STATE ESTIMATION APPROACHES FOR CONTROL SYSTEMS WITH STATE CONSTRAINTS AND UNCERTAINTY

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Abstract
The state estimating procedures for nonlinear control dynamical system with uncertainty in the initial data are studied. We assume here that the coefficients of the matrix in linear terms of system state velocities are not exactly known, but belong to the given compact set in the corresponding space. The right hand sides of differential equations of dynamical system may contain also additional nonlinearities defined by quadratic (with respect to state vectors) functions. The emphasis in this paper is on the problem of estimating the nonlinear dynamics of systems of this type, complicated by the presence of an additional state constraint. We present here new approaches and algorithms allowing to find external ellipsoidal estimates of reachable sets of nonlinear control system of the studied type.

Key words
Control systems, Nonlinear dynamics, Estimation problem, Set-membership uncertainty, Ellipsoidal calculus.

1 Introduction

The paper is a further contribution to the study of estimation problems for uncertain systems in the case when a probabilistic description of noise and errors is not available, but only bounds on them are known [Kurzhanski and Valyi, 1997; Kurzhanski and Varaiya, 2014; Chernousko, 1994; Chernousko, 1996; Schweppe, 1973; Bertsekas, 1995; Walter and Pronzato, 1997]. Such models may be found in many applied areas ranged from engineering problems from physics to economics as well as to biological and ecological modeling when it occurs that a stochastic nature of the errors is questionable. The key issue in nonlinear set-membership estimation is to find suitable techniques, which produce related bounds for the set of unknown system states without being too computationally demanding, some of such approaches may be found e.g. in [Baier, Gerdts and Xausa, 2013; Dontchev and Lempio, 1992; Kurzhanski and Filippova, 1993; Mazurenko, 2012; Filippova and Lisin, 2000; Matviychuk, 2016; Polyak, Nazin, Durieu and Walter, 2004].

In this paper the modified state estimation approaches which use the special structure of nonlinearity of studied control system and also take into account state constraints are presented. We assume here that the system nonlinearity is generated by the combination of two types of functions in related differential equations, one of which is bilinear and the other one is quadratic. We find here the set-valued estimates of related reachable sets of such nonlinear uncertain control system under additional complication when we assume that unknown states of the system should belong to a prescribed region in the state space (we consider here the case when this region is defined by an ellipsoid in related space). It should be noted here that state constraints appear in a very natural way when modeling many real life engineering applications in robotics, aeronautics, medicine and other branches [Apreutesei, 2009; August and Koeppl, 2012; Ceccarelli, Di Marco, Garulli, and Giannitrapani, 2004]. Therefore the results of the paper may be of interest not only for the mathematical control theory and the fundamental theory of dynamical systems, but also may present interest for the study of corresponding physical models and for other applications in the areas noted above.

2 Preliminaries and Problem Formulation

We need to define first some auxiliary constructions and results which will be used in the following.

2.1 Basic Notations and Definitions

We will start by introducing the following basic notations. Let $\mathbb{R}^n$ be the n–dimensional Euclidean space and $x'y$ be the usual inner product of $x, y \in \mathbb{R}^n$ with the prime as a transpose, with $\|x\| = (x'x)^{1/2}$. Denote comp $\mathbb{R}^n$ to be the variety of all compact subsets $A \subset \mathbb{R}^n$ and conv $\mathbb{R}^n$ to be the variety of all compact convex subsets $A \subset \mathbb{R}^n$. Let us denote the variety of all closed convex subsets $A \subseteq \mathbb{R}^n$ by the symbol clconv $\mathbb{R}^n$. Let $\mathbb{R}^{n \times m}$ stands for the set of all real $n \times m$-matrices, $I \in \mathbb{R}^{n \times n}$ be the identity matrix, $\mathrm{tr}\,(A)$ be the trace of $n \times n$-matrix A (the sum of its diagonal elements). We denote by $B(a, r) = \{x \in \mathbb{R}^n : \|x - a\| \le r\}$ the ball in $\mathbb{R}^n$ with a center $a \in \mathbb{R}^n$ and a radius $r > 0$ and by

$$E(a, Q) = \{x \in \mathbb{R}^n : \ (Q^{-1}(x - a), (x - a)) \le 1\}$$

the *ellipsoid* in $\mathbb{R}^n$ with a center $a \in \mathbb{R}^n$ and with a symmetric positive definite $n \times n$-matrix Q.

