A DUMB-BELL SATELLITE WITH A CABIN. EXISTENCE AND STABILITY OF RELATIVE EQUILIBRIA

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Abstract
The problem of motion of a dumb-bell satellite in an orbital plane passing through the attracting center is considered. It is assumed that the satellite is equipped with a cabin that is allowed to slide along the straight connection of the two endbodies. The structure of the set of stationary configurations depending on the parameter, which is the position of the cabin, is studied both analytically and numerically. Especially bifurcation of trivial configurations for which the dumb-bell is located along the local vertical is considered. It is shown that this bifurcation is accompanied by the appearance or disappearance of "oblique" configurations, for which all massive points composing the satellite are not located at the common vertical.

Key words
dumb-bell satellite dynamics, bifurcation diagrams, stability

1 Introduction

The objective of the investigation is to show that the possible bifurcation of relative equilibrium configurations which can be obtained as the result of a slow change of the cabin's position will not lead to the loss of stability of the "working", vertical position of the satellite. These results are important for the deployment of large-scale tethered satellite systems.

Stationary configurations are obtained by Routh's method (see [Routh, 1892], [Karapetyan, 1998]). The same method allows to investigate the sufficient conditions of stability of the considered motions. Bifurcations of the stationary motions are studied with the methods of bifurcation theory initiated by Poincaré (see, for example, [Chetaev, 1961]).

The dynamics of space objects, including moving parts, has been investigated by many authors. This usually has been connected with the necessity to estimate the influence of relative motions of moving parts (crew motion, circulation of liquids, etc.) on the attitude dynamics of a spacecraft [Thomson, Fung, 1965; Amin, Newton, 2000; Moisseyev, Rumyantsev, 1965]. Investigations in dynamics of space dumb-bells arise to [Okunev, 1971] (see also [Beletsky, Ponomareva, 1990]).

The possibilities of loss of stability of the symmetric motions were pointed out in [Burov, Karapetyan, 1995], [Burov, 1996] for cross-shaped satellites. The possibilities of sudden overturn of the satellite due to loss of tension in the tethered elements were pointed out in [Burov, Troger, 2000] (see also [Kosenko, Stepanov, 2006]).

The modern development of large-scale space systems, in particular, of satellite systems with tethered elements and space elevators, have posed problems related to their dynamics. The considered system belongs to the mentioned class of systems and represents by itself one of the simplest systems allowing both analytical and numerical treatment without supplementary simplifying assumptions such as the so-called "satellite approximation". Another set of applied problems is related to orientation keeping of the system for deployment and retrieval of tethered sub-satellites as well as for relative cabin motions of space elevators. In particular the problem of the possibility of stabilization (and destabilization) of the given state of motion due to rapid oscillations of the cabin exists. This could be the subject of separate investigation.

In this paper we present an investigations of the existence, stability and bifurcation of stationary motions in dependence on the position of the cabin.

2 Description of motion

Let us describe dynamics with Lagrange equations.

2.1 Lagrange equations

We consider the dynamics of a dumb-bell composed of two points A and B of masses m_A and m_B, respectively, connected with a massless rod of length ℓ. We further assume that there exists a third point C of the mass m_C which is constrained to move along this rod according to a given rule, for example, periodically. Let us introduce an absolute frame $OX_1X_2X_3$ (AF) with the origin at the point O coincident to the attracting center.

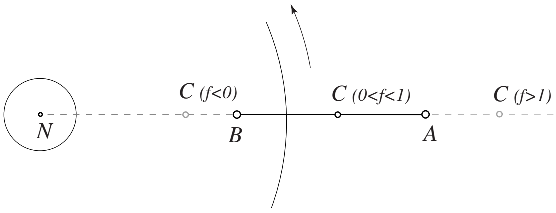

Figure 1. The dumb-bell configuration

Let $\mathbf{P} = (P_1, P_2, P_3)^T$, $P \in \mathcal{I} = \{A, B, C\}$ be a vector giving the position of the system with respect to the AF. The position of the point C can be described as $\mathbf{C} = f\mathbf{A} + (1-f)\mathbf{B}$, $f \in R$, where f is a variable parameter. Then the velocities of the mentioned points are given as $\mathbf{v}_A = \dot{\mathbf{A}}$, $\mathbf{v}_B = \dot{\mathbf{B}}$, $\mathbf{v}_C = \dot{f}\mathbf{A} + f\dot{\mathbf{A}} - \dot{f}\mathbf{B} + (1-f)\dot{\mathbf{B}} = \dot{f}(\mathbf{A}-\mathbf{B}) + f\mathbf{v}_A + (1-f)\mathbf{v}_B$. This allows to write the expression for the kinetic energy as

