COMPARISON OF NON-STANDARD FINITE DIFFERENCE METHODS FOR VIBRO-IMPACT SYSTEMS

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Abstract

The aim of this paper is to continue our investigation in the application of the non-standard finite difference methods in non-smooth mechanics. We focus our study on N-dimensional vibro-impact systems. In particular, we construct non-standard schemes associated to well known schemes developed and studied by Paoli and Schatzman (called here after the PS-scheme), Moreau, Monteiro-Marques and Mabrouk (called here after the MMM-scheme). We compare them and show that the non-standard MMM-scheme can give spurious numerical solutions, while the non-standard PS-scheme give satisfactory results.

Key words

Non-standard finite difference method, impact, mechanical systems

1 Introduction

We continue our investigation started in [Dumont and Lubuma, 2005; Dumont and Lubuma, 2007] in the application of the non-standard finite difference (shortly NSFD) method to non-smooth mechanical problems.

Here, we propose and discuss the numerical treatment, with the NSFD method, of vibro-impact systems. We are also mainly concerned into the numerical approximation of the solution of the following type of problems coming from the mechanics: find $q : [0,T] \longrightarrow \mathbb{R}^{\mathbb{N}}$, $t \longmapsto q(t)$ such that

$$\begin{array}{c} \mathcal{M}\ddot{q}(t) + \mathcal{K}q(t) + \partial\Psi_{K_N}(q) \ni h(t), \\ \text{a.e. } t \in (0,T), \end{array} \tag{1}$$

where $\mathcal{M}$ and $\mathcal{K}$ are positive definite matrix, $h : \mathbb{R} \longrightarrow \mathbb{R}^N$ is a vector-valued function related to the given forces acting on the system and $\partial\Psi_{K_N}$ is the subdifferential of the indicator function of the closed convex set $K_N = \prod_{i=1}^{N}[q_i^{\min}, q_i^{\max}] \subseteq \mathbb{R}^N$ with $q_i^{\max} \in]0, +\infty]$ and $q_i^{\min} \in [-\infty, 0[$. We complete the formulation of the model with initial conditions such as

$$q(0) = q_0, \qquad \dot{q}(0) = q_1. \tag{2}$$

For a complete description at the impacts we make a supplementary assumption: for each impact, we consider the constitutive law of impacts

$$\dot{q}_i(t+0) = -e_i\dot{q}_i(t-0), \quad \text{if } q_i(t) = q_i^{\max} \text{ or } q_i^{\min}, \tag{3}$$

with $e_i \in [0,1]$ for $i = 1, ..., N$. Following [Moreau, 1983; Moreau, 1988], we can rewrite (1) into the form

$$f(t,q) - \mathcal{M}\ddot{q} \in \partial\Psi_{T_K(q)}(\dot{q}), \tag{4}$$

where $T_K(z)$ is the tangent cone of K to z.

In general, there are only a few cases for which analytical solutions to Eqs. (1)-(2) can be found. Thus, the need of numerical methods. Several numerical approaches exist: the event-driven method, the compliant method and the time-stepping method. We are mainly interested in the third approach and, in particular, in the schemes developed by Paoli and Schatzman (see [Paoli and Schatzman, 1993; Paoli, 2001]) and those developed by Moreau, Monteiro Marques and Mabrouk (see [Mabrouk, 1998; Monteiro Marques, 1993; Moreau, 1983]). Up to now, most of these numerical methods are based on standard or classical finite difference methods, and they usually fail to replicate a number of essential physical properties. In this paper, we propose the non-standard finite difference method to improve the numerical simulations. Since the pioneer works of R.E. Mickens in the mid 1980s, the non-standard approach has shown great potentials in the design of reliable schemes that preserve significant properties of solutions of differential models in science and engineering (See, e.g. [Anguelov and Lubuma, 2000], [Mickens, 1994] and [Mickens, 2000]). The paper is organised as follows. In the next section, we give a very short

self-contained presentation of the non-standard finite difference method. Section 2 is devoted to the design of non-standard finite difference schemes for friction-less impact oscillators. Then, in section 3, we present and discuss some simulations for a 2DOF vibro-impact system.

