STABILITY ANALISYS OF LUR'E SYSTEMS WITH A PULSE-MODULATED FEEDBACK

Alexander N. Churilov
Faculty of Mathematics and Mechanics,
St. Petersburg State University, St. Petersburg, Russia
a_churilov@mail.ru

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Abstract

A nonlinear system with a sector bound nonlinearity is considered. The system is subject to a stabilizing sampled feedback with finite width impulses. An impulsive counterpart of the circle criterion for absolute stability is obtained with the help of the Gelig's averaging method.

Key words

hybrid systems; nonlinear systems; pulse modulation; integral quadratic constraints; linear matrix inequalities

1 Introduction

In the recent decades a great popularity was gained by the study of hybrid systems that combine continuous-time and impulsive dynamics. While processes in continuous physical and biological systems are often rather slow, the interactions between these systems sometimes exhibit fast behaviors that may be interpreted as an impulsive signal. (Signals are understood as some portions of energy that are used for interaction and information exchange [Basiladze, 2009].) An example is the processes in physiology, where body organs are governed by neural impulses of the brain (a fast process), while some hormones secretion in these organs modulates impulsive activity of the brain (a slow process). In modern engineering, a continuous behavior (a slow motion) may be controlled by fastly running impulsive modulators or digital devices. A natural way to handle these so different time scales is to average the fast processes in time and thus simplify the analysis. Essentially this is what we do in this paper.

The problem of system's stabilization with the help of a sampled feedback has focused the attention of many researches (see, e. g., [Fridman et al., 2004; Mirkin, 2007; Naghshtabrizi et al., 2008; Fudjioka, 2009; Fridman, 2010; Seuret and Peet, 2013; Kao, 2016; Hetel et al., 2017]). The majority of works in this field employ a zero-order hold (ZOH) (see [Åström and Wittenmark, 2011]), when a control value calculated at the beginning of a sampling interval is kept constant throughout all this interval. Stabilization of a Lur'e system by a sampled ZOH feedback was addressed in [Seifullaev and Fradkov, 2015a; Seifullaev and Fradkov, 2015b; Seifullaev and Fradkov, 2015c; Seifullaev and Fradkov, 2016; Seifullaev et al., 2017; Zhang et al., 2017; Bryntseva and Fradkov, 2018], Some other types of nonlinear systems under a ZOH event-based control were treated in [Wang et al., 2018; Proskurnikov and Mazo Jr., 2018]. The applications included pendulum and cart-pendulum controlled systems, wheeled robots, a robotic arm and Chua's oscillator. In the previous works much effort was made to minimize the upper bound of the sampling period, i. e. to keep the sampling rate as low as possible (see an extensive review by [Hetel et al., 2017]). From a practical perspective, this requirement is motivated by applications to networked control (see [Hespanha et al., 2007; Lu et al., 2012; Liu et al., 2017; Liu et al., 2019]) and allows to save capacity of a communication channel.

The main disadvantage of the ZOH technique is the time delay introduced by the hold (see [Fridman, 2010]), and this delay is the greater, the longer the sampling period. This negative effect can be decreased by using generalized sampled-data hold functions (GSHF) (see [Kabamba, 1987; Sala, 2007; Briat, 2013]) with the hold restricted to shorter intervals.

In this paper we consider a nonlinear Lur'e system whose nonlinearity satisfies a sectoral constraint (see [Luré, 1957; Khalil, 2002; Yakubovich et al., 2004; Haddad and Chellaboina, 2006]). The system is controlled by a feedback impulsive signal with rectangular (finite width) impulses that are amplitude modulated. As for widths of impulses and their sampling periods, they are considered uncertain and bounded in some finite ranges. Our aim is to obtain a sampled-data stabilizing control with a sampling period as large as possible. The numerical example demonstrates that this can be attained by an admissible choice of the duty ratio (i. e. the ratio of the pulse duration to the sampling period).

Stability considerations are based on the Gelig's averaging method introduced by [Gelig, 1982] (see also [Gelig and Churilov, 1998]) that will be discussed in detail in Section 4. The main idea of this method is a substitution of the initial train of pulses for a sequence of the average values of these pulses, with a supposition that these averages satisfy sectoral constraints not everywhere, but at some discrete time instants. We also employ mathematical technique that is conventional for the absolute stability theory, namely S-procedure (see [Yakubovich, 1971; Yakubovich et al., 2004]) and Integral Quadratic Constraint (IQC) that were originally introduced by [Yakubovich, 1968] to study pulse-width modulated control systems (see [Megretski and Rantzer, 1997] for further results). The Gelig's averaging improves the results by Yakubovich and extends the IQC approach for any type of pulse modulation. In particular we use the IQC based on Wirtinger inequality (see [Hardy et al., 1951], notice that this IQC was employed systematically in [Gelig and Churilov, 1993; Gelig and Churilov, 1998]). Unlike other averaging methods, the Gelig's stability criteria are not asymptotical, they can be used for an estimation of the sampling frequency. At sufficiently high sampling rates they reduce to the conventional absolute stability conditions (circle criterion, Popov criterion and some others).

This paper continues discussion started in [Churilov, 2018], where an application of the absolute stability theory to the ZOH control was examined. We also base on the stability results that were previously formulated in terms of frequency-domain inequalities in [Gelig and Churilov, 1998]. Following [Boyd et al., 1994], the new stability criterion is stated as a feasibility problem for Linear Matrix Inequalities (LMI) which allow numerical solution using standard software packages. In the numerical example it is shown how a reasonable choice of a duty ratio can significantly increase the admissible sampling period when compared with ZOH (up to a tripling for high gains).

The paper is organized as follows. Firstly we describe a model that comprises a Lur'e type nonlinear system under an impulsive control. Then the concept of averaging is discussed and a discrete-time sectoral bounds (Gelig's type sectoral constraints) are introduced. In the main part of the paper we demonstrate how an impulsive system with a nonuniform sampling and noninstantaneous impulses can be treated with the help of the Gelig—Yakubovich approach to the absolute stability theory. The stability result is formulated in terms of LMI feasibility problems. Finally, the main result is illustrated by numerical examples.

