MULTISTABILITY OF TORUS IN MODEL OF LASER WITH LARGE DELAY

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Abstract
We study the dynamics of semiconductor laser with delayed optoelectronic feedback on the base of single-mode rate equations. Bifurcations are analyzed in the case of large delay. Continued sets of quasi-normal system are derived in the form of multi-dimensional space-time equations. Possible coexistence is demonstrated of large number of multi-frequency attractors.

Key words
Delayed Feedback, Lasers, Asymptotic Analysis

1 Introduction

Phenomenon of coexistence of two or more attractors in nonlinear system is under permanent studying in view of various technical applications. Recently delay-induced multistability is once discussed [Yanchuk and Perlikowski, 2009]– [Erneux, 2008]. The peculiarity of the effect is that such systems demonstrate a number of the stable oscillating regimes. Moreover one can control the effect as a number of cycles increases with the delay increases.

In this paper we apply local bifurcation analysis in order to derive quasi-normal forms in the case of the sufficiently large delay in optoelectronic feedback (FB) controlling the pumping. In this way multistability is shown of one-frequency cycles like demonstrated in [Yanchuk and Perlikowski, 2009] that can be called FB modes. In addition, coexistence is demonstrated a wide set of multi-frequency attractors (torus). The quasi-normal forms derived are the space-time equations of universal structure.

Dynamics of the semiconductor laser with delayed optoelectronic FB is governed by the equations

$$\begin{aligned}\dot{u} =& vu(y-1),\\ \dot{y} =& q-y-yu+\alpha u(t-T),\end{aligned} \tag{1}$$

where u_i and y_i are proportional to the photon density in the cavity and to the population inversion in the laser; v is the ratio of photon damping time in the cavity to the relaxation time of inversion of population; q is the constant part of the pumping rate determined by constant injection current; $\alpha u(t-T)$ is the part of the pumping rate modulated by the intensity at the delayed moment $(t-T)$. The delay T is the time of propagation and transformation signal in optoelectronic FB, α is the FB coefficient. In laser systems it is reasonable to get $-1<\alpha<1$. Mathematical model (1) is valid for the dynamical regimes with the characteristic time of intensity changing much more than the time propagation over the diode resonator.

Optoelectronic FB scheme in the pumping has been firstly applied with the aim of stabilization of spike frequency in laser diodes. The method has been also used for dynamical control in a gas laser, and in solid state microchip lasers.

In papers [Grigorieva, 1993] on the base of special asymptotic technics spiking in the delayed system (1) has been studied in the case of the large parameter v (in class-B lasers 10^3-10^4). In papers [Grigorieva, 1999; Bestehorn, 2000] local dynamics has been considered under large v in the vicinity of the stationary state (u_0, v_0) with

$$u_0=\frac{q-1}{1-\alpha}, \qquad y_0=1. \tag{2}$$

It has been found that the critical cases in the stability problem are of asymptotically infinite dimension. With the method [Kaschenko, 1996] quasinormal forms have been constructed that determine the dynamical evolution of the order parameters. In dependence on the relation between large values T and v these quasinormal forms can be delayed equations or space-time parabolic equations.

In this paper the main assumption is that the only time delay is large,

$$T\gg 1 \text{ or } \varepsilon=T^{-1}\ll 1. \tag{3}$$

while other parameters are of the order of unit.

We consider the solutions to Eqs.(1) with the initial conditions from the sufficiently small ε-independed vicinity of the stationary state (2). Applying the normalizing method given in [Kaschenko, 1996], we construct asymptotically continued sets of nonlinear evolution equations. Their nonlocal dynamics determine the local dynamics of the original system (1). Each representative from the set determines the time behavior of special space like structure. Thus multistability can occure.

In the small vicinity of the stationary state let substitute $u = u_0 + u_1, \qquad y = y_0 + y_1$ into the Eqs.(1). Dropping the index 1 we get the system with zeroth stationary state:

$$\begin{cases} \dot{u} = vu_0 y + vuy, \\ \dot{y} = -u - (1+u_0)y + \alpha u(t-T) - uy. \end{cases} \tag{4}$$

2 Linear analysis

Here we consider the linearized system

$$\dot{w} = Aw + \alpha Bw(t-T), \tag{5}$$

where $w = (u,y)^T$, $A = \begin{pmatrix} 0 & b \\ -1 & -a \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $a = 1 + u_0 = \frac{q-\alpha}{1-\alpha}$, $b = vu_0 = \frac{v(q-1)}{1-\alpha}$.

