximate the fractional differentiator, is the curve $\{G(j\omega) \mid -\infty < \omega < \infty\}$ in the complex plane. This point emphasizes the importance of the phase accurateness of the approximating filter in the stability preservation of the frequency domain methods of finding rational approximation for fractional order systems. It was demonstrated that if the approximating filter is not chosen properly, the original system and its approximated model may be different in the sense of stability. In this case, the numerical simulation results are not validated and may lead to wrong consequences. Some of these wrong conclusions presented in the previous literature have been reported in [Tavazoei and Haeri, 2007a; Tavazoei and Haeri, 2007b]. Also, in this paper we analyzed the Charef's method in the sense of stability preservation. We showed that when number of poles and zeros in approximating filters resulted from the Charef's method tends to infinity, the approximation model of a stable fractional order system is necessarily stable.

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References

Aoun, M., Malti, R., Levron, F., and Oustaloup, A., (2004). Numerical simulations of fractional systems: An overview of existing methods and improvements, *Nonlinear Dynamics*, **38**, pp. 117-131.

Brown, J.W., and Churchill, R.V., (2004). *Complex Variables and Applications*, 7th Ed, McGraw-Hill, New York.

Calderon A.J., Vinagre B.M. and Feliu V., (2006). Fractional-order control strategies for power electronic buck converters, *Signal Processing,* **86**, pp. 2803-2819.

Charef A., (2006). Analogue realization of fractional order integrator, differentiator and fractional PI$^{\lambda}$D$^{\mu}$ controller, *IEE Proc. Control Theory Appl.*, **153(6)**, pp. 714-720.

Charef A., Sun H.H., Tsao Y.Y. and Onaral B., (1992). Fractal system as represented by singularity function, *IEEE Transactions on Automatic Control*, **37(9)**, pp. 1465-1470.

Feliu-Batlle V., Rivas Perez R., and Sanchez Rodriguez L., (2007). Fractional robust control of main irrigation canals with variable dynamic parameters, *Control Engineering Practice*, **15**, pp. 673-686.

Hartley T.T., Lorenzo C.F. and Qammer H.K., (1995). Chaos in a fractional order Chua's system, *IEEE Trans. Circuits Syst. I,* **42**, pp. 485-490.

Hilfer, R. (Editor), (2001). *Applications of Fractional Calculus in Physics*, World Scientific Pub Co, New

Jersey.

Li C. and Chen G., (2004). Chaos and hyperchaos in the fractional order Rössler equations, *Physica A: Statistical Mechanics and its Applications,* **341**, pp. 55-61.

Matignon, D. (1996). Stability result on fractional differential equations with applications to control processing, In *IMACS-SMC Proceedings, Lille, France,* pp. 963-968.

Oustaloup A. (1995). *La Dérivation Non Entière: Théorie, Synthèse et Applications*. Paris, France: Editions Hermès.

Petras I., Podlubny I., O'Leary P., Dorcak L. and Vinagre B., (2002). Analogue Realization of Fractional Order Controllers, *FBERG, Technical University of Kosice, Kosice*.

Podlubny, I., (1999a). Fractional-order systems and $PI^{\lambda}D^{\mu}$-controllers, *IEEE Transactions on Automatic Control*, **44 (1),** pp. 208-214.

Podlubny, I., (1999b). *Fractional differential equations*, Academic Press, New York.

Tavazoei M.S., and Haeri M., (2008). Chaos control via a simple fractional order controller, *Physics Letters A,* **372**, pp. 798-807.

Tavazoei M.S., and Haeri M., (2007a) Unreliability of frequency domain approximation in recognizing chaos in fractional order systems, *IET Signal Processing,* **1(4)**, pp. 171-181.

Tavazoei, M.S. and Haeri, M., (2007b). A necessary condition for double scroll attractor existence in fractional order systems, *Physics Letters A*, **367(1-2),** pp. 102-113.

Vinagre B.M., Podlubny I., Hernandez A., Feliu V., (2000) Some approximations of fractional order operators used in control theory and applications, *Fractional Calculus & Applied Analysis,* **3(3)**, pp. 945-950.