$$
\begin{aligned}
&T = \tfrac{1}{2}[m_A\mathbf{v}_A^2 + m_B\mathbf{v}_B^2 + m_C(\dot{f}\mathbf{A} + f\mathbf{v}_A - \dot{f}\mathbf{B}+ \\
&+(1-f)\mathbf{v}_B)^2] = \mathcal{T}(\mathbf{v}_A, \mathbf{v}_B; \mathbf{A}, \mathbf{B}; f, \dot{f})
\end{aligned} \tag{1}
$$

The introduction of the distances $r_P = (\mathbf{P}, \mathbf{P})^{1/2}$, $P \in \mathcal{I}$ allows to write down the expression for the potential energy as

$$
U = -GM\left(\frac{m_A}{r_A} + \frac{m_B}{r_B} + \frac{m_C}{r_C}\right) \tag{2}
$$

Finally, the expression for the constraint reads

$$
\varphi = (\mathbf{A}-\mathbf{B}, \mathbf{A}-\mathbf{B}) - \ell^2 = 0 \tag{3}
$$

which expresses the inextensibility of the rod.

If the rule of motion of the third point is prescribed, the Lagrange equations are:

$$
\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}_P} = \frac{\partial L}{\partial P}, \quad P \in \{A, B\}. \tag{4}
$$

The Lagrange function reads

$$
L = \mathcal{T} - U - \frac{1}{2}\lambda\varphi, \tag{5}
$$

where λ is the Lagrange multiplier. These equations have to be analysed together with the constraint equation (3). We further assume $f = const$.

Suppose Z is the mass center of the whole system. Then

$$
\overrightarrow{ZA} = \mu_1\overrightarrow{BA}, \qquad \overrightarrow{ZB} = \mu_2\overrightarrow{BA}, \quad \overrightarrow{ZC} = \mu_3\overrightarrow{BA},
$$

$$
\mu_1 = \frac{m_B + (1-f)m_C}{m},\ \mu_2 = -\frac{m_A + fm_C}{m},\ \mu_3 = \frac{m_A(f-1) + m_Bf}{m} \tag{6}
$$

$$
m = m_A + m_B + m_C
$$

Assume that the points move in the orbital plane passing through the attracting center. Let us introduce polar coordinates with the angle measured starting from the axis OX_1. Then

$$
\overrightarrow{OZ} = [r\cos\nu, r\sin\nu]^T. \tag{7}
$$

Assume that

$$
\overrightarrow{BA} = [\ell\cos(\nu+\varphi), \ell\sin(\nu+\varphi)]^T, \tag{8}
$$

i.e. φ is the angle between $\overrightarrow{OZ}$ and $\overrightarrow{ZA}$. Then the kinetic energy can be represented as

$$
\mathcal{T}'(\dot{r}, \dot{\varphi}, \dot{\nu}, r, \varphi) = T_Z + T_r \tag{9}
$$

where $T_Z = \frac{m}{2}(\dot{r}^2 + r^2\dot{\nu}^2)$, $T_r = \frac{1}{2}J_z(\dot{\nu} + \dot{\varphi})^2$, $J_z = m^{-1}(m_Am_B\overrightarrow{AB}^2 + m_Bm_C\overrightarrow{BC}^2 + m_Cm_A\overrightarrow{AC}^2)$. The use of (6), (7) allows to put

$$
\begin{aligned}
r_A &= \sqrt{r^2 + \mu_1 l(\mu_1 l + 2r\cos\varphi)}, \\
r_B &= \sqrt{r^2 + \mu_2 l(\mu_2 l + 2r\cos\varphi)}, \\
r_C &= \sqrt{r^2 + \mu_3 l(\mu_3 l + 2r\cos\varphi)}
\end{aligned} \tag{10}
$$

in the expression for the gravitational potential (2) and rewrite it as $\mathcal{U}' = \mathcal{U}'(r, \varphi)$.