Dynamics of Eqs. (5) is determined by the roots of the characteristic equation

$$\lambda^2 + a\lambda + b = \alpha b \exp(-\lambda T). \tag{6}$$

Taking in Eq.(6) $\lambda = i\omega$ we get

$$-\omega^2 + i\omega a + b = \alpha b \exp(-i\omega T), \tag{7}$$

from which the equation

$$(b-z)^2 + a^2 z = \alpha^2 b^2 \tag{8}$$

follows where $z = \omega^2 \geq 0$. Evidently, Eq.(8) has solution $z \geq 0$ under some α. In particular, $z = \omega^2 = 0$ under $\alpha = \pm 1$ but there is no solutions under $\alpha = 0$. Hence, for $\alpha = 0$ all roots of Eq. (6) have negative real parts and, in turn, Eqs. (4) have the solutions tend to $u = y = 0$ under $t \to \infty$.

The bifurcation boundary corresponds to the first positive $\alpha = \alpha_+ \leq 1$ and the first negative $\alpha = \alpha_- \geq -1$ for which Eq.(8) can be solved while for $\alpha \in [0, \alpha_+)$ and $\alpha \in (\alpha_-, 0)$ Eq. (8) has no roots.

In order to find $\alpha_+ \ \ \alpha_-$ we rewrite Eq.(8) as

$$z^2 - (2b - a^2)z + b^2(1-\alpha^2) = 0. \tag{9}$$

Denote discriminant of Eq. (9)

$$d(\alpha) = P(\alpha)(1-\alpha)^{-4},$$

where $P(\alpha) = (q-\alpha)^4 - 4v(q-1)(q-\alpha)^2(1-\alpha) + 4v^2(q-1)^2\alpha^2(1-\alpha)^2$. Let $\alpha_+ = \min(\widetilde{\alpha}_+, 1)$ and $\alpha_- = \max(\widetilde{\alpha}_-, -1)$ are the minimal positive (maximal negative) roots of the equation

$$P(\widetilde{\alpha}_\pm) = 0.$$

For sufficiently large values v one can get

$$\alpha_\pm = \pm v^{-1/2} \frac{q}{\sqrt{q-1}} + O(v^{-1}) \tag{10}$$

The roots $z_\pm = \omega_\pm^2$ of Eq.(9) under $\alpha = \alpha_+ \ \ \alpha = \alpha_-$ are

$$z_\pm = \frac{v(q-1)}{(1-\alpha_\pm)} - \frac{(q-\alpha_\pm)^2(1-\alpha_\pm)^2}{2}. \tag{11}$$

Below we set α_0 is α_+ or α_- and ω_0 is ω_+ or ω_-.

Let us introduce the function $\Theta = \Theta(\varepsilon) \in [0, 2\pi)$ that makes the value $(\omega_0 \varepsilon^{-1} + \Theta)$ to be multiple 2π and the parameter $\varkappa \in [0, 2\pi)$ for which the equation

$$\alpha_0 b_0 \exp(-i\varkappa) = b_0 - \omega_0^2 + i\omega_0 a_0, \tag{12}$$

is valid where $a_0 = (q - \alpha_0)(1-\alpha_0)^{-1}$, $b_0 = v(q-1)(1-\alpha_0)^{-1}$. With these notations and under the condition

$$\alpha = \alpha_0 + \varepsilon^2 \alpha_1 \tag{13}$$

we represent asymptotically some group of the roots $\lambda_k(\varepsilon)$, $(k = 0, \pm 1, \pm 2, \ldots)$ to the characteristic Eq.(6) which fall into the small vicinity of the imaginary axis

$$\lambda_k(\varepsilon) = i\omega_0 + i\varepsilon(\Theta + 2k\pi + \varkappa) + \varepsilon^2 \lambda_{k1} + \varepsilon^3 \lambda_{k2} + O(\varepsilon^4), \tag{14}$$