2 A brief Introduction to the non-standard finite difference method for differential system

To simplify the exposition in this part, we consider the following N-dimensional second order system

$$\begin{cases} \mathcal{M}^2\ddot{X} + \mathcal{A}^2 X = 0; \\ X(0) = X_0 \text{ and } \dot{X}(0) = V_0. \end{cases} \tag{5}$$

where, $\mathcal{A}^2$ is a symmetric square matrix of order N that is positive definite and diagonalizable, and $\mathcal{M}^2$ is the diagonal mass-matrix with positive coefficients. Then setting $\mathcal{B}^2 = \mathcal{M}^{-2}\mathcal{A}^2$, it is well known from classical Analysis and Algebra theories that the unique solution of system (5) is

$$X(t) = \cos(t\mathcal{B})\, X_0 + B^{-1}\sin(t\mathcal{B})\, V_0, \tag{6}$$

where $\mathcal{B}$ is a square root of $\mathcal{B}^2$. Here we consider the matrix trigonometric functions, cosine and sine (see the remark below for a general definition of matrix function)

Remark 1. *Let $\mathcal{B}^2$ a square matrix of order N that is positive definite and diagonalizable. Thus, there exists a transition matrix Λ such that*

$$\Lambda^{-1}\mathcal{B}^2\Lambda = \Omega^2 = diag(\omega_1^2, \cdots, \omega_N^2)) \tag{7}$$

where $\omega_1^2, \cdots, \omega_N^2$ are the eigenvalues of $\mathcal{B}^2$ counted according to their multiplicities. The representation (7) is essential in the understanding of matrix functions. More precisely, given an arbitrary real-valued function g of real variable, a matrix function $\mathcal{B}^2 \mapsto g(\mathcal{B}^2)$ is well-defined, via (7), by

$$g(\mathcal{B}^2) := \Lambda diag(g(\omega_1^2), \cdots, g(\omega_n^2))\Lambda^{-1}. \tag{8}$$

Remark 2. *Let $\mathcal{M}$ be a square matrix of order n such that $\Lambda^{-1}\mathcal{M}\Lambda$ is diagonal, Λ being the same transition matrix as in (7) and (8). Since the product of two diagonal matrices is commutative, Eq. (8) readily yields the commutativity property*

$$\mathcal{M}g(\mathcal{B}) = g(\mathcal{B})\mathcal{M}, \tag{9}$$

which will be used at several occasions in what follows.

Let us emphasize that the solution of the problem (5) satisfies the principle of conservation of energy

$$||\mathcal{M}\dot{X}(t)||^2 + \|\mathcal{A}X(t)\|^2 = \text{constant}, \tag{10}$$

where, with $||\cdot.||$ being the Euclidean norm on $\mathbb{R}^n$, the energy of the system is given by

$$\mathcal{E} \equiv \mathcal{E}(X,t) := \frac{1}{2}\left[\left\|\mathcal{M}\dot{X}(t)\right\|^2 + \|\mathcal{A}X(t)\|^2\right]. \tag{11}$$

Denoting $(X^n)_{n\geq 0}$ a sequence of approximations to a solution X of (5) at the discrete time $t_n = n\Delta t$, where Δt is the step size, standard numerical techniques leads to the following standard approximation

$$\frac{X^{n+1} - 2X^n + X^{n-1}}{(\Delta t)^2} + \mathcal{B}^2 X^n = 0. \tag{12}$$

In [Dumont and Lubuma, 2007], we only consider $\mathcal{M} = Id$ and we show that the non-standard approach consists in replacing the traditional denominator Δt with a matrix-function $\Phi_{\mathcal{B}}(\Delta t)$, which satisfy

$$\Phi_{\mathcal{B}}(\Delta t) = \Delta t^2\mathcal{I} + O\left(\Delta t^4\right) \text{ as } \Delta t \to 0.$$

In fact, using (6) and some properties of trigonometric matrix function, we have

$$\begin{aligned} X(t+\Delta t) &= \cos((t+\Delta t)\mathcal{B})\, X_0 + \mathcal{B}^{-1}\sin((t+\Delta t)\mathcal{B})\, V_0 \\ &= (\cos(t\mathcal{B})\cos\Delta t\mathcal{B} - \sin t\mathcal{B}\sin\Delta t\mathcal{B})X_0 \\ &\quad + \mathcal{B}^{-1}(\sin t\mathcal{B}\cos\Delta t\mathcal{B} + \cos t\mathcal{B}\sin\Delta t\mathcal{B})\, V_0 \end{aligned}$$

$$\begin{aligned} X(t-\Delta t) &= \cos((t-\Delta t)\mathcal{B})\, X_0 + B^{-1}\sin((t-\Delta t)\mathcal{B})\, V_0 \\ &= (\cos(t\mathcal{B})\cos\Delta t\mathcal{B} + \sin t\mathcal{B}\sin\Delta t\mathcal{B})X_0 \\ &\quad + \mathcal{B}^{-1}(\sin t\mathcal{B}\cos\Delta t\mathcal{B} - \cos t\mathcal{B}\sin\Delta t\mathcal{B})\, V_0. \end{aligned}$$

