STABILITY OF STEADY STATE MODES AT PRIMARY RESONANCE IN NONLINEAR 2DOF SYSTEMS WITH CLOSE NATURAL FREQUENCIES

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Abstract

The interaction of external and 1:1 internal resonances in 2DOF nonlinear systems with symmetric cubic characteristics is studied. Steady–state modes in undamped systems are studied, with an emphasis on stability properties of coupled steady-state modes (CSMs) that appear in a vicinity of the primary resonance.

Key words

2DOF nonlinear systems, internal resonance, stability.

1 Introduction

Dynamic behavior of nonlinear systems with closed eigenfrequencies for at least two natural modes (circular plates, disks, cylindrical shells and other structures) was studied in a few works (see, e. g., [Williams and Tobias, 1963; Evensen, 1966; Sridhar, Mook and Nayfeh, 1978; Yasuda and Asano, 1986; Nayfeh and Balachandran, 1989; Nayfeh and Balachandran, 1993; Ribeiro and Petyt, 1999; Touze, Thomas and Chaigne, 2002; Vakakis, 1992; Manevich A. I., Manevich L. I., 2005]. It has been established that 1:1 internal resonance results in appearance of coupled (two-mode) steady-state and non-stationary oscillations under external resonance condition.

The stability properties of these modes until now were studied only on illustrative examples or for particular structures, and some general results have been obtained only for autonomous systems [10]. In this paper the interaction of 1:1 internal resonance with primary external resonance under harmonic excitation in 2DOF cubic symmetric systems is investigated with focusing on stability properties of the steady-state coupled modes for undamped systems.

2 Governing Equations and Solution by the Multiple Scales Method

2.1 Equations of motion

In the main cubic approximation for symmetric systems the unharmonic part of the potential can be assumed in the form

$$V = \varepsilon\left(\frac{1}{4}b_{11}U_1^4 + \frac{1}{2}b_{12}U_1^2U_2^2 + \frac{1}{4}b_{22}U_2^4\right), \quad (2.1)$$

where U_1, U_2 are principal coordinates of the linearized system, ε is a small parameter. An external force F (per unit of mass) is assumed to act only on the first degree of freedom; damping coefficients are supposed to have the order of ε, these coefficients for both principal coordinates being the same. So equations of motion are as follows:

$$\left.\begin{aligned} &\frac{d^2U_1}{dt^2} + 2\varepsilon\mu\frac{dU_1}{dt} + \omega_1^2U_1 + \\ &\quad \varepsilon\,(b_{11}U_1^3 + b_{12}U_1U_2^2) = F\sin\Omega t, \\ &\frac{d^2U_2}{dt^2} + 2\varepsilon\mu\frac{dU_2}{dt} + \omega_2^2U_2 + \\ &\quad \varepsilon\,(b_{12}U_1^2U_2 + b_{22}U_2^3) = 0 \end{aligned}\right\}. \quad (2.2)$$

The eigenfrequencies are assumed to be close, and the difference between the external force frequency and the first eigenfrequency is also assumed to be small. These conditions are specified by the following expressions:

$$\omega_1 \equiv \omega, \quad \omega_2 \equiv \omega + \varepsilon\sigma, \quad \Omega = \omega + \varepsilon\delta \quad (2.3)$$

where σ and δ are detuning parameters. After introducing dimensionless variables $\tau = \Omega t$, $u = U/U^0$, where U^0 is a characteristic scale, the set (2.2) with account of (2.3) and smallness of ε takes the form:

$$\begin{rcases}\ddot{u}_1+u_1=\varepsilon\,(f\,\sin t-2\tilde{\mu}\dot{u}_1+2\,\tilde{\delta}u_1-c_{11}u_1^3-c_{12}\,u_1u_2^2),\\ \ddot{u}_2+u_2=\varepsilon\,[-2\tilde{\mu}\dot{u}_2+2(\tilde{\delta}-\tilde{\sigma})\;u_2-c_{12}u_1^2u_2-c_{22}u_2^3\,],\end{rcases} \tag{2.4}$$

where the upper dot denotes differentiation with respect to τ and following notations are introduced:

$$\tilde{\mu}=\frac{\mu}{\Omega},\;\tilde{\delta}=\frac{\delta}{\Omega},\;\tilde{\sigma}=\frac{\sigma}{\Omega},\,c_{ij}=\frac{b_{ij}\,(U^0)^2}{\Omega^2},\;f=\frac{F}{\Omega^2\,U^0} \tag{2.5}$$