Consider the ordinary differential equation

$$\dot{x} = f(t, x, u(t)) \tag{1}$$

with function $f : T \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ measurable in t and continuous in other variables. Here x stands for the state space vector, t stands for time ($t \in T = [t_0, t_1]$) and $u(t)$ is a control function,

$$u(t) \in Q(t) \tag{2}$$

where $Q(t)$ is a set-valued map ($Q : T \to \text{comp}\,\mathbb{R}^m$) measurable in t. The given data allows to consider a set-valued function

$$\mathcal{F}(t, x) = \bigcup \{ \ f(t, x, u) \ \mid \ u \in Q(t) \ \} \tag{3}$$

and further on, a differential inclusion [Aubin and Frankowska, 1990; Filippov, 1985]

$$\dot{x} \in \mathcal{F}(t, x) \tag{4}$$

that reflects the variety of all models of type (1)-(2).

Let us assume that the initial condition to the system (1) (or to the differential inclusion (4)) is unknown but bounded

$$x(t_0) = x_0, \ \ x_0 \in X_0 \in \text{comp}\,\mathbb{R}^n \tag{5}$$

One of the principal points of interest of the theory of control under uncertainty conditions [Kurzhanski and Valyi, 1997; Kurzhanski and Varaiya, 2014] is to study the set of all solutions $x[t] = x(t, t_0, x_0)$ to (1)-(5) (respectively, (4)-(5)) and furthermore the subset of those trajectories $x[t] = x(t, t_0, x_0)$ that satisfy both (4)-(5) and a restriction on the state vector (the "viability" constraint [Kurzhanski and Filippova, 1993])

$$x[s] \in Y(s), \quad s \in [t_0, t] \tag{6}$$

where $Y(\cdot)$ ($Y(t) \in \text{conv}\,\mathbb{R}^p$) is a convex compact valued multifunction.

The viability constraint (6) may be induced by state constraints defined for a given plant model or by the so-called measurement equation

$$y(t) = G(t)x + w, \tag{7}$$

where y is the measurement, $G(t)$ is a matrix function, w is an unknown but bounded "noise" and

$$w \in Q(t), \ \ Q(t) \in \text{comp}\,\mathbb{R}^p.$$

The problem consists in describing the set $X[\cdot] = \{x[\cdot] = x(\cdot, t_0, x_0)\}$ of solutions to the system (4)-(6) (the viable solution bundle or "viability bundle"). The point of special interest is to describe the t – cross-section $X[t]$ of this set that is actually the attainability domain of system (4)-(6) at the moment t. Unfortunately, the exact determination of the reachable set $X[t]$ is a difficult problem and the problem of finding its estimating sets is of interest.

2.2 First Approach: Evolution Equations

In this section we formulate necessary techniques and results. We assume that the notions of continuity and measurability of set-valued maps are taken in the sense of [Filippov, 1985; Aubin and Frankowska, 1990].