Then summing the previous equalities, leads to

$$\begin{aligned} X(t+\Delta t) + X(t-\Delta t) &= 2\cos t\mathcal{B}\cos\Delta t\mathcal{B}X_0 \\ &\quad + \mathcal{B}^{-1}\sin t\mathcal{B}\cos\Delta t\mathcal{B}V_0. \end{aligned}$$

The using the commutativity property, we finally obtain

$$X(t+\Delta t) + X(t-\Delta t) = 2\cos(\Delta t\mathcal{B})\, X(t)$$

Thus we obtain the following exact schemes of (5), i.e. $X^n = X(t_n)$, for such that we obtain

$$X^{n+1} - 2\cos(\Delta t\mathcal{B})\, X^n + X^{n-1} = 0, \tag{13}$$

or equivalently, using the fact that $\cos(\Delta t\mathcal{B}) = Id - 2 * \sin^2(\Delta t\mathcal{B}/2)$

$$\Phi_{\mathcal{B}}^{-1}\left(X^{n+1} - 2X^n + X^{n-1}\right) + \mathcal{M}^{-2}\mathcal{A}^2X^n = 0, \tag{14}$$

with $\Phi_{\mathcal{B}} \equiv \Phi_{\mathcal{B}}(\Delta t) = 4\sin^2\left(\frac{\Delta t}{2}\mathcal{B}\right)\mathcal{B}^{-2}$.

It is well known that (5) is equivalent to the following first-order system of $2N$ differential equations

$$\begin{cases} \dot{X} = V \\ \mathcal{M}^2\dot{V} = -\mathcal{A}^2X \end{cases} \tag{15}$$

with dependent variables (X, V) and independent variable t.

The second-order system of N difference equations (12) is equivalent to the following one-step system of $2N$ difference equations

$$\begin{cases} X^{k+1} - X^k = \Delta tV^k \\ \mathcal{M}^2\left(V^{k+1} - V^k\right) = -\Delta t\mathcal{A}^2X^{k+1}. \end{cases} \tag{16}$$

In the same way, (14) is equivalent to the one-step system of $2n$ difference equations

$$\begin{cases} X^{k+1} - \cos(\Delta t\mathcal{B})X^k = \sin(\Delta t\mathcal{B})\mathcal{B}^{-1}V^k \\ V^{k+1} - \cos(\Delta t\mathcal{B})V^k = -\sin(\Delta t\mathcal{B})\mathcal{B}X^k, \end{cases} \tag{17}$$

which is an exact scheme of the system of differential equations (15).

Finally, it is not always possible to construct exact schemes like the ones before. An other way to construct non-standard schemes is a discrete form of the principle of conservation of energy. The main idea is to approximate the derivative in (11) by using Mickens' rule regarding the denominator. We consider the approximation

$$\dot{X}(t_n) \approx \Phi_{\mathcal{B}}^{-1/2}(X^{n+1} - X^n).$$

The discrete energy $\mathcal{E}^n$ at the time t_n is then

$$\begin{aligned}\mathcal{E}^n = \tfrac{1}{2}\big((X^{n+1} - X^n)^T\mathcal{M}^2\Phi_B^{-1}(X^{n+1} - X^n) \\ +(X^{n+1})^T\mathcal{A}^2X^n\big).\end{aligned} \tag{18}$$

Thus, we can prove that

Proposition 3. *The equivalent schemes (14) and (13) satisfy the following discrete principle of conservation energy for every $n \geq 1$:*

$$\mathcal{E}^n = \mathcal{E}^{n-1}. \tag{19}$$

Proof: in order to show the previous principle, we need some additionnal results. We know that $\mathcal{B}^2 = \mathcal{M}^{-2}\mathcal{A}^2$, thus $\mathcal{M}^2\mathcal{B}^2 = \mathcal{A}^2$ and

$$\left(\mathcal{M}^2\mathcal{B}^2\right)^T = \left(\mathcal{B}^2\right)^T\mathcal{M}^2 = \mathcal{M}^2\mathcal{B}^2$$