2.2 Solution by the multiple scales method

The set of equations (2.4) has been solved by the multiple scales method. Introducing the fast and slow times $T_0=t$, $T_1=\varepsilon\,T_0$, ..., expanding the solution in series by ε

$$u_k=u_{ko}(T_0,T_1,...)+\varepsilon u_{k1}(T_0,T_1,...)+...(k=1,2),$$

and applying the standard procedure of the method, one obtains the equations of the first and second order approximations. Solution of the first order equations can be taken in the form

$$u_{k0}=a_k\cos(\Omega t+\theta_k)\quad(k=1,2) \tag{2.6}$$

Then in the second order approximation the conditions of vanishing secular terms results in the following set of equations governing the slow modulation of amplitudes a_1,a_2 and phase components θ_1, θ_2 of two linear modes:

$$\begin{rcases}\frac{da_1}{dT_1}+\tilde{\mu}a_1-\frac{c_{12}}{8}\,a_1a_2^2\sin2\gamma=-\frac{f}{2}\cos\theta_1,\\ a_1\frac{d\theta_1}{dT_1}-\tilde{\delta}a_1+\frac{3c_{11}}{8}a_1^3+\frac{c_{12}}{8}\,a_1a_2^2\left(2+\cos2\gamma\right)=\frac{f}{2}\sin\theta_1,\\ \frac{da_2}{dT_1}+\tilde{\mu}\,a_2+\frac{c_{12}}{8}a_1^2a_2\sin2\gamma=0,\\ a_2\frac{d\theta_2}{dT_1}-(\tilde{\delta}-\tilde{\sigma})a_2+\frac{3c_{22}}{8}a_2^3+\frac{c_{12}}{8}a_1^2\,a_2\left(2+\cos2\gamma\right)=0,\end{rcases} \tag{2.7}$$

where $\gamma=\theta_2-\theta_1$. It can be seen that equations (2.7) in the case $\mu=0$ may be written in a "quasihamiltonian" form

$$a_k\frac{d\,\theta_k}{d\,T_1}=\frac{\partial H}{\partial a_k},\;a_k\frac{d\,a_k}{d\,T_1}=-\frac{\partial H}{\partial\theta_k}\;(k=1,2), \tag{2.8}$$

where Hamiltonian H is as follows:

$$H=\frac{1}{2}\tilde{\delta}a_1^2+\frac{1}{2}(\tilde{\delta}-\tilde{\sigma})\,a_2^2-\frac{3}{32}\left(c_{11}\,a_1^4+c_{22}\,a_2^4\right)-\frac{1}{16}c_{12}\,a_1^2\,a_2^2\left[2+\cos2(\theta_2-\theta_1)\right]+\frac{1}{2}a_1f\sin\theta_1 \tag{2.9}$$

So the first integral of set (2.7) for $\mu=0$ is

$$H(a_1,a_2,\theta_1,\theta_2)=C \tag{2.10}$$

3 Steady-state modes in undamped systems

Consider steady-state (stationary) modes in *undamped* systems. For $a_k=const$, $\theta_k=const$ ($k=1,2$), one obtains the following set of equations:

$$\begin{rcases}a_1\,a_2^2\sin2\gamma=4\,f_0\cos\theta_1,\\ 3\alpha_1a_1^3+a_1a_2^2\left(2+\cos2\gamma\right)-8\,\delta_*\,a_1=4\,f_0\sin\theta_1,\\ a_1^2a_2\sin2\gamma=0,\\ 3\alpha_2\,a_2^3+a_1^2a_2\left(\,2+\cos2\gamma\right)-8\,(\delta_*-\sigma_*)\,a_2=0,\end{rcases} \tag{3.1}$$

where the nonlinear coefficients, detuning parameters and the external force are normalized by dividing them by c_{12}:

$$\alpha_1=\frac{c_{11}}{c_{12}},\;\alpha_2=\frac{c_{22}}{c_{12}},\;\delta_*=\frac{\tilde{\delta}}{c_{12}},\;\sigma_*=\frac{\tilde{\sigma}}{c_{12}},\;f_0=\frac{f}{c_{12}} \tag{3.2}$$

