STOCHASTIC SENSITIVITY AND CONTROL OF CHAOS

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Abstract: We suggest a new technique for analysis and control of the forced nonlinear oscillations based on stochastic sensitivity function (SSF). This function describes the dispersion of random trajectories near deterministic attractor. The possibilities of SSF to predict some peculiarities of dynamics for stochastically and periodically forced oscillators are shown. The thin effects observed in Brusselator and stochastic Lorenz model near chaos in a period-doubling bifurcations zone are presented. The problem of stochastic cycles control based on SSF is considered. The possibilities for formation of stochastic attractor with desired features by feedback regulator are presented. An example of controlling chaos for Brusselator is considered.

Keywords: Sensitivity analysis, control, stochastic systems, chaos

1. INTRODUCTION

Stochastic fluctuations of nonlinear oscillations play an important role for understanding of the corresponding dynamical phenomena for electronic generators, lasers, mechanical, chemical and biological systems. The various noise-induced transitions through periodic to more complicated regimes are a central problem in modern nonlinear dynamics stochastic theory. The sensitivity analysis of random forced oscillations is a key for investigation of these transitions. Control of stochastic and chaotic oscillations is challenging and fundamental problem of nonlinear engineering (Chen and Yu (2003), Fradkov and Pogromsky (1998)).

Analysis of nonlinear oscillations under the stochastic disturbances was started by Pontryagin *et al.* (1933) and continued by many researchers. The random trajectories of forced system leave the closed curve of deterministic limit cycle and due to cycle stability form some bundle around it. Stochastic cycles were considered both near and far from Hopf bifurcation point. A qualitative effect of external fluctuations on the Hopf scenario was found and investigated by Moss and McClintock (1989) and Arnold *et al.* (1997). Small external noises acting on limit cycles may give rise of local phase-dependent response of the oscillations. Local instability of cycle is a reason of its significant sensitive dependence and can cause of noise-induced transition to chaos (Bashkirtseva and Ryashko (2000)). A variance of stochastic bundles perpendicular to the deterministic orbit is a natural measure of limit cycles sensitivity. Kolmogorov-Fokker-Planck (KFP) equation gives the most detailed probabilistic description. However, the direct using of this equation is very difficult even for the simplest situations. Under these circumstances asymptotics and approximations are used. Asymptotic analysis of distribution density for small noises based on quasipotential function is actively developed (Freidlin and Wentzell (1984), Naeh *et al.* (1990), Dembo and

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Zeitouni (1995), Roy (1997), Smelyanskiy *et al.* (1997)).

Quasipotential gives exponential asymptotics for stationary probability density. In Section 2 we give the first approximation of quasipotential in the vicinity of limit cycle. This approximation is an orbital quadratic form. Matrix of this quadratic form defines a covariance of the normal deviations of random trajectories for any point on a cycle. This matrix function plays a role of stochastic sensitivity function (SSF) of a cycle. SSF is a natural probabilistic measure of stochastic cycles response to small random disturbances (Bashkirtseva and Ryashko (2002), Bashkirtseva and Ryashko (2004)).

For the case of cycle on a plane (2D-cycle) a scalar analytical representation of this matrix SSF is given. The possibilities of this scalar SSF to predict some peculiarities of 2D-cycles for forced Brusselator are demonstrated. New critical values of Brusselator parameters with the help of SSF are shown. For these values very small disturbances transfers Brusselator to chaotic regime.

Now the main interest of researchers is concentrated on the analysis of nonlinear systems with 3D-cycles. These systems demonstrate the period-doubling bifurcation and transition to chaos. In this paper, we present new effective method of the SSF construction for 3D-cycles. On the basis of singular expansion, the matrix differential equation is reduced a system of three scalar equations only. A periodic solution of this simple system can be found by stabilization method.

The results of the investigation of multiscroll Lorenz cycles near chaos in a period-doubling bifurcations zone are presented.

In Section 3, we consider the problem of stochastic cycles control based on SSF. The possibilities for formation of stochastic attractor with desired features by feedback regulator are discussed. Controllability analysis and effective algorithms for regulators synthesis are presented. An example of controlling chaos for Brusselator is considered.