Consider the differential inclusion (4), where $x \in \mathbb{R}^n$, $\mathcal{F}$ is a continuous set-valued map ($\mathcal{F} : [t_0, t_1] \times \mathbb{R}^n \to \text{conv}\mathbb{R}^n$) that satisfies the Lipschitz condition with constant $L > 0$, namely

$$h(\mathcal{F}(t, x), \mathcal{F}(t, y)) \le L \parallel x - y \parallel, \ \ \forall x, y \in \mathbb{R}^n$$

where $h(A, B)$ is the Hausdorff distance for $A, B \subseteq \mathbb{R}^n$, i.e.

$$h(A, B) = \max\{h^+(A, B), h^-(A, B)\},$$

with $h^+(A, B), h^-(A, B)$ being the Hausdorff semidistances between the sets A, B,

$$h^+(A, B) = \sup\{d(x, B) \mid x \in A\},$$

$$h^-(A, B) = h^+(B, A),$$

$$d(x, A) = \inf\{\parallel x - y \parallel \ \mid y \in A\}.$$

Assuming a set $X_0 \in \text{comp}\,\mathbb{R}^n$ to be given, denote $x[t] = x(t, t_0, x_0)$ ($t \in T = [t_0, t_1]$) to be a solution to (4) (an isolated trajectory) that starts at point $x[t_0] = x_0 \in X_0$.

We take here the Caratheodory–type trajectory $x[\cdot]$, i.e. as an absolutely continuous function $x[t]$ $(t \in T)$ that satisfies the inclusion

$$\frac{d}{dt}\, x[t] = \dot{x}[t] \in \mathcal{F}(t, x[t]) \tag{8}$$

for almost every $t \in T$.

We require all the solutions $\{x[t] = x(t, t_0, x_0) \mid x_0 \in X_0\}$ to be extendable up to the instant t_1 that is possible under some additional assumptions [Filippova and Berezina, 2008].

Let $Y(t)$ be a continuous set-valued map ($Y : T \rightarrow \operatorname{conv} \mathbb{R}^n$), $X_0 \subseteq Y(t_0)$.

Definition 1. [Kurzhanski and Filippova, 1993] *A trajectory $x[t] = x(t, t_0, x_0)$ $(x_0 \in X_0,\ t \in T)$ of the differential inclusion (8) will be called viable on $[t_0, \tau]$ if*

$$x[t] \in Y(t) \quad \textit{for all} \quad t \in [t_0, \tau]. \tag{9}$$

We will assume that there exists at least one solution $x^*[t] = x^*(t, t_0, x_0^*)$ of (8) (together with a starting point $x^*[t_0] = x_0^* \in X_0$) that satisfies the condition (9) with $\tau = t_1$.

Let $\mathcal{X}(\cdot, t_0, X_0)$ be the set of all solutions to the inclusion (8) that emerge from X_0 (the "trajectory bundle"). Denote $\mathcal{X}[t] = \mathcal{X}(t, t_0, X_0)$ its crossection at instant t.

The subset of $\mathcal{X}(\cdot, t_0, X_0)$ that consists of all solutions to (8) viable on $[t_0, \tau]$ will be further denoted as $X(\cdot, \tau, t_0, X_0)$ (the "viable trajectory bundle") with its s – cross-sections as $X(s, \tau, t_0, X_0)$, $s \in [t_0, \tau]$. We introduce symbol $X[\tau]$ for these cross-sections at instant τ, namely

$$X[\tau] = X(\tau, t_0, X_0) = X(\tau, \tau, t_0, X_0).$$

The set-valued functions $\mathcal{X}[t]$ and $X[t]$ $(t \in T)$ will be referred to as the *trajectory tube* and *viable trajectory tube* (or *viability tube*) respectively. They may be considered as the set-valued analogies of the classical isolated trajectories constructed now under uncertainty conditions.