Since the system is invariant with respect to rotations about the point N, the coordinate ν – the "true anomaly" – is cyclic: the Lagrange function $\mathcal{L}' = \mathcal{T}' - \mathcal{U}'$ does not have any ν dependency, and

$$
\begin{aligned}
&\mathcal{J}_\nu = \frac{\partial \mathcal{L}'}{\partial \dot{\nu}} = mr^2\dot{\nu} + \tilde{m}l^2(\dot{\nu} + \dot{\varphi}) = p = \text{const}, \\
&\tilde{m} = \mu_1^2 m_A + \mu_2^2 m_B + \mu_3^2 m_C
\end{aligned} \tag{11}
$$

The function (11) is an appropriate first integral, and the order of the system can be reduced by means of the *Routh reduction*.

This reduction yields Routh's function:

$$
\begin{aligned}
R &= R(\dot{r}, \dot{\varphi}, r, \varphi; p) = [\mathcal{L}' - p\dot{\nu}] = \\
&= \tfrac{1}{2}\left(m\dot{r}^2 + \tfrac{mr^2l^2\tilde{m}\dot{\varphi}^2}{J_0} + \tfrac{pl^2\tilde{m}\dot{\varphi}}{J_0} - \tfrac{p^2}{J_0}\right) - U_a \\
J_0 &= mr^2 + l^2\tilde{m}
\end{aligned} \tag{12}
$$

and the equations of motion can be written as

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{y}} = \frac{\partial R}{\partial y}, \quad y \in \{r, \varphi\} \tag{13}$$

3 Stationary motions and their stability

Here we consider stationary motions and obtain the conditions of their stability.

3.1 Stationary configurations: Existence

In order to investigate the stability of the stationary motions we consider the critical points of the amended (reduced) potential

$$U_a(r,\varphi;p) = -R(0,0,r,\varphi,p) = \frac{p^2}{2(mr^2 + l^2\tilde{m})} - \\ -GM\left(\frac{m_A}{r_A} + \frac{m_B}{r_B} + \frac{m_C}{r_C}\right) = \frac{p^2}{2J_0} + U \tag{14}$$

Critical points of the amended potential are the solutions of the equations:

$$\begin{aligned} \frac{\partial U_a}{\partial r} &= -\frac{p^2 mr}{J_0^2} + GM(rP_0(r,\varphi;f) + \\ &\quad + l\cos\varphi P_1(r,\varphi;f)) = 0 \\ \frac{\partial U_a}{\partial \varphi} &= -GMlr\sin\varphi P_1(\varphi,r;f) = 0 \end{aligned} \tag{15}$$

where

$$P_k(r,\varphi;f) = \frac{m_A\mu_1^k}{r_A^3} + \frac{m_B\mu_2^k}{r_B^3} + \frac{m_C\mu_3^k}{r_C^3} \tag{16}$$

The second equation of (15) possesses solutions:

I. $\varphi = 0$,
II. $\varphi = \pi$,
III. $(r,\varphi) : P_1(r,\varphi;f) = 0$,

For the solutions I and II the points N, A, C and B are located on the same straight line radially oriented and passing through the attraction point (center of the Earth) (see Fig. 2).

For the solution III the system moves in one of the so-called "oblique" configurations. They are symmetric with respect to the local vertical, as shown in Fig. 3.

The dependence of the integral's constant p on the altitude r and the angle φ on can be presented as:

$$p^2 = \frac{GMJ_0^2}{mr}(rP_0 + l\cos\varphi P_1) \tag{17}$$

For every fixed value of f and p the corresponding curve on the plane (r,φ) intersects the sets of the curves $I \bigcup II \bigcup III = S$, at the points of appropriate stationary motions.