2 Preliminaries

The concept of pulse modulation is of significant importance in engineering and in mathematical biology. Let us introduce some general definitions following [Gelig and Churilov, 1998] (see also [Skoog and Blankenship, 1970; Kuntsevich and Chekhovoi, 1970]). In mathematical terms, pulse modulator is an operator that acts on the space of continuous functions (called *modulating signals*) and converts every such function into a train of noninstantaneous impulses (pulses):

$$M \; : \; \sigma(t) \mapsto f(t).$$

The most general characterization of a train of pulses is a sequence of times $t_0 = 0 < t_1 < t_2 < \ldots$ called *sampling instants*. It is assumed that this sequence is strictly increasing and has no accumulation points (the latter assumption excludes Zeno behavior (see [Pogromsky et al., 2003])). The time interval between successive samples (t_n, t_{n+1}) is called *the nth sampling interval*, with its length $T_n = t_{n+1} - t_n$ termed *the nth sampling period* (see [Kalman and Bertram, 1959]). The real valued function $f(t)$ defined for $t \geqslant t \geqslant t_0$ will be called a train of pulses if it is represented as

$$f(t) = \hat{f}(P_n, t), \quad t_n \leqslant t < t_{n+1}, \quad n = 0, 1, \ldots,$$

where the function $\hat{f}(\cdot, \cdot)$ describes the form of a pulse and P_n is a vector of parameters. The most common pulse form is rectangular. Such a pulse is characterized by its amplitude (with polarity), width (duration), phase (displacement of its leading edge from t_n) and instantaneous frequency (equal to $1/T_n$). Some of these parameters are fixed, while the others depend on the modulating signal. Signal-dependent parameters are called *modulated* with $f(t)$ being *a modulated impulsive signal*. From the mathematical perspective, modulated parameters are functionals of $\sigma(\cdot)$. They carry information about the modulating signal and can be used for control. Unlike switched or relay systems (see [Tsypkin, 1984; Liberzon, 2003]), the memory of a modulator resets at the end of each sampling interval.

Of a great interest is the case when T_n varies in certain limits. This may be due to pulse-frequency modulation (PFM) (that is also called "signal dependent sampling" by [Jury, 1961]), when the sampling period T_n is a functional of the modulating signal. The theory of PFM has a long history going back to 1940s-1970s, see, among others, [Ross, 1949; Jones et al., 1961; Dorf et al., 1962; Li and Jones, 1963; Pavlidis, 1965; Pavlidis and Jury, 1965; Bombi and Ciscato, 1967; Jury and Blanchard, 1967; Skoog and Blankenship, 1970; Kuntsevich and Chekhovoi, 1971a; Kuntsevich and Chekhovoi, 1971b; Varadarajan, 1971; Gülçür and Meyer, 1973]. A renewed interest in PFM was inspired by the emergence of the concept of event based control that was put forward in [Åström and Bernhardsson, 1999; Årzén, 1999] and related papers by [Åström and Bernhardsson, 2002; Åström, 2008]. Another reason for variability of T_n

may be an uncertainty in sampling times. Communication technology requires that the sampling frequency be high enough, so that $\sigma(\cdot)$ can be recovered (demodulated) from $f(\cdot)$, however for control purposes requirements to sampling are less stringent.

3 Impulsive Lur'e system

Consider a system comprised of the following three parts. Further, it will be called *impulsive Lur'e system.*

3.1 Linear time-invariant subsystem

Assume that the linear part is represented by the equations

$$\dot{x}(t) = Ax(t) + B_0 f_0(t) + Bf(t), \quad (1)$$

$$\sigma_0(t) = C_0 x(t), \quad \sigma(t) = Cx(t). \quad (2)$$

Here $x(t)$ is a p-dimensional state vector. The functions $f_0(t)$, $f(t)$ are the inputs of the linear part (1), (2), while $\sigma_0(t)$, $\sigma(t)$ are its outputs. Here A, B, B_0, C, C_0 are constant coefficients, A is a $p \times p$-matrix, B, B_0 are p-dimensional columns and C, C_0 are p-dimensional rows.

3.2 Continuous nonlinear subsystem

The internal nonlinear feedback $\sigma_0 \mapsto f_0$ is given by the nonlinearity

$$f_0(t) = \varphi_0(\sigma_0(t), t), \quad (3)$$

where the function $\varphi_0(\sigma_0, t)$ is continuous and obeys the Lur'e type sectoral bound

$$\nu_1 \leqslant \frac{\varphi_0(\sigma_0, t)}{\sigma_0} \leqslant \nu_2 \quad (4)$$

for all σ_0, t. Here ν_1, ν_2 are given scalars.

The above two subsystems together make up a Lur'e-type control system, whose zero equilibrium will be stabilized by a signal $f(t)$.

3.3 Impulsive subsystem

The external impulsive feedback $\sigma \mapsto f$ is obtained by sampling the continuous signal $\sigma(t)$ at times t_n, satisfying the recurrence

$$t_{n+1} = t_n + T_n, \quad n \geqslant 0. \quad (5)$$

The function $f(t)$ is given by

$$f(t) = \begin{cases} \lambda_n, & t_n \leqslant t < t_n + \tau_n, \\ 0, & t_n + \tau_n \leqslant t < t_{n+1}, \end{cases} \quad (6)$$

where λ_n, τ_n, T_n are pulse amplitude, pulse width and sampling period, respectively. Here we are not interested in the exact form of τ_n, T_n, only their bounds will be used.

Assume that amplitudes (with polarities) are modulated by the signal $\sigma(t)$:

$$\lambda_n = F(\sigma(t_n)), \quad (7)$$

where $F(\cdot)$ is a nondecreasing function with $F(0) = 0$.