where

$$\lambda_{k1} = -i\Delta[\Theta + 2k\pi + \varkappa], \tag{15}$$

$$\begin{aligned} \Delta &= -i(ia_0 - 2\omega_0)(\alpha_0 b_0)^{-1}\exp(i\varkappa), \\ \lambda_{k2} &= \sigma(\Theta + 2k\pi + \varkappa)^2 + \Delta(\Theta + 2k\pi + \varkappa) + C, \\ \sigma &= \frac{1}{2}\Delta^2 + [a_0\Delta + 2i\Delta\omega_0][4\omega_0^2 + a_0^2]^{-1}, \\ C &= \Big[i\omega_0 a_1 + (1-\alpha_0)b_1 + \alpha_1 b_0 \exp(-i\varkappa)\Big] \\ &\quad \cdot \Delta(a_0 - 2i\omega_0)(4\omega_0^2 + a_0^2)^{-1}, \\ a_1 &= \alpha_1(q-1)(1-\alpha_0)^{-2}, \\ b_1 &= \alpha_1 v(q-1)(1-\alpha_0)^{-2}. \end{aligned} \tag{16}$$

Note, $\Im\Delta = 0$, hence the solutions do not leave local vicinity of the stationary state under the condition

$$\Re\sigma > 0 \tag{17}$$

that is always valid for sufficiently large v.

The representations $\lambda_k(\varepsilon)$ are discontinues relative to ε as the function $\Theta(\varepsilon)$ is discontinues and $|\lambda_k(\varepsilon)| \to \infty$ as $\varepsilon \to 0$. It is also important that asymptotical representations (14) are non-uniform relative to number $k = 0, \pm 1, \pm 2, \ldots$.

Some groups of the characteristic roots an be represented in another form. To this end we introduce the parameter $\gamma \in (0,1)$ and choose n arbitrary positive values $\omega_1, \ldots, \omega_n$. Let us also introduce the function $\Theta_j = \Theta_j(\omega, \varepsilon) \in [0, 2\pi)$ that makes the expression $\omega_j \varepsilon^{-\gamma} + \Theta_j(\omega, \varepsilon)$ to be multiple 2π. Let $\Omega = (\omega_1, \ldots, \omega_n)$, $\Theta = (\Theta_1, \ldots, \Theta_n)$, $K = (k_1, \ldots, k_n)$ where $k_j \in Z,\ j = 0, \ldots, n$. Then in rather wide vicinity of the bifurcation point

$$\alpha = \alpha_0 + \varepsilon^{2\gamma}\alpha_1 \tag{18}$$

the group of the characteristic roots to Eq.(6) with vanishing real part under $\varepsilon \to 0$ can be asymptotically represented as

$$\begin{aligned}\lambda_k(\Omega, \varepsilon) = {} & i\omega_0 + i(\Omega, K)\varepsilon^{1-\gamma} + \\ & i\varepsilon\Big(\Theta_0 + (\Theta, K) + 2\pi K_0 + \varkappa\Big) - \\ & i2\varepsilon^{2-\gamma}\frac{(\Omega, K)}{\sigma} + 2\varepsilon^{3-2\gamma}(\Omega, K)^2\sigma + \\ & \varepsilon^{2\gamma}C + o(\varepsilon^{2\gamma}, \varepsilon^{2(1-\gamma)}).\end{aligned} \tag{19}$$

Note, to eigenvalues $\lambda_k(\varepsilon)$ and $\lambda_k(\Omega, \varepsilon)$ there correspond the eigenvectors of the linearized Eqs.(6)

$$w(t, \varepsilon) = [w_0 + o(1)] \exp\Big[i\omega_0\varepsilon^{-1}(1 + o(1))t\Big]$$

where $w_0 = (b_0, i\omega_0)$ and the vector functions $w_0 \exp(i\omega_0 t)$ and $\overline{w}_0 \exp(-i\omega_0 t)$ are the solutions of the linear system

$$\dot{w} = Cw, \tag{20}$$

where $C = A_0 + \alpha_0 \exp(-i\varkappa)B_0$.

3 Quasinormal form in small vicinity of equilibrium

It follows from Eq.(14) that in the small vicinity of the bifurcation point

$$\alpha = \alpha_0 + \varepsilon^2\alpha_1$$

there is the (asymptotically) infinite set of the characteristic roots with vanishing real part under $\varepsilon \to 0$. Hence, the critical case of infinite dimension is realized. In order to construct the normal form we introduce the formal expansion

$$\begin{aligned}U(t, \varepsilon) = {} & \varepsilon\Big[w(\tau, \tau_1)\exp(i\omega_0 t)w_0 + c.c.\Big] + \\ & + \varepsilon^2 U_2(t, \tau, \tau_1) + \varepsilon^3 U_3(t, \tau, \tau_1) + \ldots,\end{aligned} \tag{21}$$

where $\tau = \varepsilon^3 t,\ \tau_1 = \varepsilon(1 + \varepsilon\Delta)t,\ U_2(t, \tau, \tau_1)$, $U_3(t, \tau, \tau_1)$ are $(2\pi/\omega_0)$-periodic functions relative to the first argument, and

$$w(\tau, \tau_1) = \sum_{k=-\infty}^{\infty} \xi_k(\tau)\exp\Big(ir_k\tau_1\Big).$$

with $r_k = (\Theta + 2k\pi + \varkappa)$.