which implies that $\left(\mathcal{B}^2\right)^T = \mathcal{M}^2\mathcal{B}^2\mathcal{M}^{-2}$. Using the fact that $(\sin(\mathcal{B}))^T = \sin\mathcal{B}^T$, and using the previous result, we deduce that $\left(\left(\sin\frac{\Delta t}{2}\mathcal{B}\right)^{-2}\right)^T = \mathcal{M}^2\left(\sin\frac{\Delta t}{2}\mathcal{B}\right)^{-2}\mathcal{M}^{-2}$. Setting $F^{n+1} = \frac{1}{2}\left((X^{n+1} - X^n)^T\mathcal{M}^2\Phi_B^{-1}(X^{n+1} - X^n)\right)$ and multiplying (14) by $\left(\mathcal{M}^2\left(X^{n+1} - X^{n-1}\right)\right)^T$, we obtain

$$\begin{aligned}&F^{n+1} - F^n - \\ &\left(X^{n+1} - X^n\right)^T\mathcal{M}^2\mathcal{B}^2\left(\sin^2\left(\tfrac{\Delta t}{2}\right)\mathcal{B}\right)^{-1}\left(X^n - X^{n-1}\right) + \\ &\left(X^n - X^{n-1}\right)^T\mathcal{M}^2\mathcal{B}^2\left(\sin^2\left(\tfrac{\Delta t}{2}\right)\mathcal{B}\right)^{-1}\left(X^{n+1} - X^n\right) \\ &= 0\end{aligned}$$

The last term can be rewritten as follows

$$\begin{aligned}&\left(X^{n+1} - X^n\right)^T\left(\sin^2\left(\tfrac{\Delta t}{2}\right)\mathcal{B}\right)^{-T}\left(\mathcal{B}^2\right)^T\mathcal{M}^2(X^n - X^{n-1}) = \\ &\left(X^{n+1} - X^n\right)^T\left(\sin^2\left(\tfrac{\Delta t}{2}\right)\mathcal{B}\right)^{-T}\mathcal{M}^2\mathcal{B}^2\left(X^n - X^{n-1}\right) = \\ &\left(X^{n+1} - X^n\right)^T\mathcal{M}^2\left(\sin^2\left(\tfrac{\Delta t}{2}\right)\mathcal{B}\right)^{-1}\mathcal{B}^2\left(X^n - X^{n-1}\right) = \\ &\left(X^{n+1} - X^n\right)^T\mathcal{M}^2\mathcal{B}^2\left(\sin^2\left(\tfrac{\Delta t}{2}\right)\mathcal{B}\right)^{-1}\left(X^n - X^{n-1}\right).\end{aligned}$$

Finally, we have

$$\begin{aligned}&F^{n+1} - F^n + \left(X^{n+1} - X^{n-1}\right)^T\mathcal{M}^2\mathcal{B}^2X^n = 0, \\ &F^{n+1} + \left(X^{n+1}\right)^TA^2X^n = F^n + \left(X^n\right)^TA^2X^{n-1}\end{aligned}$$

which implies the result.

Then, it is possible to consider the discrete energy, E^n at the time t_n, associated to (17) is

$$E^n = \frac{1}{2}\left(\|\mathcal{M}V^n\|^2 + \|\mathcal{A}X^n\|^2\right). \tag{20}$$

Thus, we can prove that

Proposition 4. *The scheme (17) satisfy the following discrete principle of conservation energy for every $n \geq 1$ and for all $\Delta t > 0$:*

$$E^n = E^{n-1}. \tag{21}$$

Proof. Set $\mathcal{B} = \mathcal{M}^{-1}\mathcal{A}$. We first compute

$$\begin{aligned}\left\|\mathcal{M}V^{n+1}\right\|^2 = &\|\mathcal{M}cos(\Delta t\mathcal{B})V^n\|^2 + \|\mathcal{M}sin(\Delta t\mathcal{B})\mathcal{B}X^n\|^2 \\ &-(V^n)^T(cos(\Delta t\mathcal{B})^T\mathcal{M}\mathcal{A}\sin(\Delta t\mathcal{B})\mathcal{B}X^n - \\ &-(X^n)^T\sin(\Delta t\mathcal{B})^T\mathcal{A}\mathcal{M}\cos(\Delta t\mathcal{B})V^n\end{aligned}$$

$$\begin{aligned}\left\|\mathcal{A}X^{n+1}\right\|^2 = &\|\mathcal{M}\cos(\Delta t\mathcal{B})X^n\|^2 + \left\|\mathcal{M}sin(\Delta t\mathcal{B})\mathcal{B}^{-1}V^n\right\|^2 - \\ &-(X^n)^T(cos(\Delta t\mathcal{B})^T\mathcal{M}\mathcal{A}\sin(\Delta t\mathcal{B})V^n - \\ &-(V^n)^T\sin(\Delta t\mathcal{B})^T\mathcal{A}\mathcal{M}\cos(\Delta t\mathcal{B})X^n\end{aligned}$$