Let $F(\cdot)$ satisfy a sectoral bound

$$k_1 \leqslant \frac{F(\sigma)}{\sigma} \leqslant k_2 \quad \text{for all} \quad \sigma, \quad (8)$$

where k_1, k_2 are some numbers, $0 < k_1 \leqslant k_2$.

The *duty ratio* of the nth pulse is defined as

$$d_n = \tau_n / T_n.$$

Assume that

$$0 < T_* \leqslant T_n \leqslant T, \quad 0 \leqslant d_* \leqslant d_n \leqslant d \leqslant 1 \quad (9)$$

for all n, where T_*, T, d_*, d are some given numbers. Inequality $d \leqslant 1$ ensures that pulses do not overlap.

From (5), (9) we get $t_n \geqslant t_0 + nT_*$, so $t_n \to +\infty$ as $n \to \infty$.

4 Gelig's averaging

The concept of pulse averaging goes back to the principle of equivalent areas (PEA) put forward by [Andeen, 1960a; Andeen, 1960b] for systems with pulse-width modulation. The ideas of averaging were often used for digital redesign when a continuous-time impulsive system is converted to a discrete-time system that captures the main properties of the original model (see e. g. [Friedland, 1976; Ieko et al., 2001]). Further we will employ the technique of pulse averaging introduced in [Gelig, 1982] and refined in [Gelig and Churilov, 1993; Gelig and Churilov, 1998].

Here we will explain the idea of the Gelig's pulse averaging.

For an integrable function $g(t)$ and any integer $n \geqslant 0$ define a linear functional

$$S_n(g) = \int_{t_n}^{t_{n+1}} g(t)\,dt = \int_0^{T_n} g(t_n + t)\,dt. \quad (10)$$

Let v_n be the averaged value of the signal $f(t)$ over the nth sampling interval (t_n, t_{n+1}), namely

$$v_n = \frac{1}{T_n} S_n(f) = \frac{\lambda_n \tau_n}{T_n} = \lambda_n d_n, \quad n \geqslant 0. \quad (11)$$

From (11) and (8) we have

$$k_1 d_n \leqslant \frac{v_n}{\sigma(t_n)} \leqslant k_2 d_n, \quad n \geqslant 0. \quad (12)$$

Then (12) and (9) imply a discrete-time quadratic constraint (a Gelig's type constraint)

$$\mu_1 \leqslant \frac{v_n}{\sigma(t_n)} \leqslant \mu_2, \quad n \geqslant 0, \quad (13)$$

where

$$\mu_1 = k_1 d_*, \quad \mu_2 = k_2 d\,. \tag{14}$$

Define two piecewise constant functions

$$v(t) = v_n, \quad \tilde{\sigma}(t) = \sigma(t_n), \quad t_n \leqslant t < t_{n+1}. \tag{15}$$

Then (13) can be rewritten as

$$\mu_1 \leqslant \frac{v(t)}{\tilde{\sigma}(t)} \leqslant \mu_2, \quad t \geqslant t_0. \tag{16}$$

Let $u(t)$ be the averaged error of the replacement of $f(t)$ for $v(t)$:

$$u(t) = \int_{t_0}^{t} (f(s) - v(s))\, ds, \quad t \geqslant t_0. \tag{17}$$

Obviously, $u(t)$ is continuous for $t \geqslant t_0$. Moreover, $u(t_n) = 0$ for $n \geqslant 0$ and

$$u(t) = \int_{t_n}^{t} (f(s) - v(s))\, ds, \quad t_n \leqslant t, \quad n \geqslant 0.$$

Then

$$f(t) = v(t) + \dot{u}(t), \quad t_n < t < t_{n+1},\ n \geqslant 0. \tag{18}$$

Let us take a function $w(t) = C\mathrm{e}^{At}B$ that is an impulse response of the linear part of system (1), (2) from input f to output σ. With the help of (18) we obtain

$$\int_{t_n}^{t_{n+1}} w(t_{n+1} - s) f(s)\, ds = \int_{t_n}^{t_{n+1}} w(t_{n+1} - s) v(s)\, ds - \int_{t_n}^{t_{n+1}} \dot{w}(t_{n+1} - s) u(s)\, ds.$$

Thus taking into account only averaged responses, we can replace the signal $f(t)$ for two signals $v(t)$, $u(t)$, however applied to different points of the control circuit. The advantage of such a replacement is that $v(t)$, $u(t)$ allow for bounds (quadratic and integral) that are conventional for the absolute stability theory.

Notice that in the case of ZOH we have $\tau_n \equiv T_n$, $f(t) \equiv v(t)$ and $u(t) \equiv 0$ (see [Churilov, 2018]).

5 The main statement

Recall that the numbers μ_1, μ_2 are defined by (14).

The following theorem presents LMI conditions for asymptotic to zero of the impulsive Lur'e system.

Theorem 1. *Assume that there exist a symmetric $p \times p$ matrix H and nonnegative scalars ε_i, $0 \leqslant i \leqslant 4$, such that the following system of matrix inequalities (understood in terms of positive and negative definiteness of quadratic forms) is feasible:*

$$H > 0, \quad \Pi < 0 \tag{19}$$

where Π is a symmetric matrix with the block components

$$\begin{aligned}
\Pi_{11} &= HA + A^\top H - \mu_1\mu_2 C^\top C - \varepsilon_0\nu_1\nu_2 C_0^\top C_0 \\
&\quad + \varepsilon_1 A^\top C^\top C A, \\
\Pi_{12} &= HB_0 + \tfrac{1}{2}\varepsilon_0(\nu_1+\nu_2)C_0^\top + \varepsilon_1 A^\top C^\top C B_0, \\
\Pi_{13} &= HB + \tfrac{1}{2}(\mu_1+\mu_2)C^\top + \varepsilon_1 A^\top C^\top C B, \\
\Pi_{14} &= -A^\top H B + \mu_1\mu_2 CB\, C^\top, \\
\Pi_{15} &= \mu_1\mu_2 C^\top + \varepsilon_4\, A^\top C^\top, \\
\Pi_{22} &= -\varepsilon_0 + \varepsilon_1 (CB_0)^2, \quad \Pi_{23} = \varepsilon_1 CB_0\, CB, \\
\Pi_{24} &= -B^\top H B_0, \quad \Pi_{25} = \varepsilon_4\, CB_0,
\end{aligned}$$