For the solution I-III the relation between radius r and inclination φ can be drawn as in Fig. 4 for $f < 0$, as

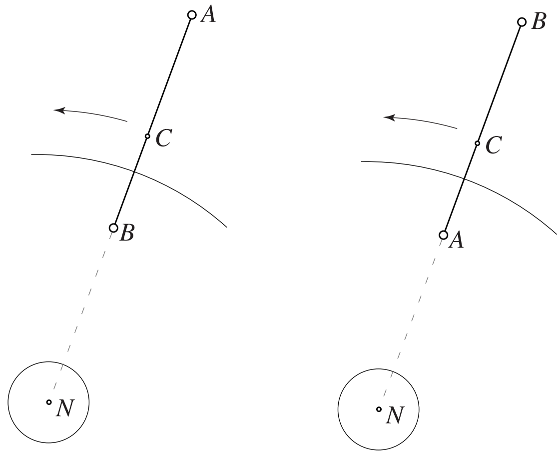

Figure 2. Solutions I and II

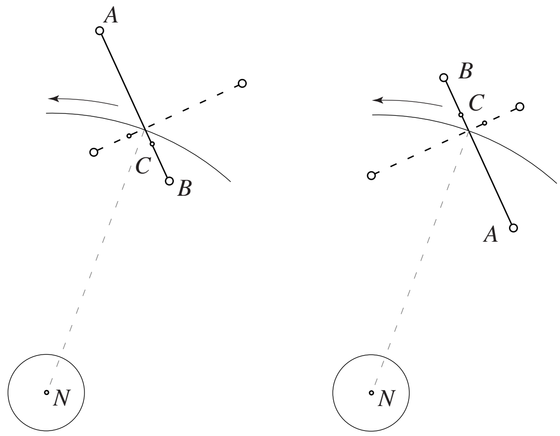

Figure 3. Solutions III

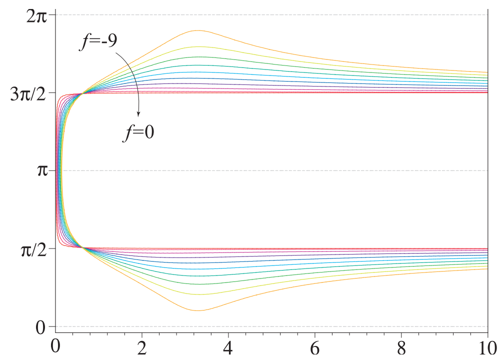

Figure 4. Curves I-III for $f < 0$ at (r,φ)

in Fig. 5 for $f \in [0,1]$ and as in Fig. 6 for $f > 1$, respectively.

Here and below in the figures we use dimensionless parameters, chosen in such a way that $M = 1$, $G = 1$, $l = 1$. For all figures the following values of masses are used: $m_A = m_B = 20/41$, $m_C = 1/41$.

If we add to the Figs 4-6the curves, determined by (17) and parameterized by the integral constant p, then for every p we obtain the solutions as intersection with

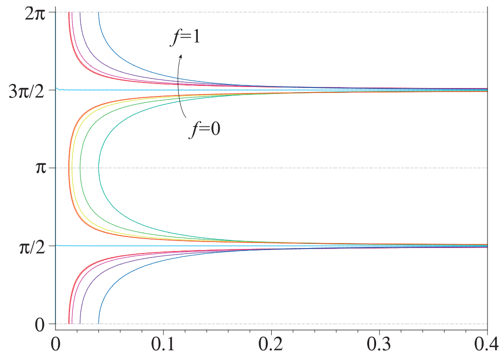

Figure 5. Curves I-III for $f \in [0, 1]$ at (r, φ)

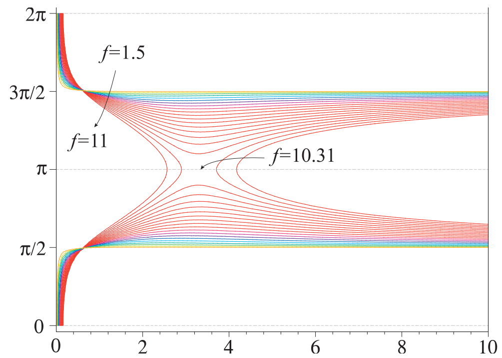

Figure 6. Curves I-III for $f > 1$ at (r, φ)

the curves I-III.

One can focus on the branching of solutions I, II accompanied by the birth (or disappearance) of solution III. It is instructive that for certain, sufficiently large values of f the branch starts for example at I and tends asymptotically to $\varphi = \pm\pi/2$ for $r \to \infty$ However, for other sufficiently moderate values of f the branch III connects solutions I and II.