Then it is easy to check that

$$\|\mathcal{M}\cos(\Delta t\mathcal{B})\mathcal{B}X^n\|^2+\|\mathcal{M}\sin(\Delta t\mathcal{B})\mathcal{B}X^n\|^2=\|\mathcal{A}X^n\|^2,$$

$$\|\mathcal{M}\sin(\Delta t\mathcal{B})V^n\|^2+\|\mathcal{M}\cos(\Delta t\mathcal{B})V^n\|^2=\|\mathcal{M}V^n\|^2.$$

Finally, we deduce that $\left\|\mathcal{M}V^{n+1}\right\|^2+\left\|\mathcal{A}X^{n+1}\right\|^2=\|\mathcal{M}V^n\|^2+\|\mathcal{A}X^n\|^2$ for all $n\in\mathbb{N}$. $\square$

Although, the use of matrix function is not easy to handle because we need to know the eigenvalues w_i^2, with $i=1,...,N$, of the matrix $\mathcal{M}^{-1}\mathcal{A}$and to find the transition matrix. An alternative it is possible to consider $q\geq\max\left\{\omega_1^2,..,\omega_N^2\right\}$and to replace the usual denominator $(\Delta t)^2$by a real-valued function ϕ^2, whic satisfies $\phi^2(z)\leq 4$for $z\geq 0$as well as the asymptotic relation

$$\phi(z)=z+O\left(z^2\right).$$

Thus, we obtain the following new non-standard method

$$\mathcal{M}^2\frac{X^{n+1}-2X^n+X^{n-1}}{\phi^2\left(\sqrt{q}\Delta t\right)/q}+\mathcal{A}^2X^n=0. \qquad (22)$$

Although the non-standard scheme (22) is not exact, this scheme has the following properties, which makes it more interesting than (12).

Proposition 5 ([Dumont and Lubuma, 2007]). *The non-standard scheme (22) is unconditionally stable in the sense of Lax Richtmyer. Furthermore, this scheme satisfies the discrete principle of conservation energy (19) provided that the matrix function $\Phi_A^{-1/2}$ in (18) is replaced with the scalar function$\sqrt{q}/\phi(\sqrt{q}\Delta t)$.*

Remark 6. *Since $\mathcal{M}=Id$, it is possible to construct exact schemes associated to N-dimensional damped systems, like*

$$\ddot{q}+2\mathcal{C}\dot{q}+\mathcal{K}q=0, \qquad (23)$$

such that $\mathcal{C}$ and $\mathcal{K}$ are symmetric positive definite and have the same transition matrix Λ (see [Dumont and Lubuma, 2007]).

3 Applications to a 2DOF vibro-impact system.

Following [Dumont and Lubuma, 2007], we consider the following two-degree-of-freedom mass spring damper system, depicted in figure 1.

Setting $\mu_k=\frac{k_2}{k_1}$ and making the change of independant variable $t=\sqrt{\frac{k_1}{m_1}}\tau$, the previous system can be

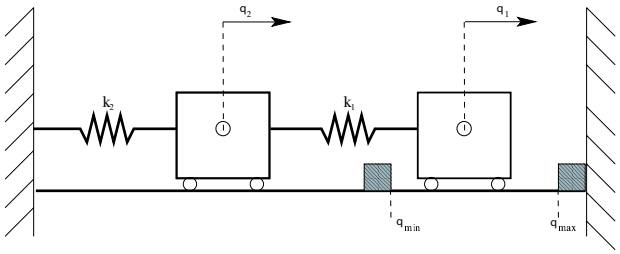

Figure 1. 2DOF mass-spring-damper system with impacts

rewritten in the dimensionless form in the following manner with $q=(q_1,q_2)^T$

$$\begin{cases}\mathcal{M}\ddot{q}+\mathcal{K}q+\partial\Psi_{K_2}(q)\ni 0,\\ \dot{\overline{q}}(t+0)=\text{Proj}_{\overline{u}(t)}\left(T_{\overline{K}_2}(\overline{u}(t)),\dot{\overline{u}}(t-0)\right)\end{cases} \qquad (24)$$

where $K_2=[q_{\min},q_{\max}]\times\mathbb{R}$,

$$\mathcal{M}^2=\begin{pmatrix}1&0\\0&\mu_m\end{pmatrix},\quad \mathcal{A}^2=\mathcal{K}=\begin{pmatrix}1&-1\\-1&1+\mu_k\end{pmatrix}.$$