It is suitable to rewrite Eqs.(4) in vector form for $U = (u_1, u_2)$, $(u_1 = u,\ u_2 = y)$:

$$\dot{U} = AU + \alpha BU(t - \tau) + u_1u_2V,\ V = \begin{pmatrix} v \\ -1 \end{pmatrix}. \tag{22}$$

We introduce Eq.(21) into Eq.(22) and collect the coefficients at the same order of small ε. On the second step we get the system to determine $U_2 = U_{21}\exp(i\omega_0 t) + \overline{U}_{21}\exp(-i\omega_0 t) + U_{22}\exp(2i\omega_0 t) + \overline{U}_{22}\exp(-2i\omega_0 t)$:

$$\begin{aligned}CU_{21} = {} & -i\left(\sum_{k=-\infty}^{\infty}\xi_k(\tau)r_k e^{ir_k\tau_1}\right)w_0 + \\ & + i\alpha_0\Delta e^{-i\varkappa}\left(\sum_{k=-\infty}^{\infty}\xi_k(\tau)r_k e^{ir_k\tau_1}\right)B_0w_0,\end{aligned} \tag{23}$$

$$C_0U_{22} = i\omega_0 b_0 w^2(\tau, \tau_1)V, \tag{24}$$

where $C_0 = 2i\omega_0 T - A_0 - \alpha_0\exp(-2i\varkappa)B_0$.

From Eqs.(24) one finds

$$U_{22} = i\omega_0 b_0 w^2(\tau, \tau_1)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The solution U_{21} exists if and only if the right-hand vector is orthogonal to the vector

$$w_1 = (a_0 - i\omega_0, b_0)^T \tag{25}$$

that is valid in the vicinity of the bifurcation point. Hence we get

$$U_{21} = \left(\sum_{k=-\infty}^{\infty}\xi_k(\tau)r_k\exp\left[ir_k\tau_1\right]\right)w_{00}. \tag{26}$$

Here w_{00} is the solution to the system

$$Cw_{00} = i[\alpha_0 \Delta \exp(-i\varkappa)B_0 - I]w_0. \quad (27)$$

On the third step we collect coefficients at ε^3. The system obtained is linear relatively to $U_3(t, \tau, \tau_1)$. It includes inhomogeneous terms $\exp(in\omega_0 t)$, $n = 0, \pm 1, \pm 2, \pm 3$, hence the solution includes these harmonics too, $U_{3n}(\tau, \tau_1)$ for which we get

$$CU_3 = \Gamma, \quad (28)$$

where

$$\begin{aligned}\Gamma =& w[A_1 + \alpha_1 e^{-i\varkappa} B_0]w_0 + \alpha_0 \frac{dw}{d\tau} e^{-i\varkappa} B_0 w_0 - \\ & i\delta \left(\sum_{k=-\infty}^{\infty} \xi_k r_k^2 e^{ir_k \varepsilon_1} \right) w_{00} - \\ & \frac{1}{2}\Delta \left(\sum_{k=-\infty}^{\infty} \xi_k r_k^2 e^{ir_k \tau_1} \right) \cdot \alpha_0 e^{-i\varkappa} B_0 w_0 + \\ & \omega_0 b_0 [\omega_0 v_1 + i b_0 v_2] |w|^2 w V, \\ A_1 =& (q-1)(1-\alpha_0)^{-2} \alpha_1 \begin{pmatrix} 0 & v \\ 0 & 1 \end{pmatrix}.\end{aligned} \quad (29)$$