and

$$\begin{aligned}
\Pi_{33} &= -1 + (1-d_*)T\varepsilon_3 + \varepsilon_1 (CB)^2, \\
\Pi_{34} &= -B^\top H B - \tfrac{1}{2}(\mu_1+\mu_2) CB \\
&\quad + \tfrac{1}{3}(1-d_*)T\varepsilon_2 - \varepsilon_3, \\
\Pi_{35} &= -\tfrac{1}{2}(\mu_1+\mu_2) + \varepsilon_4\, CB, \\
\Pi_{44} &= -\varepsilon_2 - \mu_1\mu_2 (CB)^2, \\
\Pi_{45} &= -\mu_1\mu_2\, CB, \quad \Pi_{55} = -\mu_1\mu_2 - \tfrac{1}{4}\varepsilon_1\pi^2 T^{-2}.
\end{aligned}$$

Here $\Pi_{ij} = \Pi_{ji}$ ($1 \leqslant i < j \leqslant 5$), $\top$ denotes matrix transpose. Then any solution of the impulsive Lur'e system obeys $x(\cdot) \in L^2([t_0, +\infty))$ and $\lambda_n \to 0$ as $n \to +\infty$. If we additionally assume that there exists a scalar ν_0 such that the function

$$\hat{\varphi}_0(\sigma_0, t) = \varphi_0(\sigma_0, t) - \nu_0\sigma \tag{20}$$

is bounded for all σ, t, then $x(t) \to 0$ as $t \to +\infty$.

Here $L^2([t_0, +\infty))$ is the space of square integrable vector valued functions $x(\cdot)$.

The blocks Π_{ij} contain parameters of the controlled Lur'e system (A, B_0, C_0, ν_1, ν_2) as well as parameters related to the impulsive control ($B, C, T, d_*, \mu_1 = k_1 d_*$, $\mu_2 = k_2 d$). The main design parameters are the bounds for the sampling period, the duty ratio and the nonlinear control gain.

6 Proof of the main statements

We begin with some auxiliary statements.

6.1 The main lemma

Let H be a symmetric matrix of order p. Define a quadratic form

$$L_H(x, f_0, u, v) = (x - Bu)^\top H(Ax + B_0 f_0 + Bv),$$

where x is a p-dimensional vector and u, v, f_0 are scalars. The following proposition will be used to derive the main statements of this paper.

Lemma 1. *Let there exist a positive definite matrix H and a number $\delta > 0$ such that for every solution of the impulsive Lur'e system we can find a function $\Phi(t)$ satisfying the inequality*

$$\begin{aligned} &L_H(x(t), f_0(t), u(t), v(t)) + \Phi(t) \\ &\quad \leqslant -\delta\big(\|x(t)\|^2 + v(t)^2\big), \quad t \geqslant t_0, \end{aligned} \tag{21}$$

with $v(t)$, $u(t)$ defined by (15), (17)*, and*

$$S_n(\Phi) \geqslant 0 \quad \text{for all} \quad n \geqslant 0. \tag{22}$$

Then $x(\cdot) \in L^2([t_0, +\infty))$ and $v_n \to 0$ as $n \to +\infty$. If in addition we assume that there exists a number ν_0 such that the function defined by (20) *is bounded for all σ_0, t, then we get $x(t) \to 0$ as $t \to +\infty$.*

Proof. Let us consider a quadratic form

$$V(x, u) = (x - Bu)^\top H(x - Bu)$$

of a vector x and a scalar u. Using (18) along the solutions of an impulsive Lur'e system we get

$$\dot V(x(t), u(t)) = L_H(x(t), f_0(t), u(t), v(t)). \tag{23}$$

Then (21), (22) and (23) imply

$$\begin{aligned} V(x(t_n), 0) - V(x(t_0), 0) \leqslant &- \delta \int_{t_0}^{t_n} \|x(t)\|^2\, dt \\ &- \delta \sum_{k=0}^{n-1} T_k v_k^2 \end{aligned}$$

for $n \geqslant 1$. Hence

$$\int_{t_0}^{t} \|x(s)\|^2 ds + T_* \sum_{k=0}^{n} v_k^2 \leqslant -\frac{1}{\delta} V(x(t_0), 0)$$

for all $t \geqslant t_{n+1}$, which implies $v_n \to 0$ as $n \to \infty$ and $x(\cdot) \in L^2([t_0, +\infty))$. The rest of the proof is obtained with the help of the Barbalat's lemma along the lines of the proof of Theorem 1 in [Churilov, 2018]. □

Further we will construct a suitable $\Phi(t)$ to satisfy (21), (22). This will be achieved by using the Yakubovich's S-procedure and the IQC technique. We will represent $\Phi(t)$ as a sum of quadratic forms considered along solution of the impulsive Lur'e system.

6.2 Quadratic constraint for functions $\sigma_0(t)$, $f_0(t)$

The simplest constraint follows from the sectoral bound (4) with (3). If we take a quadratic form

$$W_0(x, f_0) = (\nu_2 C_0 x - f_0)(f_0 - \nu_1 C_0 x)$$

then along the solutions we have $W_0(x(t), f_0(t)) \geqslant 0$. This inequality is valid for all t.

6.3 IQC for functions $u(t)$, $v(t)$

Direct calculation show that

$$u(t) = \begin{cases} \frac{T_n - \tau_n}{\tau_n}(t - t_n)v_n, & t_n \leqslant t \leqslant t_n + \tau_n, \\ (t_{n+1} - t)v_n, & t_n + \tau_n \leqslant t \leqslant t_{n+1}. \end{cases} \tag{24}$$

Thus if the sampling frequency $1/T_n$ is high, then the error function $u(t)$ is negligible small when compared with $v(t)$.