Note also that $\mathcal{K}$ is symmetric and positive definite. First, we consider $\mathcal{C}=0$. The eigenvalues of $\mathcal{M}^{-1}\mathcal{K}$ are

$$w_1^2=\frac{1+\mu_k+\mu_m-\sqrt{(1+\mu_k+\mu_m)^2-4\mu_k\mu_m}}{2\mu_m},$$
$$w_2^2=\frac{1+\mu_k+\mu_m+\sqrt{(1+\mu_k+\mu_m)^2-4\mu_k\mu_m}}{2\mu_m},$$

and the transition matrix is $\Lambda=\begin{pmatrix}\frac{1}{\sqrt{1+(1-w_1^2)^2}}&-\frac{1}{\sqrt{1+(1-w_2^2)^2}}\\ \frac{w_1^2-1}{\sqrt{1+(1-w_1^2)^2}}&\frac{1-w_2^2}{\sqrt{1+(1-w_2^2)^2}}\end{pmatrix}$. We have existence of a Lipschitz continuous solution [Paoli and Schatzman, 1993]. Based on the exact schemes in Section 2, we now design, for the constrained problem (24), (2) & (3), non-standard schemes. Following [Dumont and Lubuma, 2005], we consider the approach proposed and studied by Paoli and Schatzman [Paoli and Schatzman, 1993; Paoli, 2001]. To discretize $\partial\Psi_{K_2}(q)$, we assume that the e-weighted position $\frac{q^{n+1}+eq^{n-1}}{1+e}$ has to meet the constraint $\frac{q^{n+1}+eq^{n-1}}{1+e}\in K_2$. Thus, the exact schemes (13) and (14) of the free motion lead to the non-standard finite difference schemes

$$\Phi_{\mathcal{B}}^{-1}\left(q^{n+1}-2q^n+q^{n-1}\right)+\partial\Psi_{(1+e)K_2}\left(q^{n+1}+eq^{n-1}\right)\ni-\mathcal{M}^{-2}\mathcal{A}^2q^n \qquad (25)$$

and

$$q^{n+1}-2\left(\cos\left(\Delta t\mathcal{M}^{-1}\mathcal{A}\right)\right)q^k+q^{n-1}+\partial\Psi_{(1+e)K_2}\left(q^{n+1}+eq^{n-1}\right)\ni 0. \qquad (26)$$

Of course, the last one is the easiest to implement. Following Moreau, problem (24) can be rewritten in the following way

$$\begin{cases} -\mathcal{M}^2\ddot{q} - \mathcal{A}^2 q \in \partial\psi_{T_{\overline{K}_2}(q)}\left(\dot{q}\right), \\ \dot{q}\left(t+0\right) = \text{Proj}_{q(t)}\left(T_{\overline{K}_2}\left(q(t)\right), \dot{q}\left(t-0\right)\right) \end{cases}$$

We now consider numerical schemes adapted to the previous formulation [Moreau, 1983; Monteiro Marques, 1993; Mabrouk, 1998]

$$\begin{cases} q^{n+1} = q^n + \Delta t v_n, \\ -\left(v^{n+1} - v^n\right) - \Delta t \mathcal{M}^{-2}\mathcal{A}^2 q^{n+1} \in \partial\psi_{T_{\overline{K}_2}(q^{n+1})}\left(q^{n+1}\right) \end{cases} \tag{27}$$

For all $q \in \mathbb{R}^N$, $T_{\overline{K}_2}(q)$ is a closed, non-empty convex set. Thus the previous inclusion is equivalent to: for all $n \geq 0$

$$\begin{cases} q^{n+1} = q^n + \Delta t v_n, \\ v^{n+1} = -ev^n + (1+e)\,\text{Proj}_{q^{n+1}}\left(T_{\overline{K}_2}\left(\overline{u}^{n+1}\right), v^n \right. \\ \qquad \left. -\frac{\Delta t}{1+e}\mathcal{M}^{-2}\mathcal{A}^2 q^{n+1}\right). \end{cases} \tag{28}$$

where $\text{Proj}_q\left(T_{K_2}(q), .\right)$ denotes the projection on $T_{\overline{K}}(q)$ for the kinetic metric