The ***uncoupled*** (driven) mode, for which $a_2=0$, is governed by the equation (from the second equation (3.1))

$$3\alpha_1\,a_1^3-8\,\delta_*a_1=4\,f_0\sin\theta_1, \tag{3.3}$$

where (from the first equation (3.1)) $\theta_1=(2m-1)\pi/2$, $m=0,1,...$, so $\sin\theta_1=\pm1$. Equation (3.3) provides the frequency response curve (f. r. c.) for uncoupled steady-state modes with δ_* as a frequency parameter. Alternation of sign of $\sin\theta_1$ is equivalent to alternating a_1 sign, so one may assume $\sin\theta_1=1$, but allow negative a_1, see Fig. 1 (here $\alpha_1=0.5$, $f_0=0.5$).

The fourth equation (3.1) with $a_1=0$ determines the "backbone curve" for the second mode ("companion mode"):

$$3\,\alpha_2\,a_2^2-8\,(\delta_*-\sigma_*)=0\,. \tag{3.4}$$

For the **coupled** steady-state modes (CSM), from the third and the first equation (3.1) we obtain

$\sin 2(\theta_2 - \theta_1) = 0$, $\cos\theta_1 = 0$, so the following relations for the phase differences between two degrees of freedom γ and for θ_1 hold for undamped systems:

$$\gamma \equiv \theta_2 - \theta_1 = n\frac{\pi}{2} \;\; (n = 0, 1, 2, \dots),$$

$$\theta_1 = (2m-1)\frac{\pi}{2} \;\; (m = 0, 1, \dots). \quad (3.5)$$

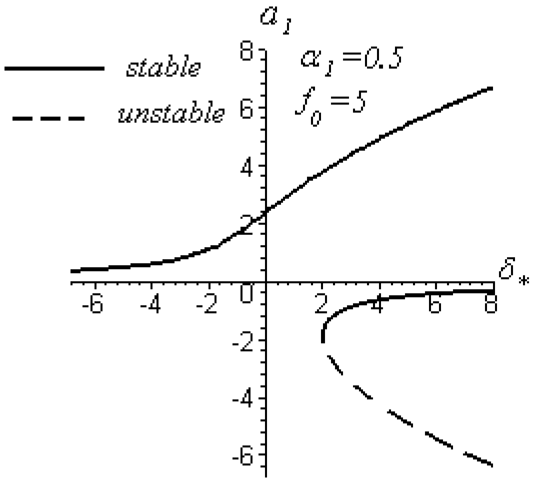


Fig. 1. Frequency response curves for uncoupled steady-state modes with allowing negative a_1.

For $n = 0$ and *even* n the phase difference γ between two linear modes can be assumed to be equal to 0 or π. These are **normal** modes (NM) – in-phase or anti-phase oscillations. For *odd* n the phase difference γ can be assumed equal to $\pi/2$ or $-\pi/2$. These are **elliptic** modes (EM) for which maximal deflection in one linear mode corresponds to zero in another one, and inversely. For both normal and elliptic modes the phase difference θ_1 between the external force and the driven mode can be assumed equal to $\pm\pi/2$.

With these θ_1, θ_2 values the set of equations with respect to amplitudes a_1, a_2 takes the form:

$$\left.\begin{aligned} &-8\,\delta_* a_1 + 3\,\alpha_1 a_1^3 + a_1 a_2^2 (2\pm1) = 4\, f_0 \sin\theta_1, \\ &a_2\left[-8\,(\delta_* - \sigma_*) + 3\,\alpha_2 a_2^2 + a_1^2\,(2\pm1)\right] = 0, \end{aligned}\right\} \quad (3.6)$$

where the upper sign relates to the normal modes and the lower sign to the elliptic modes. For coupled modes the expression in square brackets in the second equation (3.6) equals to 0, whence

$$a_2^2 = \frac{1}{3\,\alpha_2}\left[8(\delta_* - \sigma_*) - (2\pm1)a_1^2\right]. \quad (3.7)$$