Eqs.(28) can be solved if $(\Gamma, z_0) = 0$ is vald. Takig into account Eqs.(25)– (29) we get the system for $w(\tau, \tau_1)$. Moreover one can see that

$$\begin{aligned}\frac{\partial w}{\partial \tau_1} &= i \sum_{k=-\infty}^{\infty} \xi_k(\tau) r_k \exp\left[i r_k \tau_1\right], \\ \frac{\partial^2 w}{\partial \tau_1^2} &= -\sum_{k=-\infty}^{\infty} \xi_k(\tau) r_k^2 \exp\left[i r_k \tau_1\right].\end{aligned}$$

Thus the equation for $w(\tau, x)$ can be represented as parabolic problem

$$\begin{aligned}\frac{\partial w}{\partial \tau} = \sigma \frac{\partial^2 w}{\partial x^2} - 2\sigma(\Theta + \varkappa) \frac{\partial w}{\partial x} + \\ + [\sigma(\Theta + \varkappa)^2 + c]w + \delta |w|^2 w\end{aligned} \quad (30)$$

with periodic boundary conditions

$$w(\tau, x+1) \equiv w(\tau, x). \quad (31)$$

The coefficient δ in Eq.(30) reads

$$\delta = \omega_0(\omega_0 v_1 + i b_0 v_2)[v(a_0 + i\omega_0) + b_0](\alpha_0 b_0^{-1}) e^{i\varkappa}.$$

The coefficients of Eq.(30) depend through Θ from the small parameter ε. If $\varepsilon \to 0$ the function $\Theta(\varepsilon)$ infinitely change its value from $[0, 2\pi)$. Hence, the dynamics is sensitive to variation of the small ε. Let $\varepsilon_n(\Theta_0)$ is the sequence $\varepsilon_n(\Theta_0) \to 0$ then $\Theta(\varepsilon_n(\Theta_0)) = \Theta_0$.

Finally, if the boundary problem (30), (31) has the solution $u^*(\tau, x)$ then the original system (4) under $\varepsilon = \varepsilon_n(\Theta_0)$ has the asymptotic solution

$$x(t, \varepsilon) = \varepsilon u^* \left(\varepsilon^3 t, \varepsilon \left(1 + \varepsilon\Delta + \Theta(\varepsilon^2)\right) t \right) e^{ir_\varepsilon t} + c.c. \quad (32)$$

where $r(\varepsilon) = \omega_0 \varepsilon^{-1} + \Theta_0 + \varkappa + \Theta(\varepsilon)$ In particular, if $u^*(\tau, x)$ is a stable periodic solution then the original system has the stable torus of given by Eq.(32).

The boundary problem (31), (32) is close to Ginzburg-Landau equation that demonstrates complex dynamics, in particular coexistence of travelling waves. Thus we expect such multistability in Eqs.(4). The quantity of such solutions grow with the delay increases [Yanchuk and Perlikowski, 2009].

4 Quasinormal form in wide vicinity of equilibrium

It follows from Eq.(19) that in the rather wide vicinity of the bifurcation point

$$\alpha = \alpha_0 + \varepsilon^{2\gamma} \alpha_1, \quad \gamma \in (0, 1) \quad (33)$$

there is the (asymptotically) infinite set of the characteristic roots with vanishing real part under $\varepsilon \to 0$.

In such a wide vicinity we seek the solution in the form of the series

$$\begin{aligned}U =& \varepsilon^\gamma \left[e^{ih(\varepsilon)t} \sum_k \xi_k(\tau) e^{i(K, T_1)} w_0 + c.c. \right] + \\ & + \varepsilon^{2\gamma} U_2 + \varepsilon^{3\gamma} U_3 + \ldots .\end{aligned} \quad (34)$$

Here $h(\varepsilon) = \omega_0 + \varepsilon(\Theta + \varkappa)$, $\tau = \varepsilon^{2\gamma+1} t$, $T_1 = (t_1, \ldots, t_n)$, $t_j = \left(\omega \varepsilon^{1-\gamma} + \varepsilon \Theta_j + \varepsilon^{\gamma+1} \omega_j \Delta\right) t$. In Eq.(34) we sum by any integer sets $K = (k_1, \ldots, k_n)$. Te vector functions $U_j = U_j(t, \tau, t_1, \ldots, t_n)$ ($j = 2, 3$) are 2π-periodic relatively to t_i. Introducing the formal expansion (34) into (22) and collecting coefficients at the same order of ε we get quasinormal forms

$$\frac{\partial w}{\partial \tau} = \sigma \left(\frac{\partial}{\partial x_1} + \ldots + \frac{\partial}{\partial x_n} \right)^2 w + cw + \gamma |w|^2 w \quad (35)$$

with periodic boundary conditions for each space argument $x_j,\ j = 1, \ldots, n$:

$$w(\tau, x_1, \ldots, x_j + \frac{2\pi}{\omega_j}, x_{j+1}, \ldots, x_n) \equiv w(\tau, x_1, \ldots, x_n). \quad (36)$$