Let us consider the so-called funnel equation

$$\lim_{\sigma \rightarrow +0} \sigma^{-1} h\big(\mathcal{X}[t+\sigma], \bigcup_{x \in \mathcal{X}[t]} (x + \sigma \mathcal{F}(t, x)) \big) = 0, \quad t_0 \leq t \leq t_1,\ \mathcal{X}[t_0] = X_0. \tag{10}$$

Theorem 1. [Panasyuk, 1990; Kurzhanski and Filippova, 1993] *The multifunction $\mathcal{X}[t] = \mathcal{X}(t, t_0, X_0)$ is the unique set–valued solution to the evolution equation (10).*

Now consider the analogy of the funnel equation (10) but now for the viable trajectory tubes $X[t] = X(t, t_0, X_0)$:

$$\lim_{\sigma \rightarrow +0} \sigma^{-1} h\Big(X[t+\sigma], \bigcup_{x \in X[t]} (x + \sigma \mathcal{F}(t, x)) \bigcap Y(t+\sigma) \Big) = 0,\ t \in T,\ X[t_0] = X_0. \tag{11}$$

The following result is valid (under some additional assumptions on $\mathcal{F}(t, x)$ and $Y(t)$ [Kurzhanski and Filippova, 1993; Filippova, 2001]).

Theorem 2. [Kurzhanski and Filippova, 1993] *The set-valued function $X[t] = X(t, t_0, X_0)$ is the unique solution to the evolution equation (11).*

2.3 Second Order Approximations

The above theorems produce the first order approximation of the solution tubes $X[t]$, $\mathcal{X}[t]$. The second order approximations for differential inclusions and control systems were studied in [Dontchev and Lempio, 1992; Baier, Gerdts and Xausa, 2013] (but without a viability condition of type (9)). We formulate here one of the results which yields the second order approximation scheme (the set-valued analogy of the Runge-Kutta scheme) for $\mathcal{X}[t]$. Consider the evolution equation

$$\lim_{\sigma \rightarrow +0} \sigma^{-2} h\Big(\mathcal{X}[t+\sigma], \big(\bigcup_{x \in X[t]} (x + 0.5\, \sigma \times \bigcup_{z \in \mathcal{F}(t, \mathcal{X}[t])} (z + \mathcal{F}(t+\sigma, x+\sigma z)))) \Big) = 0, \quad t_0 \leq t \leq t_1,\ \mathcal{X}[t_0] = X_0. \tag{12}$$

Certainly the higher order approximations require more assumptions on the data. We will assume in addition that the map $\mathcal{F}$ has strongly convex values $\mathcal{F}(t, x)$ and that the support function

$$f(l, t, x) = \max_{u \in \mathcal{F}(t, x)} l' u$$

and the (unique) support vector-function $y(l, t, x)$ defined as

$$l' y(l, t, x) = f(l, t, x)$$

are both continuously differentiable in l, t, x (for $l \neq 0$).

Theorem 3. [Filippova, 2001] *The multifunction $\mathcal{X}[t] = \mathcal{X}(t, t_0, X_0)$ is the unique set–valued solution to the evolution equation (12).*

Find now the equation that produces the second order approximation for the viability tubes $X[\tau] = X(\tau, t_0, X_0)$ of (8)-(9).

Define now an auxiliary notion.

Definition 2. *Given two set-valued functions $W_1(\cdot)$, $W_2(\cdot)$, a symbol $\oint\limits_{\alpha}^{\beta} W_1(s) * W_2(s)ds$ denotes the set-valued convolution integral of $W_1(\cdot)$ and $W_2(\cdot)$ where*

$$\oint\limits_{\alpha}^{\beta} W_1(s) * W_2(s)ds = \bigcap_{M(\cdot)} \{ \int\limits_{\alpha}^{\beta} \Big((I- M(s))W_1(s) + M(s)W_2(s)\Big)ds \} \tag{13}$$

where the intersection in (13) is taken over all continuous $n \times n$-matrix-functions $M(s)$ defined on $[\alpha, \beta]$ and the integral is understood as the Aumann integral.