These two classes of curves are separated from each other by a saddle-like curve, existing for $f = f^*$. This value of f can be found from the equation

$$P_1(\pi, r; f^*) = 0 \tag{18}$$

as a condition of existence of the unique solution of (18) with respect to r. This equation reads explicitly as

$$\frac{m_A \mu_1}{|r + \mu_1 l|^3} + \frac{m_B \mu_2}{|r + \mu_2 l|^3} + \frac{m_C \mu_3}{|r + \mu_3 l|^3} = 0. \tag{19}$$

For fixed values of m_A, m_B and m_C the set (19) in the plane (r, φ) is given in Figs.8-11. The curve (17) in the same plane (r, φ) for fixed values of f and p allows to find geometrically the appropriate orientation of the considered system.

According to the theory of bifurcation by following along a connected component of the curve (17) one can observe the change of stability properties of appropriate steady motions.

Fig. 7 illustrates the set I-III for $f = -5$ and its intersection with the curve (17). The intersection for

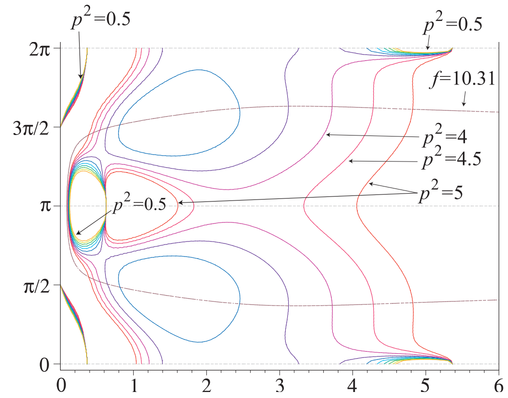

Figure 7. Curves I-III and (17) for $f = -5$ at (r, φ)

$f = 1/2$ is shown in Fig. 8. Intersections of appro-

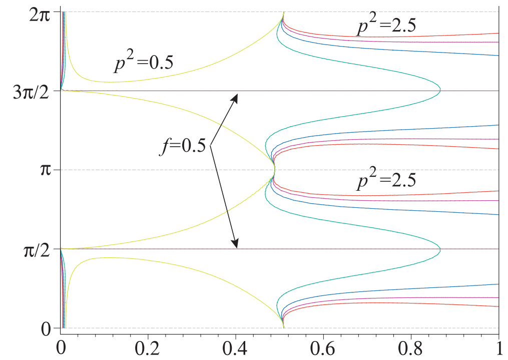

Figure 8. Curves I-III and (17) for $f = 1/2$ at (r, φ)

priate curves for $f = 5$, $f = 10.32$ and $f = 12$ respectively are shown in Figs. 9 through 11.

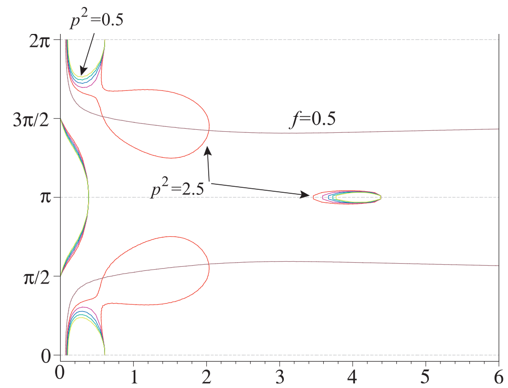

Figure 9. Curves I-III and (17) for $f = 5$ at (r, φ)

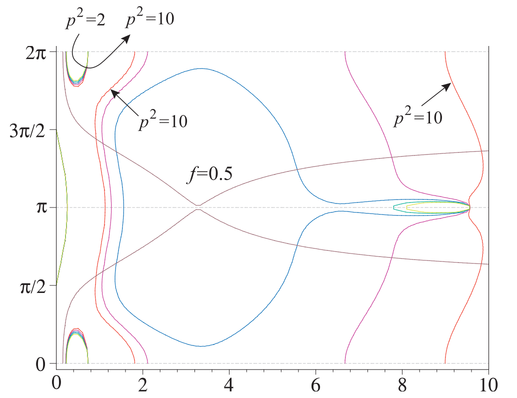

Figure 10. Curves I-III and (17) for $f = 10.32$ at (r, φ)