Excluding then a_2 from the first equation (3.6) one obtains cubic equations governing amplitudes of the driven mode for NMs and EMs:

$$\frac{3}{8}a_1^3\left(\alpha_1 - \frac{(2\pm1)^2}{9\alpha_2}\right) - a_1\left(\delta_* - \frac{(2\pm1)(\delta_* - \sigma_*)}{3\alpha_2}\right) = \frac{1}{2} f_0 \sin\theta_1 \quad (3.8)$$

From (3.7) the condition of existence of CSMs follows:

$$\frac{1}{\alpha_2}\left[8(\delta_* - \sigma_*) - (2\pm1)\,a_1^2\right] \geq 0 \quad (3.9)$$

Equations (3.8) (together with (3.9)) determine the f. r. curves for CSMs, where δ_* plays role of a frequency parameter. Note that the f. r. c. for coupled modes should be constructed in 3D space (a_1, a_2, δ_*). Uncoupled modes are depicted by curves lying in the planes (a_1, δ_*) or (a_2, δ_*)

If the CSM's frequency response curve is branching off the f. r. c. for the driven uncoupled mode then for the bifurcation (branching) point $a_2 = 0$, so along with equation (3.3) this point satisfies condition (3.9) as equality. After exclusion of δ_* from these equations one obtains the following equations for the branching points (for NMs and EMs, index "b" stands for branching) :

$$\left[3\alpha_1 - (2\pm1)\right]a_{1b}^3 - 8\sigma_* a_{1b} - 4\, f_0 \sin\theta_1 = 0, \quad (3.10)$$

$$\delta_{*b} = \sigma_* + \frac{2\pm1}{8}a_{1b}^2$$

So there exist one or three branching points on the f. r. c. (for a given σ_*), separately for NMs and EMs.

It follows from (3.10) that $\delta_{*b} > \sigma_*$. With account of denotations (2.3) and (3.2) one can conclude that *in case b_{12} >0 (b_{12} <0) bifurcation points on the f. r. c. for driven uncoupled mode appear when the excitation frequency is larger (lesser) than the natural frequency of the second uncoupled mode.*

In Fig. 2 the frequency response curves for the system with parameters: $\alpha_1 = 1.1$, $\alpha_2 = -0.1$, $\sigma_* = 0$, $f_0 = 5$ (exact internal resonance) are presented. Frequency response curves for uncoupled modes, coupled normal modes and coupled elliptic ones are depicted in 3D space (a_1, a_2, δ_*). Black curves 1 relate to uncoupled driven modes; blue one 2 are backbone curve for the companion modes. These curves lie in the planes (a_1, δ_*) and (a_2, δ_*), relatively. Red curve 3 correspond to NMs, and brown one 4 – to EMs (both these curves are spatial). (the curves are symmetric with respect to (a_1, δ_*) plane; negative a_2 values are not shown here). In this case only one bifurcation point exists for NMs and

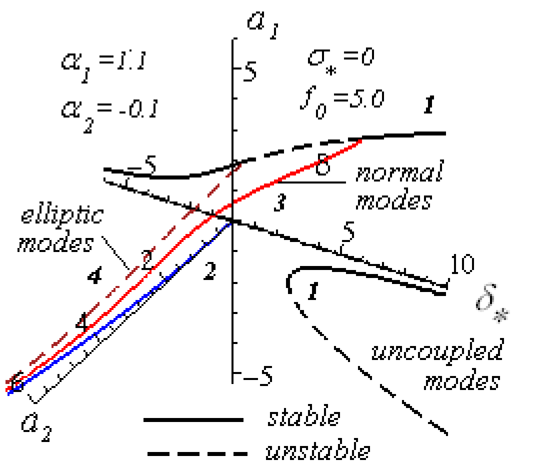


Fig.2. Frequency response curves for coupled steady-state modes (one bifurcation point for NMs and EMs)

EMs, respectively, on the f. r. c. for the driven mode Each point gives rise to one branch for the coupled NMs or EMs. Each branch approaches the backbone curve for the companion mode