2. STOCHASTIC SENSITIVITY OF LIMIT CYCLES

For many dynamical processes with regular oscillations, the basic mathematical model is the nonlinear deterministic system

$$\dot{x} = f(x) \tag{1}$$

with T-periodic solution $x = \xi(t)$. Here x is $n-$vector, $f(x)$ is $n-$vector function. Let γ be a phase curve (limit cycle) of solution $\xi(t)$ satisfying the following stability property.

Definition 1. The cycle γ is called exponentially stable if for small neighbourhood Γ of cycle γ there exist constants $K > 0$, $l > 0$ such that for any solution $x(t)$ of system (1) with $x(0) = x_0 \in \Gamma$ the following inequality holds

$$\|\Delta(x(t))\| \le Ke^{-lt}\|\Delta(x_0)\|.$$

Here $\Delta(x) = x - \gamma(x)$ is a deviation of a point x from a cycle γ, $\gamma(x)$ is the point on cycle γ that is nearest to x.

A system of stochastic differential equations (in Ito's or Stratonovich's sense)

$$\dot{x} = f(x) + \varepsilon\sigma(x)\dot{w}, \tag{2}$$

is a traditional mathematical model allowing to study quantitative description of results of external disturbances. Here $w(t)$ is a $n-$dimensional Wiener process, $\sigma(x)$ is $n \times n-$matrix function of disturbances with intensity ε.

The random trajectories of forced system (2) leave the closed curve of deterministic cycle γ and due to cycle stability form some bundle around it.

The detailed description of random distribution dynamics of this bundle is given by Kolmogorov-Fokker-Planck (KFP) equation. If the character of transient is inessential and main interest is connected with regime of steady-state stochastic auto-oscillations then it is possible to restrict the research by analysis of a stationary density function $\rho(x, \varepsilon)$. Analytical research of stationary KFP equation for stochastic limit cycles considered here is a very difficult problem. Under these circumstances asymptotics and approximations based on quasipotential $v(x) = -\lim_{\varepsilon\to 0}\varepsilon^2 \log \rho(x, \varepsilon)$ are actively used.

The probabilistic distribution for the bundle of random trajectories localized near cycle has Gaussian approximation

$$\rho \approx Ke^{-\frac{v(x)}{\varepsilon^2}} \approx K \exp\left(-\frac{(\Delta(x), \Phi^{+}(\gamma(x))\Delta(x))}{2\varepsilon^2}\right)$$

with covariance matrix $\varepsilon^2\Phi(\gamma)$. This covariance matrix characterizes a dispersion of the points of intersection of random trajectories with hyperplane orthogonal to cycle at the point γ. A function $\Phi(\gamma)$ is a stochastic sensitivity function (SSF) of limit cycle. This function allows to describe non-uniformity of a bundle width along cycle for all directions. It gives the simple way to indicate the most and the least sensitive parts of cycle to external noises.

It is convenient to search for a function $\Phi(\gamma)$ in parametric form. The solution $\xi(t)$ connecting the points of cycle γ with points of an interval $[0, T)$ gives the natural parametrization $\Phi(\xi(t)) = W(t)$. Matrix function $W(t)$ is a solution of Lyapunov equation (Bashkirtseva and Ryashko (2004))

$$\dot{W} = F(t)W + WF^\top(t) + P(t)S(t)P(t), \quad (3)$$

with conditions

$$W(0) = W(T) \quad (4)$$

$$W(t)r(t) \equiv 0, \quad (5)$$

Here

$$F(t) = \frac{\partial f}{\partial x}(\xi(t)), \;\; S(t) = \sigma(\xi(t))\sigma^\top(\xi(t)),$$

$$r(t) = f(\xi(t)), \;\; P(t) = P_{r(t)}, \;\; P_r = I - rr^\top / r^\top r,$$

where P_r is a projection matrix onto the subspace orthogonal to the vector $r \neq 0$.