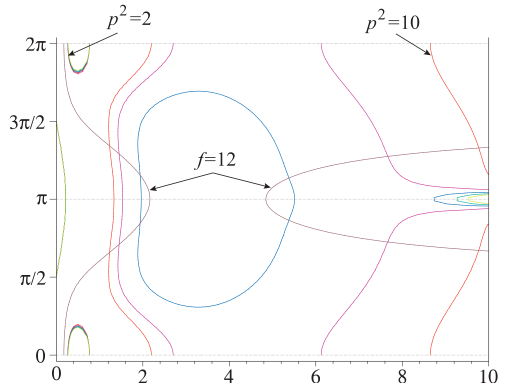

Figure 11. Curves I-III and (17) for $f = 12$ at (r, φ)

3.2 Stationary configurations: Stability

For the investigation of sufficient conditions of stability of the stationary motions we must consider the second derivatives of the reduced potential U_a:

$$\begin{aligned}
\frac{\partial^2 U_a}{\partial r^2} &= \frac{2p^2 mr}{J_0^3}\frac{\partial J_0}{\partial r} - \frac{p^2 m}{J_0^2} + GM(P_0 - 3S) \\
\frac{\partial^2 U_a}{\partial \varphi \partial r} &= -GMl \sin\varphi (P_1 + r\frac{\partial P_1}{\partial r}) \\
\frac{\partial^2 U_a}{\partial \varphi^2} &= -GMl(\cos\varphi P_1 + l\sin^2\varphi \frac{\partial P_1}{\partial \varphi})
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
S(r,\varphi) &= \frac{m_A(r+\mu_1 l\cos\varphi)^2}{r_A^5} + \frac{m_B(r+\mu_2 l\cos\varphi)^2}{r_B^5} + \\
&+ \frac{m_C(r+\mu_3 l\cos\varphi)^2}{r_C^5} \\
\frac{\partial P_1}{\partial r} &= -3\left(\frac{m_A\mu_1}{r_A^4}\frac{\partial r_A}{\partial r} + \frac{m_B\mu_2}{r_B^4}\frac{\partial r_B}{\partial r} + \frac{m_C\mu_3}{r_C^4}\frac{\partial r_C}{\partial r}\right) = \\
&= -3(Q_1 r + Q_2 l\cos\varphi) \\
\frac{\partial P_1}{\partial \varphi} &= -3lr\sin\varphi\left(\frac{m_A\mu_1^2}{r_A^5} + \frac{m_B\mu_2^2}{r_B^5} + \frac{m_C\mu_3^2}{r_C^5}\right) = \\
&= -3lr\sin\varphi Q_2 \\
Q_k(r,\varphi;f) &= \frac{m_A\mu_1^k}{r_A^5} + \frac{m_B\mu_2^k}{r_B^5} + \frac{m_C\mu_3^k}{r_C^5}
\end{aligned}$$

We now consider separately the cases $\varphi = 0$, $\varphi = \pi$, $P_1(r,\varphi) = 0$.

3.3 Stability of the class I. motions

Let $\varphi = 0$, $\cos\varphi = 1$. Hence the mixed derivative of the amended potential vanishes, and sufficient conditions of stability read:

$$\begin{aligned}
C_{rr} &= \frac{p^2 m(3mr^2 - l^2\tilde{m})}{(mr^2 + l^2\tilde{m})^3} - 2GM\left[\frac{m_A}{|r+\mu_1 l|^3} + \frac{m_B}{|r+\mu_2 l|^3} + \right. \\
&\left. + \frac{m_C}{|r+\mu_3 l|^3}\right] > 0 \\
C_{\varphi\varphi} &= -GMl\left(\frac{m_A\mu_1}{|r+\mu_1 l|^3} + \frac{m_B\mu_2}{|r+\mu_2 l|^3} + \right. \\
&\left. + \frac{m_C\mu_3}{|r+\mu_3 l|^3}\right) > 0
\end{aligned} \tag{21}$$

The relative equilibria are drawn on the plane (p^2, r) (see Fig. 12). The first condition provides stability of the system with respect to the radial variable. This condition depends on the value of the integral constant p. The second condition provides stability of the system with respect to its deviations from the local vertical. This condition does not depend on p. It is remarkable, that $C_{\varphi\varphi}$ has the term $P_1(0, r; f)$ as a multiplier. It means the birth of oblique motions of type III and is accompanied by a change of the degree of instability of the motion from class I. The relative equilibria which are located above the curve Γ_1 are Lyapunov stable. For the equilibria below this curve the degree of instability is equal to 1 or 2, and they are Lyapunov unstable.