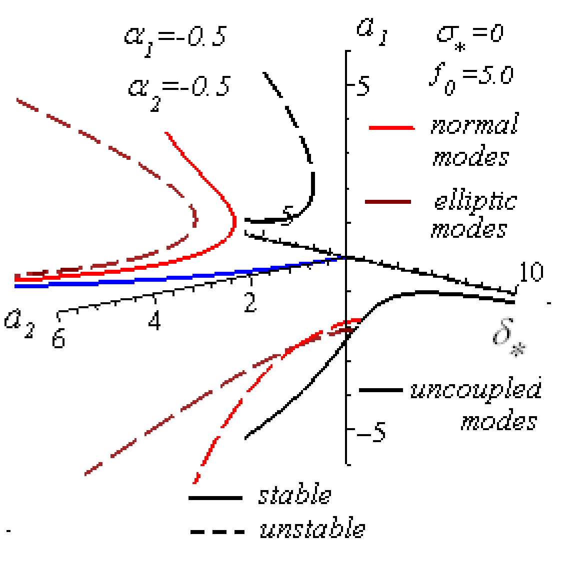
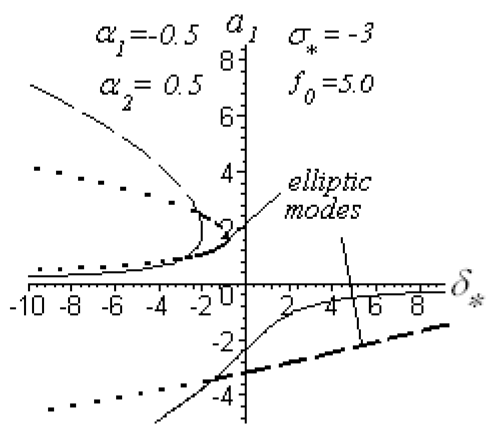


Fig. 3. Spatial freq. response curves for uncoupled driven (black curves) and companion (blue) modes; coupled NMs (red curves) and EMs (brown curves)

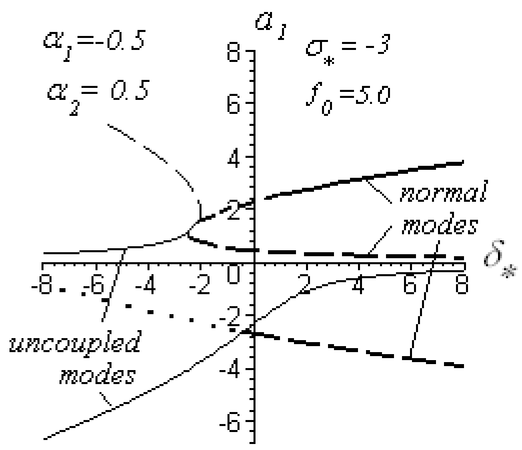
In Fig. 3 the frequency response curves are depicted for case $\alpha_1 = -0.5$, $\alpha_2 = -0.5$, $\sigma_* = 0$, $f_0 = 5$. Here also only one bifurcation point exists for NMs and EMs, respectively.

But along with these branches there exist additional curves for the NMs and for EMs, which *are not branching off the driven modes curve*. These curves approach the backbone curve for the companion mode 2.

In Figs. 4, 5 results for variant $\alpha_1 = -0.5$, $\alpha_2 = 0.5$, $\sigma_* = -3$, $f_0 = 5$ are presented. In Fig. 4, a, b, the projections of the f. r. c. for NMs and EMs, respectively, on the plane (a_1, δ_*) are depicted (together with the f. r. c. for uncoupled driven modes depicted by thin curves).



(*a*)



(*b*)

Fig. 4. Projections of the frequency response curves for NMs and EMs on the plane (a_1, δ_*) (the case of three bifurcational points)

The dotted parts of the curves relate to portions of the f. r. c., where CSMs disappear (condition (3.9) is not satisfied). In this case three bifurcation points exist both for NMs and EMs on the f. r. c. for the driven mode. For the NMs these points give rise to three infinite curves, but for EMs one finite curve connects two bifurcation points (forming a loop), along with one infinite curve branching off the uncoupled driven mode f. r. c.

In Fig. 5 spatial f. r. curves are depicted for NMs and EMs.