Using (17), the corresponding non-standard fully explicit scheme is

$$\begin{cases} q^{n+1} = \Psi q^n + \Phi v_n, \\ v^{n+1} = -ev^n + (\Psi + eId)\,\text{Proj}_{q^{n+1}}\left(T_{\overline{K}_2}\left(\overline{u}^{n+1}\right), v^n - \right. \\ \qquad \left. -\left(\Psi + eId\right)^{-1}\Phi\mathcal{M}^{-2}\mathcal{A}^2 q^n\right). \end{cases} \tag{29}$$

with $\Psi = \cos(\Delta t \mathcal{M}^{-1}A)$ and $\Phi = \sin(\Delta t \mathcal{M}^{-1}A)A^{-1}\mathcal{M}$ (see also [Dumont, 2005] for application of MMM-scheme to one-degree of freedom vibro-impact oscillators).

Remark 7. *The non-standard schemes (26) or (29) satisfies both the continuous and the discrete principles of conservation of energy (19) and (21) between, two consecutive impact times.*

3.1 Numerical experiments

Like in [Dumont and Lubuma, 2007], we first consider the special case, where $m = 10^{12}$, i.e the mass of the second spring is very large. In this case the second body can be considered as a "rigid wall"; thus, only the first body will move. We consider the following data:

Δt	e	(q_1, v_1)	(q_2, v_2)	$q_{\min}$	$q_{\max}$	μ
0.01	0.5	$(1, 0)$	$(0, 0)$	0	$+\infty$	5

We use the non-standard methods (13) or (17). In particular, we make a comparison with the standard explicit (or semi-implicit) scheme in order to show the efficiency of the non-standard schemes when no impacts occur. We obtain, with self-explanatory notation, the results:

$$\max_n \left|q_1^{PS-NS}(t_n) - q_1^{exact}(t_n)\right| = 4.18 \times 10^{-11},$$
$$\max_n \left|q_1^{MMM-NS}(t_n) - q_1^{exact}(t_n)\right| = 9.21 \times 10^{-13},$$
$$\max_n \left|q_1^{S}(t_n) - q_1^{exact}(t_n)\right| = 4.99 \times 10^{-3},$$

As expected, the non-standard schemes give very good results for q_1. If we add impacts (with $e = 0.5$), the first body bounces until it stops while the second one does not move: we obtain exactly the same result as in [Dumont and Lubuma, 2005] (see Fig. 3, page 1942), as expected. Note that if we consider a huge value for μ, for instance $\mu = 10^{12}$, the standard schemes blow-up while the non-standard ones not [Dumont and Lubuma, 2007].

Let $\Delta t = 0.005$, we now consider the general case

e	(q_1, v_1)	(q_2, v_2)	$q_{\min}$	$q_{\max}$	μ	m
0	$(1, -1)$	$(0, 5)$	0.2	1	5	1, 1000

We compute the solution with the nonstandard MMM and PS-schemes. Since $e = 0$, the motions of both bodies are regular and become periodic after a while (see Figures 2 and 4). Note the difference between

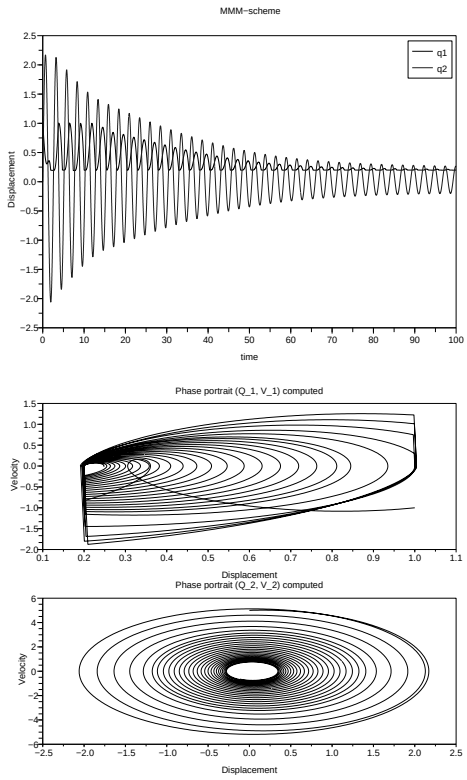

Figure 2. Displacement and Phase portrait for $e = 0$ and $m = 1$ (NSMMM-scheme)