Consider the equation

$$\lim_{\sigma\to+0} \sigma^{-2} h\Big(X[t+\sigma], \oint\limits_{t}^{t+\sigma} (\bigcup_{x\in X[t]} (x + 0.5\, s \times \bigcup_{z\in\mathcal{F}(t,X[t])} (z + \mathcal{F}(t+s, x+sz)))) * Y(s)ds\Big) = 0, \ \ X[t_0] = X_0, \ \ t_0 \le t \le t_1. \tag{14}$$

Theorem 4. [Filippova, 2001] *The viability tube $X[t] = X(t, t_0, X_0)$ is the unique set–valued solution to the evolution equation (14).*

Remark. Results of this section may be used as background for computer simulations for finding the reachable sets of uncertain dynamical systems with (or without) state constraints. Unfortunately related computer simulations require a large amount of memory and a lot of time, in fact they are grid methods, see, for example [Baier, Gerdts and Xausa, 2013]. Therefore, the question arises how to construct external (and if possible, internal) estimating sets for reachable sets, the calculation of which could turn out to be more rapid.

3 Second Estimation Approach: Main Results

Consider the following system

$$\dot{x} = A(t)x + f(x)d + u(t), \ t_0 \le t \le t_1, \quad x_0 \in X_0 = E(a_0, Q_0), \ \ u(t) \in \mathcal{U} = E(\hat{a}, \hat{Q}), \tag{15}$$

where $x, d, x_0 \in \mathbb{R}^n$, $\|x\| \le K$ $(K > 0)$, $f(x)$ is the nonlinear function, which is quadratic in x, $f(x) = x'Bx$, here we assume that the $n \times n$-matrices B, Q_0 and $\hat{Q}$ are symmetric and positive definite.

The $n \times n$-matrix function $A(t)$ in (15) is of the form

$$A(t) = A^0 + A^1(t), \tag{16}$$

where the $n \times n$-matrix A^0 is given and the measurable and $n \times n$-matrix $A^1(t)$ is unknown but bounded, $A^1(t) \in \mathcal{A}^1$ $(t \in [t_0, t_1])$,

$$A(t) \in \mathcal{A} = A^0 + \mathcal{A}^1. \tag{17}$$

Here

$$\mathcal{A}^1 = \big\{ A=\{a_{ij}\}\in R^{n\times n} : a_{ij} = 0 \ \text{for} \ i \ne j, \ \text{and} \ a_{ii} = a_i, \ i = 1, \ldots, n, \ a = (a_1, \ldots, a_n), \ a'Da \le 1 \big\}, \tag{18}$$

where $D \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix.

We assume also that we have the additional state constraint on trajectories of the system (15), namely the following inclusion should be satisfied

$$x[t] \in Y = E(\tilde{a}, \tilde{Q}), \ \ t_0 \le t \le t_1, \tag{19}$$

where the ellipsoid $E(\tilde{a}, \tilde{Q})$ is given (with the center $\tilde{a} \in \mathbb{R}^n$ and the positive definite $n \times n$–matrix $\tilde{Q}$).

Let the absolutely continuous function $x[t] = x\big(t; u(\cdot), A(\cdot), x_0\big)$ be a solution to dynamical system (15)–(19) with initial state $x_0 \in X_0$, with admissible control $u(\cdot)$ and with a matrix $A(\cdot)$. The reachable set $X[t]$ at time t $(t_0 < t \le t_1)$ of the system (15)–(19) (under *viability* constraint (19) of type (9)) is defined as

$$\begin{aligned} X[t] =& \{x \in \mathbb{R}^n : \exists\, x_0\in X_0, \exists\, u(\cdot)\in\mathcal{U}, \exists\, A(\cdot)\in\mathcal{A}, \\ & x = x[t] = x\big(t; u(\cdot), A(\cdot), x_0\big), \\ x[\tau] =& x\big(\tau; u(\cdot), A(\cdot), x_0\big) \in Y, \ \forall \tau \in [t_0, t]\}. \end{aligned} \tag{20}$$

Using the analysis of the special bilinear-quadratic type of nonlinearity of control systems with uncertain initial data and with ellipsoidal state constraints we find here the external ellipsoidal estimate $E(a^+(t), Q^+(t))$ (with respect to the inclusion of sets) of the reachable set $X[t]$ $(t_0 < t \le t_1)$.