Lemma 2. *The following IQC are valid for $n \geqslant 1$*

$$S_n\Big(\tfrac{2}{3}T(1 - d_*)uv - u^2\Big) \geqslant 0, \tag{25}$$

$$S_n\Big(T(1 - d_*)v^2 - 2uv\Big) \geqslant 0. \tag{26}$$

Proof. From (24) by a straightforward integration we obtain

$$S_n(u) = \tfrac{1}{2}(T_n - \tau_n)T_n v_n = \tfrac{1}{2}(T_n - \tau_n)S_n(v), \tag{27}$$

$$S_n(u^2) = \tfrac{1}{3}(T_n - \tau_n)^2 T_n v_n^2 = \tfrac{1}{3}(T_n - \tau_n)^2 S_n(v^2). \tag{28}$$

This implies

$$\begin{aligned} S_n(uv) &= \tfrac{1}{2}(T_n - \tau_n)S_n(v^2), \\ S_n(u^2) &= \tfrac{2}{3}(T_n - \tau_n)S_n(uv). \end{aligned} \tag{29}$$

Since inequalities (9) are valid and $u(t)v(t) \geqslant 0$ for all t, from (27), (28), (29) we come to (25), (26). □

6.4 IQC based on Wirtinger inequality

The renown Wirtinger's inequality was previously employed in the theory of sampled-data systems in a number of publications (see, e. g., [Gelig and Churilov, 1993; Gelig and Churilov, 1996; Gelig and Churilov, 1998]). We will apply its version that was proved in [Gelig and Churilov, 1998].

Using previously defined functions $\sigma(t)$, $\tilde\sigma(t)$, $u(t)$, introduce a new auxilairy function

$$\xi(t) = \sigma(t) - \tilde\sigma(t) - CB\, u(t). \tag{30}$$

Recall that the function $\tilde\sigma(t)$ is piecewise constant and its derivative is zero everywhere except points t_n, $n \geqslant 0$. We are interested in estimating the difference

$\xi_0(t) = \sigma(t) - \tilde{\sigma}(t)$ by using its derivative $\dot{\xi}_0(t)$ on the intervals $t_n < t < t_{n+1}$. The problem is that the derivative $\dot{\sigma}(t) = CAx(t) + CB_0 f_0(t) + CBf(t)$ includes the term $CBf(t)$ depending on the impulsive signal. To exclude this term we subtract the expression $CB\,u(t)$ that is small when compared with the signal $v(t)$ for high sampling rates.

Obviously, the right-sided limit $\xi(t_n^+) = 0$ and

$$\dot{\xi}(t) = CAx(t) + CB_0 f_0(t) + CBv(t) \tag{31}$$

for $t_n < t < t_{n+1},\ n \geqslant 0$. The Wirtinger inequality gives

$$\int_{t_n}^{t_{n+1}} \xi(t)^2\,dt \leqslant \frac{4T_n^2}{\pi^2} \int_{t_n}^{t_{n+1}} \dot{\xi}(t)^2 dt.$$

Then the quadratic form

$$W_w(x, f_0, v, \xi) = (CAx + CB_0 f_0 + CBv)^2 - \tfrac{1}{4}\pi^2 T^{-2}\xi^2$$

satisfies the IQC $S_n(W_w) \geqslant 0$, $n \geqslant 0$, along the solutions of the impulsive Lur'e system.

6.5 Rearrangement of the discrete-time quadratic constraint

Now we will take advantage of inequality (16) that implies

$$W_v(\tilde{\sigma}(t), v(t)) \geqslant 0, \quad t \geqslant t_0, \tag{32}$$

where

$$W_v(\tilde{\sigma}, v) = (\mu_2\tilde{\sigma} - v)(v - \mu_1\tilde{\sigma}). \tag{33}$$

Further, we will rearrange (32), (33) to replace $\tilde{\sigma}(t)$ for functions $\sigma(t)$, $\xi(t)$, $u(t)$.

Lemma 3. *Let σ, $\tilde{\sigma}$, u, ξ be any real numbers that satisfy $\xi = \sigma - \tilde{\sigma} - CB\,u$. At this relationship the quadratic form $W(\tilde{\sigma}, v)$ can be represented as a sum of three quadratic forms*

$$W_v(\tilde{\sigma}, v) = W_1(\sigma, v) + W_2(\sigma, v, \xi) + W_3(\sigma, v, \xi, u)$$

for all σ, $\tilde{\sigma}$, v, u, ξ. Here

$$\begin{aligned} W_1(\sigma, v) &= (\mu_2\sigma - v)(v - \mu_1\sigma), \\ W_2(\sigma, v, \xi) &= 2\mu_1\mu_2\sigma\xi - (\mu_1 + \mu_2)v\xi - \mu_1\mu_2\xi^2, \\ W_3(\sigma, v, \xi, u) &= CB\big[2\mu_1\mu_2\sigma u - 2\mu_1\mu_2\xi u \\ &\qquad - (\mu_1 + \mu_2)uv - \mu_1\mu_2 CB\,u^2\big]. \end{aligned}$$

Proof. Introduce an additional variable $\xi_0 = \sigma - \tilde{\sigma}$. Then

$$\begin{aligned} W_v(\tilde{\sigma}, v) = W(\sigma - \xi_0, v) &= W_1(\sigma, v) \\ &+ 2\mu_1\mu_2\sigma\xi_0 - (\mu_1 + \mu_2)v\eta - \mu_1\mu_1\xi_0^2. \end{aligned}$$

Since $\xi_0 = \xi + CB\,u$, we arrive at the statement of Lemma 3 by direct calculations. □

With the help of Lemma 3, inequality (32) can be replaced for the inequality

$$\begin{aligned} W_1(\sigma(t), v(t)) &+ W_2(\sigma(t), v(t), \xi(t)) \\ &+ W_3(\sigma(t), v(t), \xi(t), u(t)) \geqslant 0 \end{aligned}$$

along the solutions of the system.