If for some $\omega_1, \ldots, \omega_n$ the quasinormal form (35), (36) has the limited solution $w^*(\tau, x_1, \ldots, x_n)$ then

Eqs. (22) have the asymptotic solution

$$U(t,\varepsilon) = \varepsilon^{\gamma}\Big[e^{ih(\varepsilon)t}w^*(\varepsilon^{1+2\gamma}t, t_1, \dots, t_n) \\ +c.c.\Big]O(\varepsilon^{2\gamma}),$$

where $t_j = (\omega_j \varepsilon^{\gamma} + \varepsilon\Theta(\omega_j, \varepsilon) + o(\varepsilon))t$.

The system (35), (36) is degenerate parabolic problem with rather complex dynamics. In addition, the system includes arbitrary integer parameter $n = 1, 2, 3, 4, \dots$ and n arbitrary continued parameters $\omega_0, \dots, \omega_n$. To each set $(n, \omega_0, \dots, \omega_n)$ there can corresponds the space-time structure in the quasinormal form and, in turn, the quasiperiodic solution to the original system. Thus hyper-multistability can occur in sufficiently wide vicinity of the bifurcation point.

Numerical simulation support the last conclusion. Fig.1 shows that various torus coexist under the same parameters for different initial functions.

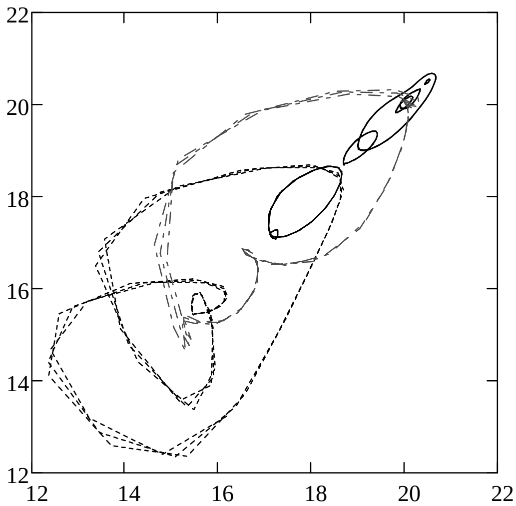

Figure 1. The map of local maxima Umax$_{i+2}$(Umax$_i$) for coexisting torus solutions obtained under different initial function $u(s) = u_s(1 + 0.5\sin(\omega_{in}s))$, $\omega_{in} = 2\pi n/T$, $n = 25, 26, ...33$ Other parameters $v = 20$, $T = 16$, $q = 3.5$, $\alpha = 0.7$

Despite seeming complexity of boundary problems (35), (36), they allow to simplify studying of local dynamics (22) because of excluding fast oscillating functions.

Dynamics of Eqs.(35), (36) depends essentially on α_1, γ. One can specify these parameter relatively to the small parameter $\varepsilon = T^{-1}$. For example, let $\varepsilon = 10^{-3}$ is sufficiently small value and α is close to the critical value, i.e. $\alpha = \alpha_0 + \mu$ where $\mu = 10^{-1}$ or $\mu = 10^{-2}$ is the small deviation from the bifurcation point. Setting $\gamma \in \left[\frac{1}{4}, \frac{1}{2}\right]$ and $\mu = \varepsilon^{2\gamma}\alpha_1$ we find

1. if $\mu = 10^{-1}$ then $\alpha_1 \in \left[\sqrt{10}, 10^2\right]$;
2. if $\mu = 10^{-2}$ then $\alpha_1 \in \left[\frac{1}{\sqrt{10}}, 10\right]$.

To each α_1 from the intervals obtained there can correspond the steady regime in Eqs.(1) under the same parameters v, q, α and T.

In conclusion, we show that the dynamics of the laser system with large time delay can be described by continued sets of quasi-normal system in the form of multidimensional space-time equations. To each solution of the quasi-normal form there corresponds the torus in the original system. Thus coexistence is demonstrated of large number of multi-frequency attractors.

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