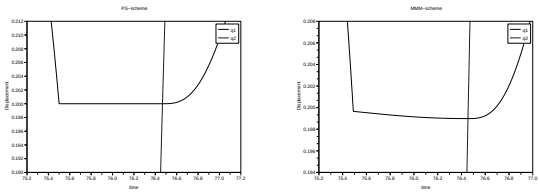

Figure 3. Sticking zone: comparison between the NSPS-scheme and the NSMMM-scheme ($m = 1$)

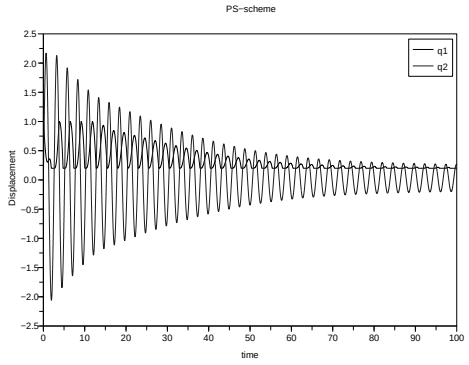

Figure 4. Displacement for $e = 0$ and $m = 1$ (NSPS-scheme)

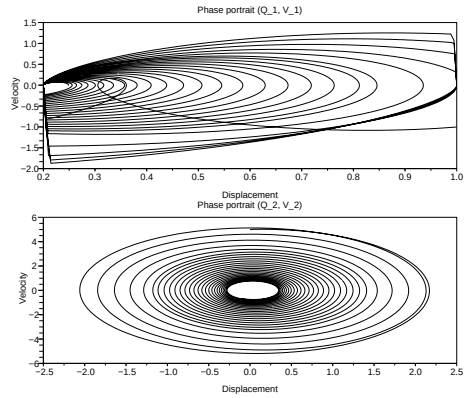

Figure 5. Displacement and Phase portait for $e = 0$ and $m = 1$ (NSPS-scheme)

the numerical solutions: the MMM-solution allows some penetration (see the phase portraits in figures 2 and 5), that may depend on Δt, while the PS-solution not. Moreover, as seen in figure (3), during the sticking phase the solution computed with the nonstandard MMM-scheme is not good (the lower rigid stop behaves like a flexible stop and allows more and more penetration), while the solution computed with the nonstandard PS-scheme do not penetrate during the sticking phase, as expected. Since $m = 1000$, the second body oscillates periodically. Unfortunately, in that case, the non-standard MMM-scheme leads to a spurious solution for the first mass: during the sticking phase, the penetration of m_1 becomes more and more important (see Figure 6). For the same data, the non-standard PS-scheme performs very good result.

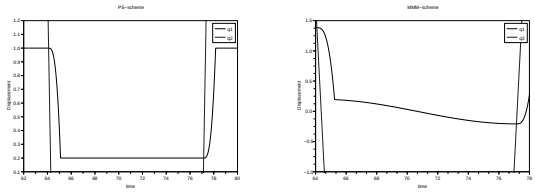

Figure 6. Sticking zone: comparison between the NSPS-scheme and the NSMMM-scheme ($m = 1000$)

See [Dumont and Lubuma, 2007] for additional examples that may be investigate with the MMM-scheme and compare with the PS-scheme.

4 Conclusion

One of the main concerns in the study of impact oscillators is that these are non-smooth systems, which could have complex behaviors. When we perform numerical computations, it is essential to design schemes, which preserve as much as possible the significant physical properties of the systems.

In this paper, we have constructed non-standard approximations for N-dimensional vibro-impact oscillators. Motivated by the structure of the exact scheme of the harmonic oscillator, the construction makes use of one of Mickens' rules. This is the renormalization of the denominator and the numerator of the discrete derivative [Mickens, 1994]. Compared to classical methods, which are based on the usual finite difference method, the scheme under consideration in this work is unconditionally stable and it replicates, a number of key physical properties of the vibro-impact system. We showed that the non-standard PS-scheme gives very satisfactory results, while the non-standard version of the MMM-scheme not really. We need to investigate further and to understand why these differences between both approaches.

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