6.6 Additional IQC for $\xi(t)$

Since the right-sided limit $\xi(t_n^+) = 0$, we get

$$\int_{t_n}^{t_{n+1}} \xi(t)\dot{\xi}(t)\,dt = \tfrac{1}{2}\xi(t_{n+1}^-)^2 \geqslant 0.$$

(Here $\xi(t_{n+1}^-)$ is the left-sided limit.) Then (31) implies

$$S_n(\xi\,CAx + \xi\,CB_0 f_0 + \xi\,CBv) \geqslant 0, \quad n \geqslant 0.$$

6.7 Proof of Theorem 1

Define

$$X = \operatorname{col}\{x,\ f_0,\ v,\ u,\ \xi\}, \tag{34}$$

where x is a p-dimensional vector, f_0, v, u, ξ are scalars. Consider the quadratic form $X^\top \Pi X$, where the matrix Π is defined in the formulation of Theorem 1. It can be easily verified that

$$X^\top \Pi X = 2(x - Bu)^\top H(Ax + B_0 f_0 + Bv) + W(X),$$

where

$$\begin{aligned} W(X) = \;& \varepsilon_0(\nu_2 C_0 x - f_0)(f_0 - \nu_1 C_0 x) \\ &+ (\mu_2 Cx - v)(v - \mu_1 Cx) \\ &+ 2\mu_1\mu_2 Cx\,\xi - (\mu_1 + \mu_2)v\xi - \mu_1\mu_2\xi^2 \\ &+ CB\big[2\mu_1\mu_2 Cx\,u - 2\mu_1\mu_2\xi u \\ &- (\mu_1 + \mu_2)uv\big] - \mu_1\mu_2 CB\,u^2 \\ &+ \varepsilon_1\big[(CAx + CB_0 f_0 + CBv)^2 \\ &- \tfrac{1}{4}\pi^2 T^{-2}\xi^2\big] + \varepsilon_2\big[-u^2 + \tfrac{2}{3}(1 - d_*)Tuv\big] \\ &+ \varepsilon_3\big[-2uv + (1 - d_*)Tv^2\big] \\ &+ 2\varepsilon_4\,\xi(CAx + CB_0 f_0 + CBv). \end{aligned} \tag{35}$$

Thus the inequality $\Pi < 0$ is equivalent to the existence of a sufficiently small number $\delta > 0$ such that

$$2(x - Bu)^\top H(Ax + B_0 f_0 + Bv) + W(X) \leqslant -\delta\|X\|^2 \tag{36}$$

for all x, f_0, v, u, ξ and for X defined by (34). With the help of constraints obtained in Subsections 6.2–6.6 we conclude that $S_n(W) \geqslant 0$ for all $n \geqslant 0$, and hence Lemma 1 is applicable.

7 Necessary conditions for feasibility of linear matrix inequalities

Consider the second LMI in (19). In the proof of Theorem 1 it is shown that $\Pi < 0$ can be reformulated as inequality (36) with the quadratic form $W(X)$ defined by (35). Let us put $u = \xi = 0$ in (36). By discarding some nonnegative terms in the left-hand side of (36), we obtain

$$\begin{aligned} &2x^\top H(Ax + B_0 f_0 + Bv) \\ &\qquad + \varepsilon_0(\nu_2 C_0 x - f_0)(f_0 - \nu_1 C_0 x) \\ &\qquad + (\mu_2 Cx - v)(v - \mu_1 Cx) \\ &\qquad \leqslant -\delta(\|x\|^2 + u^2 + v^2) \end{aligned} \tag{37}$$

for all x, f_0, v. Inequality (37), where $H > 0$, $\varepsilon_0 \geqslant 0$, presents LMI formulation of the circle criterion of absolute stability for a system with two nonlinearities

$$\begin{aligned} &\dot{x}(t) = Ax(t) + B_0\varphi_0(\sigma_0(t), t) + B\varphi(\sigma(t)), \\ &\sigma_0(t) = C_0 x(t), \quad \sigma(t) = Cx(t) \end{aligned} \tag{38}$$

satisfying sector bounds (4),

$$\mu_1 \leqslant \frac{\varphi(\sigma)}{\sigma} \leqslant \mu_2 \tag{39}$$

(see, e. g., [Yakubovich et al., 2004]). Thus under the conditions of Theorem 1 nonlinear system (38) must be absolutely stable in the classes of nonlinearities (4), (39).

In particular, let us put $f_0 = \nu C_0 x$, $v = \mu Cx$ in inequality (37), where μ, ν are arbitrary numbers such that

$$\mu_1 \leqslant \mu \leqslant \mu_2, \quad \nu_1 \leqslant \nu \leqslant \nu_2. \tag{40}$$

Then (37) implies

$$H(A+\nu B_0 C_0+\mu BC)+(A+\nu B_0 C_0+\mu BC)^\top H < 0.$$

Since $H > 0$, we conclude that the matrix $A+\nu B_0 C_0+\mu BC$ is Hurwitz stable for any numbers μ, ν satisfying (40).

8 Numerical examples

Theorem 1 will be applied to a stabilization problem for a pendulum-like system considered in [Seifullaev and Fradkov, 2015c; Seifullaev and Fradkov, 2016].

8.1 Example of an application of Theorem 1

Let the continuous part of the impulsive Lur'e system be described by equations

$$\begin{aligned} &\dot{x}_1 = -2x_1(t) + \varphi_0(x_2(t)), \\ &\dot{x}_2 = x_1(t) - x_2(t) + 2\varphi_0(x_2(t)) - f(t), \\ &\sigma_0(t) = \sigma(t) = x_2(t), \end{aligned} \tag{41}$$

where $\varphi_0(\sigma_0) = \sin\sigma_0$ and $f(t)$ is an impulsive control of form (5), (6), (7) under suppositions (8), (9). Then

$$A = \begin{bmatrix} -2 & 0 \\ 1 & -1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix},$$
$$C_0 = C = [0 \quad 1].$$

The function $\varphi_0(\sigma_0) = \sin\sigma_0$ belongs to the class of nonlinearities (2) with $\nu_1 = -0.2173$, $\nu_2 = 1$.