We will need further the following Minkowski (gauge) functional of the star-shaped sets $M \subseteq \mathbb{R}^n$ $(0 \in M)$ [Demyanov and Rubinov, 1986; Filippova and Lisin, 2000],

$$h_M(z) = \inf\{t > 0 : z \in tM, x \in \mathbb{R}^n\}.$$

The following result presents the external estimate of reachable sets of system under viability (state) constraints. First we need to formulate the following auxiliary result.

Lemma 1. ([Filippova and Lisin, 2000; Matviychuk, 2016]) *For $X_0 = E(0, Q_0)$ and $\mathcal{A}^1$ defined in (18) the Minkowski function of the set $(I + \sigma\mathcal{A}^1) * X_0$ has the form*

$$\begin{aligned} h_{(I+\sigma\mathcal{A}^1)*X_{00}}(z) = \Big(||Q_0^{-1/2}z||^2 - \\ 2\sigma\big(\sum_{i,j=1}^{n} w_i^2 (D^{-1/2})_{ij} \cdot w_j^2\big)^{1/2} \Big)^{1/2} + o(\sigma)||Q_0^{-1/2}z||, \\ w(z) = Q_0^{-1/2}z, \ \ \lim_{\sigma\to+0} \sigma^{-1} o(\sigma) = 0. \end{aligned} \tag{21}$$

Theorem 5. *Let $X_0 = E(a_0, k^2B^{-1})$, $k \neq 0$. Then for any matrix $L \in \mathbb{R}^{n\times n}$ and for all $\sigma > 0$ the following external estimate is true*

$$X[t_0+\sigma] \subseteq E(a_L^+(\sigma), Q_L^+(\sigma)) \ + \quad (22)$$
$$o(\sigma)B(0,1), \ \ \lim_{\sigma\to+0}\sigma^{-1}o(\sigma) = 0,$$

where

$$a_L^+(\sigma) = a_0 + \sigma(\hat{a} + k^2 d + a_0 B a_0 \cdot d \ +$$
$$(A^0 - L)a_0 + L\tilde{a}),$$
$$Q_L^+(\sigma) = (p^{-1}+1)Q_1(\sigma) + (p+1)\sigma^2\hat{Q}^*,$$
$$Q_1(\sigma) = \mathrm{diag}\{(p^{-1}+1)\sigma^2 a_{0i}^2 \ +$$
$$(p+1)r^2(\sigma) |\, i = 1,\dots,n\},$$
$$r(\sigma) = \max_z ||z|| \cdot (h_{(I+\sigma\mathcal{A})*X_0}(z))^{-1},$$

p is the unique positive root of the equation $\sum_{i=1}^{n}\dfrac{1}{p+\alpha_i} = \dfrac{n}{p(p+1)}$ with $\alpha_i \geq 0$ $(i = 1,...,n)$ being the roots of the following equation $|Q_1(\sigma) - \alpha\sigma^2\hat{Q}^*| = 0$, and $E(\hat{a},\hat{Q}^*)$ is the ellipsoid with minimal volume such that

$$E(\hat{a},\hat{Q}) + L\cdot E(0,\tilde{Q}) +$$
$$(2d\cdot a_0' B + A^0)\cdot E(0, k^2B^{-1}) \subseteq E(\hat{a},\hat{Q}^*). \quad (23)$$

Proof. We use here the idea of [Kurzhanski and Filippova, 1993] for elimination of state constraints in the construction of reachable sets (see also related results in [Bettiol, Bressan and Vinter, 2010; Gusev, 2016]). Consider the following differential inclusion with $n\times n$–matrix parameter L,

$$\dot{z} \in (A_0 - L + \mathcal{A}^1)z \ + \ f(z)\cdot d +$$
$$E(\hat{a},\hat{Q}) + L\cdot E(\tilde{a},\tilde{Q}), \ \ t_0 \leq t \leq T, \quad (24)$$
$$z_0 \in X_0 = E(a_0, Q_0).$$