Note that one of the CSMs curves (for NMs and for EMs) approaches the backbone curve for the companion mode, similarly to Fig. 2 and 3.

4. Stability of the coupled stationary modes in undamped systems

A CSM is stable if the corresponding stationary point $(a_{1s}, a_{2s}, \theta_{1s}, \theta_{2s})$ of system (2.7) is stable. Consider first particular cases.

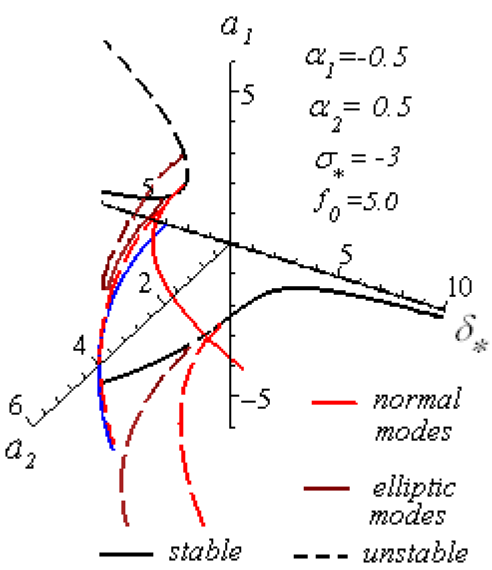


Fig. 5. Spatial freq. response curves for uncoupled driven (black curves) and companion (blue) modes; coupled NMs (red curves) and EMs (brown curves)

1. *Uncoupled* modes (forced oscillations in 1DOF systems). Hamiltonian (2.9) after dividing by c_{12} is reduced to

$$H(a_1,\theta_1) = \frac{1}{2}\delta_* a_1^2 - \frac{3}{32}\alpha_1 a_1^4 + \frac{1}{2}a_1 f_0 \sin\theta_1 \qquad (4.1)$$

where $f_0 = f/c_{12}$. The uncoupled mode is stable, if point (a_1, θ_1) is an elliptical point of surface $H(a_1,\theta_1) = C$, and is unstable, if it is a hyperbolic one. So the stability governed by hessian h of function (4.1). Accounting that $\theta_1 = (2m-1)\pi/2$, one has

$$h = \frac{1}{2}\left(\frac{9}{8}\alpha_1 a_1^2 - \delta_*\right) f_0 \sin\theta_1 \qquad (4.2)$$

Therefore the condition of stability of uncoupled mode is as follows (one may assume $f_0 > 0$):

$$9\alpha_1 a_1^2 - 8\delta_* > 0 \quad (\theta_1 = \pi/2) \qquad (4.3\text{ a})$$

$$9\alpha_1 a_1^2 - 8\delta_* < 0 \quad (\theta_1 = -\pi/2) \qquad (4.3\text{ b})$$

Note that these conditions with account of frequency response equation (3.3) are reduced to condition

$$\frac{\partial f_0}{\partial a_1} > 0 \qquad (4.4)$$

("increasing amplitude corresponds to increasing force"). As is known, condition (4.4) allows one easily to ascertain stable and unstable portions of f. r. c. (see Fig. 1, where unstable part of the f. r. c. is shown by dashed curve).

2) *Free coupled* oscillations in 2DOF systems. Putting in (2.9) $f_0 = 0$ and accounting (for $\mu = 0$) an energy integral $a_1^2 + a_2^2 = E$ (yielding from (2.7)), we reduce Hamiltonian (2.9) to a function $H(a_1^2, \gamma)$. Hessian of this function is positive under condition (it is seen from (2.7) that in this case $\gamma = n\pi/2$ with even n for NMs and odd n for EMs):

$$\mp\left[3\alpha_1 + 3\alpha_2 - 2(2 \pm 1)\right] > 0 \qquad (4.5)$$

(upper sign relates to the NMs and lower sign – to the EMs). This conditions have been derived in [Manevich A. I., Manevich L. I., 2005, p.56]. It has been proved there with use of (4.5), that parts of f. r. curves for uncoupled modes lying between bifurcation points are always unstable. Note that at free oscillations the stability of CSMs depends only on nonlinear coefficients.