Let us find necessary conditions for feasibility of inequalities (19). The matrix $A + \nu B_0 C_0 + \mu BC$, where μ, ν are scalar parameters, has the characteristical polynomial $P(z) = z^2 + (3 + \mu - 2\nu)z + (2 + 2\mu - 5\nu)$ that is Hurwitz stable iff

$$3 + \mu - 2\nu > 0, \quad 2 + 2\mu - 5\nu > 0. \tag{42}$$

Thus the matrix $A + \nu B_0 C_0$ is not Hurwitz stable for $\nu \geqslant 0.4$, so the Lur'e system (with $f(t) \equiv 0$) cannot be absolutely stable in the class of nonlinearities (2) with $\nu_2 = 1$ (see Section 7). From (42) it is seen that the matrix $A + \nu B_0 C_0 + \mu BC$ is Hurwitz stable for any ν, $-0.2173 \leqslant \nu \leqslant 1$, iff $\mu > 1.5$. Thus if the matrix inequalities of Theorem 1 are feasible, then $\mu_1 > 1.5$. Recalling (14), we obtain the necessary condition for stability

$$k_1 d_* > 1.5\,. \tag{43}$$

For further analysis, let

$$k_2 = k_1 = k, \quad \tau_n \equiv \tau \quad \text{for all} \quad n\,, \tag{44}$$

where k, τ are some numbers. The stability result depends significantly on the ratio T_*/T. We will consider two special cases: $T = 2T_*$ and $T = T_*$.

Firstly, let $T = 2T_*$. Thus the range of sampling frequencies is sufficiently wide (the maximal sampling frequency is twice as much as the minimal one), while the duration of the impulsive action τ_n is kept constant and $F(\sigma) = k\sigma$ for all σ. From (44) we obtain

$$d = 2d_* = \frac{2\tau}{T}, \qquad \mu_2 = 2\mu_1. \tag{45}$$

Since $d \leqslant 1$, (44) and (45) imply

$$d_* \leqslant 0.5, \qquad k > 3\,. \tag{46}$$

Thus in the numerical experiment we can limit considerations to parameters satisfying necessary stability conditions (43), (46). Similar reasoning shows that for a periodic sampling ($T = T_*$) we get $d_* = d \leqslant 1$ and a less stringent necessary stability condition is $k > 1.5$.

To explore feasibility of inequalities (19) we have applied MATLAB software with YALMIP package for interface and SeDuMi solver for semidefinite programming (see [Löfberg, 2004; Sturm, 1999]). It was found that in the given range of parameters only the product $kd_* = k\tau/T$ is relevant. The modelling results are

Table 1. Feasible pairs $(T,\ kd_*)$ for $4 \leqslant k \leqslant 15, T_* = 0.5\,T$

T	kd_*
0.20	[1.6, 3.8]
0.30	[1.6, 2.4]
0.35	[1.7, 2.0]
0.36	[1.8, 1.9]

Table 2. Feasible pairs (k, τ) for different values T_*, T

k	τ with $T_* = 0.175$, $T = 0.35$	τ with $T_* = T = 0.57$
2	–	[0.55, 0.57]
3	–	[0.37, 0.41]
4	[0.15, 0.17]	[0.28, 0.31]
5	[0.12, 0.14]	[0.22, 0.24]
6	[0.10, 0.11]	[0.19, 0.20]
10	[0.06, 0.07]	[0.11, 0.12]

Table 3. The case of ZOH. Upper bounds T obtained for a nonuniform sampling in [Seifullaev and Fradkov, 2015c]. Upper bounds T_s obtained by simulation for T_s-periodic sampling.

k	T ([Seifullaev and Fradkov, 2015c])	T_s
2	0.68	1.21
3	0.53	0.71
4	–	0.51
5	0.35	0.40
10	0.187	0.20

shown in Table 1. (The bounds for admissible intervals of kd_* were calculated with the accuracy one digit after the decimal point.)

Let $T = 0.35$ be fixed, $T_* = 0.5T$ and the gain k be varied. The intervals of feasible values of τ obtained with the help of Theorem 1 are given in the second column of Table 2. Similar results for a periodic sampling with a period $T = T_* = 0.57$ are given in the third column. It is seen that the greater is the feedback gain, the shorter should be duration of the impulsive action.

8.2 Comparison with the example from [Seifullaev and Fradkov, 2015c]

Let us compare the estimates from Tables 1, 2 with those obtained in [Seifullaev and Fradkov, 2015c] for system (41) with ZOH:

$$f(t) = k\sigma(t_n), \quad t_n \leqslant t < t_{n+1},$$

where $T_n = t_{n+1} - t_n \leqslant T$ for all n. The results of [Seifullaev and Fradkov, 2015c] are given in Table 3. The upper bounds T in the second column were established with the help of a Lyapunov—Krasovskii functional and the Fridman's method. (The value $k = 4$ was not considered.) For comparison, the third column (with the title T_s) contains maximal admissible values of the sampling period that were found by the direct computer simulation for periodic sampling. From Table 3 it is seen that T is close to T_s for high gains, but they differ significantly for low gains ($T_s/T = 1.78$ for $k = 2$). Notice that while the bound T is somewhat conservative, the bound T_s (obtained by simulation) is exact.

To compare these estimates with ours let us take the maximal admissible bound of the sampling period T as the main comparison index — the greater is T, the lower sampling rate is allowable for stabilization.

Since in the case $T_* = 0.5\,T$ only values $k > 3$ are feasible, Theorem 1 gives no results for $k = 2,\ 3$. From Table 1 it is seen that for $4 \leqslant k \leqslant 15$ we can achieve the value $T = 0.37$ by a suitable choice of duty ratios. By a comparison with Table 3 it is seen that for $k = 5,\ 10$ our value $T = 0.37$ exceeds the estimates obtained by the Fridman's method (0.35 and 0.187, respectively). For $k = 10$ the enlargement of T is $0.37/0.187 \approx 2$ times, so the sampling rate can be halved.