Denote by $Z(t; t_0, X_0, L)$ $(t \in [t_0, t_1])$ the trajectory tube to (24) for a fixed matrix parameter L. We have the following estimate [Kurzhanski and Filippova, 1993]

$$X[t] \subseteq \bigcap_L Z(t; t_0, X_0, L), \ \ t_0 \leq t \leq t_1. \quad (25)$$

Using results of [Filippova, 2012; Filippova, 2016; Filippova and Lisin, 2000; Filippova and Matviychuk, 2012; Filippova and Matviychuk, 2015; Matviychuk, 2016] we can find the upper ellipsoidal estimates for reachable sets $Z[t] = Z(t; t_0, X_0, L)$ of the nonlinear system (24) (we underline here that after the above elimination this new system does not have state constraints and therefore we may use some estimation results mentioned in Section 2.2). Resulting estimate (22) follows from (25) and from the above remark. □

The following algorithm is based on Theorem 5 and may be used to produce the external ellipsoidal estimates for the reachable sets of the system (15)-(19).

Fix a finite number of matrices L_s, $s = 1,\dots,r$ (r is an arbitrary integer, $r > 0$).

Algorithm. Subdivide the time segment $[t_0, t_1]$ into subsegments $[\tau_i, \tau_{i+1}]$, where $\tau_i = t_0 + i\sigma$ $(i = 1,\dots,m)$, $\sigma = (t_1 - t_0)/m$.

1. For given $X_0 = E(a_0, Q_0)$ define the smallest $k_0 > 0$ such that $E(a_0, Q_0) \subseteq E(a_0, k_0^2B^{-1})$ (k_0^2 is the maximal eigenvalue of the matrix $B^{1/2}Q_0B^{1/2}$, [Filippova, 2012; Filippova and Matviychuk, 2015]).
2. For $X_0 = E(a_0, k_0^2B^{-1})$ as an initial set define by Theorem 5 the upper estimate $E(a_{L_s}^+(\sigma), Q_{L_s}^+(\sigma))$ of the set $X(t_0+\sigma)$, $s = 1,\dots,r$.
3. Take a compact and convex set X_0^* such that $\bigcap_{1\leq s\leq r} E(a_{L_s}^+(\sigma), Q_{L_s}^+(\sigma)) \subseteq X_0^*$.
4. Consider the system on the next subsegment $[\tau_1, \tau_2]$ with the initial (at time instant τ_1) set X_0^* and with initial ellipsoid $E(a_1, k_1^2B^{-1})$ found as in step 1.
5. The next step repeats the previous iteration beginning with new initial data.

At the end of the process we will get the external estimate tube $E(a^+(t), Q^+(t))$ of the reachable sets $X(t)$ $(t_0 \leq t \leq t_1)$ of the system (15)-(19).

4 Conclusion

The paper deals with the problems of control and state estimation for a dynamical control system described by nonlinear differential equations with unknown but bounded initial states. Nonlinearity in dynamics is caused by the presence of a combination of bilinear and quadratic functions of the state of the system. The solution to the state estimation problem is studied through the techniques of trajectory tubes with their cross-sections $X[t]$ being the reachable sets at instant t to control system.

Basing on new results of ellipsoidal calculus we present the modified state estimation approaches which use the special nonlinear structure of the control system and allow to find the upper bounds of reachable sets. The applications of the problems studied in this paper are in guaranteed state estimation for nonlinear systems with unknown but bounded errors and in nonlinear control theory including applications in robotics, aeronautics, medicine, economics and other branches with uncertainty and nonlinearity in related dynamical models.

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