Let us now consider the general case of *forced coupled* oscillations in the 2DOF system. Jacobian $J = \{j_{ls}\}$ of system (2.7) (written in the normal form) in stationary points is as follows:

$$j_{13} = -\frac{(\pm 1)}{4} a_{1s} a_{2s}^2 + \frac{f_0}{2}\sin\theta_{1s}, \quad j_{14} = \frac{(\pm 1)}{4} a_{1s} a_{2s}^2,$$

$$j_{31} = -\frac{3}{4}\alpha_1 a_{1s} - \frac{f_0}{2a_{1s}^2}\sin\theta_{1s}, \quad j_{23} = -j_{24} = j_{14},$$

$$j_{32} = j_{41} = -\frac{2 + (\pm 1)}{4} a_{2s}, \quad j_{42} = -\frac{3}{4}\alpha_2 a_{2s} \qquad (4.6)$$

(other elements are zero). Here $\sin\theta_{1s} = \pm 1$, a_{2s} is given by expression (3.7), a_{1s} is obtained from (3.8).

The characteristic equation for matrix (4.6) is biquadratic one

$$\lambda^4 + 2d_1\lambda^2 + d_2 = 0, \qquad (4.7)$$

where

$$2d_1 = \pm\frac{a_{1s}^2 a_{2s}^2}{16}\left[-3\alpha_1 - 3\alpha_2 + 2(2\pm 1)\right] +$$

$$\frac{f_0 \sin\theta_{1s}}{2a_{1s}}\left[\frac{a_{1s}^2}{8}\left(9\alpha_1 - \frac{1}{\alpha_2}\right) - \delta_* + \frac{2\mp 1}{3\alpha_2}(\delta_* - \sigma_*)\right] \tag{4.8}$$

$$d_2 = \mp\frac{3f_0}{32}\sin\theta_{1s}\,\alpha_2\, a_{1s}\, a_{2s}^2\, G \tag{4.9}$$

$$G = \frac{1}{8}a_{1s}^2\left(9\alpha_1 - \frac{(2\pm 1)^2}{\alpha_2}\right) - \delta_* + \frac{2\pm 1}{3}\frac{\delta_* - \sigma_*}{\alpha_2} \tag{4.10}$$

Roots of (4.7) have no positive real parts only in the case when both roots λ^2 are real and negative. Therefore the CSMs are stable under conditions

$$d_1 > 0\,,\ d_2 > 0\,,\ d_1^2 - d_2 > 0 \tag{4.11}$$

In Figs. 2-5 unstable portions of the frequency response curves for CSMs are shown with dashed lines. Portions of f. r. c. for uncoupled modes between two bifurcation points are always unstable.

It would be interesting to clarify the physical sense of conditions (4.11). Note that G (4.10) coincides with derivative of the left hand side of equation (3.8) with respect to a_1. Expression in the first square brackets in the r. h. s. of (4.8) coincides with (4.5), and expression in the second square brackets is similar to the left hand side of equation (3.8), divided by a_1. Using these observations it can be established a linkage between the condition of stability for free oscillation (4.5), condition of type (4.4) for forced oscillation in 1DOF and conditions of stability of coupled steady-state modes in non-autonomous 2DOF systems. This enables one to ascertain certain correspondence between the shape of projection of the f. r. curve for CSMs on plane (a_1, δ_*) and stability of CSMs.

4 Conclusion

The primary resonance in 2DOF cubic symmetric systems with close eigenfrequencies has been investigated. The general analysis of steady-state oscillations, their bifurcations and stability has been presented. Alongside with one-mode (uncoupled) oscillations there exist two-mode (coupled) oscillations with synchronized motions in two degrees of freedom. These coupled oscillations are normal modes (which can be in-phase or anti-phase oscillations), or elliptic modes (for which maximal deflection in one linear mode corresponds to zero in another one, and inversely).

The spatial frequency response curves for these modes are branching off the f. r. c. for the uncoupled driven mode, and/or approach the f. r. c. for the companion uncoupled mode. There exist three or one bifurcations points (separately for normal and elliptic modes). Conditions of stability for these modes have been obtained, and it is shown that a certain correspondence can be established between stability properties and shape features of the f. r. curves.

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