Let us consider the case of a periodic sampling $T_* = T$. As it was shown above, then the feasibility region is $k > 1.5$ and includes the values $k = 2,\ 3$. Let us take the exact upper bounds $T = T_s$ for periodic ZOH control from the second column of Table 3. From Table 2 it is seen that for a bounded duty ratio the sampling period can reach the value $T = 0.57$. It is less than the values T_s for $k = 2$ and $k = 3$, but greater than T_s for $k = 4,\ 5,\ 10$. For $k = 10$ the improvement in the sampling period is especially impressive, it is $0.57/0.20 = 2.85$ times. Notice that here we compare our theoretical (partly conservative) estimate with the exact simulated estimate for ZOH.

The example demonstrates how a reasonable choice of the duty ratio can significantly reduce the sampling frequency required for stabilization.

8.3 Computer simulation

Finally, we illustrate the above results with some graphical figures obtained by computer simulation. For simulation assume that $T_n = F_f(\sigma(t_n))$, where

$$F_f(\sigma) = \begin{cases} T - \frac{T-T_*}{\sigma_*}\sigma, & 0 \leqslant \sigma \leqslant \sigma_0, \\ T_*, & \sigma \geqslant \sigma_0, \end{cases} \qquad (47)$$
$$F_f(\sigma) = F_f(|\sigma|) \quad \text{for} \quad \sigma < 0,$$

here σ_* is a given positive constant (see Fig. 1). Obviously, $T_* \leqslant F_f(\sigma) \leqslant T$ for all σ. Transients for system (41) are shown in Fig. 2. The jumps of derivative $\dot{x}_2(t)$ at sampling times are clearly visible.

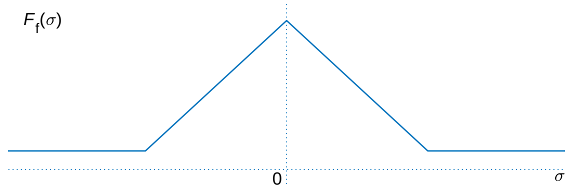

Figure 1. A graph of function $F_f(\sigma)$.

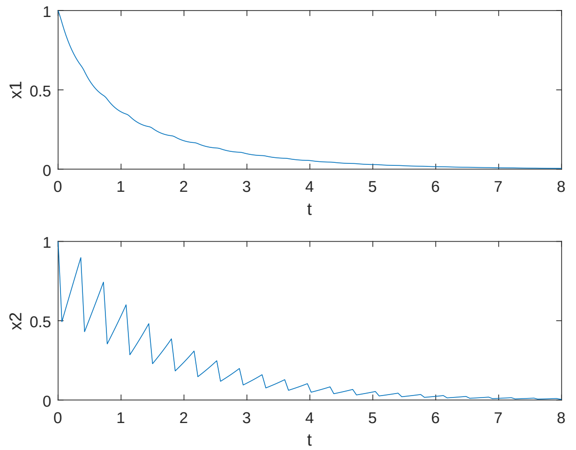

Figure 2. A transient for $k = 10$, $T = 0.36$, $T_* = 0.18$, $\tau = 0.06$ and function (47) with $\sigma_* = 0.5$.

9 Discussion

In a sampled-data control the time axis is divided into a number of sampling intervals $\Delta_n = (t_n,\, t_{n+1})$. In most cases, the control functions $f_n(t)$ considered on these intervals are independent from each other for different n. If we interpret the control signal $f_n(t)$ as a physical force, the major role is played by its total impulse over the sampling interval Δ_n, i. e., by the integral

$$\int_{t_n}^{t_{n+1}} f_n(t)\,dt.$$

In particular, the renown principle of equivalent areas (PEA) put forward in [Andeen, 1960a] suggests that if we choose another forcing function $\tilde{f}_n(t)$ that produces the same total impulse, i. e.,

$$\int_{t_n}^{t_{n+1}} \tilde{f}_n(t)\,dt = \int_{t_n}^{t_{n+1}} f_n(t)\,dt,$$

then the control effect will be generally the same. Thus when PEA is valid, the form of $f_n(t)$ is irrelevant and it can be replaced by a forcing function most convenient for analysis. In particular, $\tilde{f}_n(t)$ can be taken constant on Δ_n, then it is equal everywhere to the averaged value

$$\tilde{f}_n(t) \equiv \frac{1}{t_{n+1} - t_n} \int_{t_n}^{t_{n+1}} f_n(t)\,dt,$$

PEA is usually applicable for short sampling intervals, when the sampling frequency is sufficiently high. Here we apply the idea of PEA to middle-length time intervals, taking the advantage of some freedom of choice for the forcing function $f_n(t)$, with our main effort aimed at making the sampling interval as large as possible. Obviously, in extreme cases we can either spread the force uniformly over the whole sampling interval, or concentrate it in a single point. In the first case (as they do with ZOH) a wide sampling interval introduces a significant time delay and thus worsens dynamics of the system. In this paper we propose a simple solution to this problem — to cut the trailing edge of the control pulse and thus reduce the sampling delay. A numerical example demonstrates that on this way the length of the sampling interval can be increased in times. (Notice that this effect is related not to our mathematical technique, but to the control principle we use.) The mathematics we propose provides an alternative to the massively exploited method of Lyapunov–Krasovskii functionals.

Conclusion

In this paper we recall a stabilization feedback method based on finite pulse width sampling that has been used in electrical and power engineering for a number of years. For the stability analysis we employ a new type of Lyapunov-like functions based on the theory of Yakubovich and Gelig. The stability theorem is stated in terms of a feasibility problem for certain linear matrix inequalities that can be easily solved with standard optimization software. The illustrative example demonstrates a good agreement with the previously obtained results. It is shown that in the case of high feedback gains the sampling period can be enlarged significantly by a suitable choice of duty ratios of the control pulse train.

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