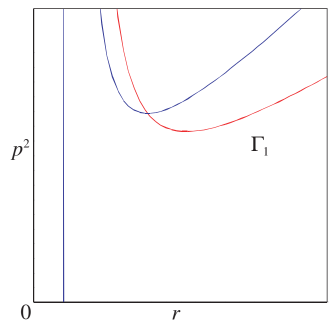

Figure 12. Curves I and (17) for $f = -5$ at (r, p^2)

3.4 Stability of the class II. motions

Let $\varphi = \pi$, $\cos\varphi = -1$. Then conditions for stability may be written as follows:

$$\begin{aligned}
C_{rr} &= \frac{p^2 m(3mr^2 - l^2\tilde{m})}{(mr^2 + l^2\tilde{m})^3} - 2GM\left[\frac{m_A}{|r-\mu_1 l|^3} + \right. \\
&\left. + \frac{m_B}{|r-\mu_2 l|^3} + \frac{m_C}{|r-\mu_3 l|^3}\right] > 0 \\
C_{\varphi\varphi} &= GMl\left(\frac{m_A\mu_1}{|r-\mu_1 l|^3} + \frac{m_B\mu_2}{|r-\mu_2 l|^3} + \right. \\
&\left. + \frac{m_C\mu_3}{|r-\mu_3 l|^3}\right) > 0
\end{aligned} \tag{22}$$

The first condition provides stability of the system with respect to the radial variable. This condition depends on the value of the integral constant p. The second condition provides stability of the system with respect to its deviations from the local vertical. This condition does not depend on p. It is remarkable, that $C_{\varphi\varphi}$ has the term P_1 as a multiplier. It means that the birth of oblique motions of the III. kind is accompanied by a change of degree of instability for motions from the class II. The situation with stability is similar as for the class I. motions. The set of motions are drawn in the plane (p^2, r) (see Fig. 13). The equilibria located above the curve G are Lyapunov stable.

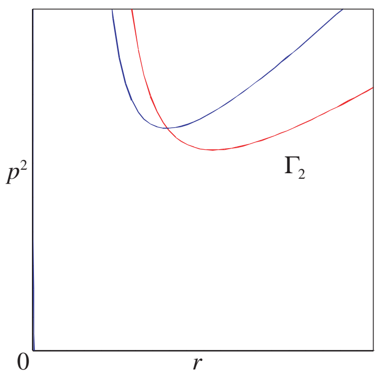

Figure 13. Curves II and (17) for $f = -5$ at (r, p^2)

3.5 Stability of the class III. motions

Let $P_1(r, \varphi) = 0$. For this class of motions the conditions of stability with respect to the radius and with respect to the angle are coupled and can be presented, for example, as

$$C_{\varphi\varphi} = \left[\frac{\partial^2 U}{\partial \varphi^2}\right]_{III} = -3GMl^2 r \sin^2\varphi Q_2 > 0 \quad (23)$$

$$\begin{aligned} D &= \left[\frac{\partial^2 U}{\partial r^2}\frac{\partial^2 U}{\partial \varphi^2} - \left(\frac{\partial^2 U}{\partial r \partial \varphi}\right)^2\right]_{III} = \\ &= 3G^2M^2l^2 \sin^2\varphi r \times \Big(Q_2 F - r\,(Q_1 r + \\ &\quad + Q_2 l \cos\varphi)^2\Big) > 0 \\ F &= P_0 m r^2 - 3\tilde{m} l^2 m r^2 + \tilde{m} l^2 + 3l^2 \sin^2\varphi Q_2 \end{aligned}$$

Here we used effectively for the simplification the condition $P_1 = 0$. One can easily see that the condition (23) fails for all values of parameters. It means that the III. motions are unstable, if $D < 0$, or one can pose the question on their gyroscopic stabilization if $D > 0$.

Acknowledgements

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