2.1 Sensitivity analysis of 2D-cycles

For the case $n = 2$ the projection matrix is given by $P(t) = p(t)p^T(t)$, where $p(t)$ is a normalized vector orthogonal to $f(\xi(t))$. As a result the matrix $W(t)$ is written as $W(t) = \mu(t)P(t)$. Here $\mu(t) > 0$ is T-periodic scalar stochastic sensitivity function (Bashkirtseva and Ryashko (2000)). The value $M = \max \mu(t), \; t \in [0, T]$ plays an important role in the analysis of stochastic dynamics about a limit cycle. We shall consider M as a *sensitivity factor* of a cycle γ response to random disturbances.

Consider forced system

$$\begin{aligned} \dot{x} &= a - (b+1)x + x^2 y + \varepsilon\theta \\ \dot{y} &= bx - x^2 y \end{aligned} \quad (6)$$

received by the addition of small disturbances $\varepsilon\theta(t)$ to classical Brusselator.

It is known that for $b > \bar{b} = 1 + a^2$ the unforced system ($\varepsilon = 0$) has a limit cycle ($\bar{b}$ is bifurcation value). We consider the results of the comparative analysis of this system cycles for a fixed $a = 0.2$ and various values of parameter $b > \bar{b} = 1.04$ from an interval [1.06,1.07].

Let disturbances in (6) be stochastic: $\theta(t) = \dot{w}$, where $w(t)$ is an independent Wiener process.

For Lyapunov exponent λ and sensitivity factor M dependence on values b is shown in Fig.1 and Fig.2.

As we can see in Fig.1a, a parameter λ monotonically decreases with growth b. This means increase of a stability degree of a cycle to disturbances of initial data. One should think it should be accompanied by the appropriate decrease in the sensitivity of a cycle to random disturbances. However, here the converse is observed. The value M behaves absolutely otherwise (see Fig.1b).

On an examined interval the function $M(b)$ is not monotonic. Its graph has sharp high peak. As a result the function $M(b)$ has an essential overfall of values. By critical value of parameter b here is $b_* = \arg\max_b M(b) = 1.064082$, $M(b_*) = 8.8 \cdot 10^{10}$.

In Fig.3 (left), the random trajectories found by a direct numerical simulation for b_* and $\varepsilon = 10^{-5}$ are demonstrated. For small stochastic disturbances the burst of response amplitude is observed.

Let disturbances in (6) be periodic: $\theta(t) = \cos(\omega t)$, where ω is a frequency.

For a critical parameter value $b_* = 1.064082$ increase in the periodic force intensity ε results in the period-doubling bifurcations of system (6) attractors: 1-cycle ($\varepsilon = 0.0005$) $\rightarrow$ 2-cycle ($\varepsilon = 0.0007$) $\rightarrow$ 4-cycle ($\varepsilon = 0.000763$) and so on. For $\varepsilon = 0.00085$ the bundle of trajectories (see Fig.4 (left)) looks chaotic. For a critical parameter value $b_* = 1.064082$ the forced Brusselator is a *generator of chaos*.

Thus, the function of sensitivity is the useful analytical tool for the prediction of singular responses of a non-linear system both to stochastic and to periodic disturbances.

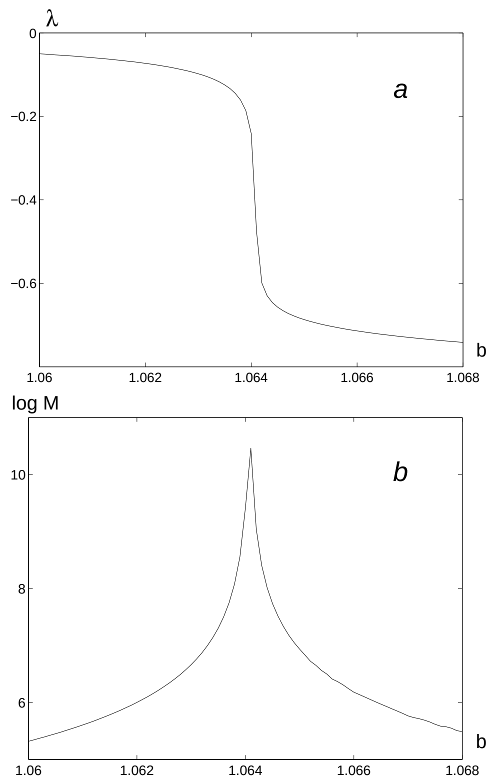

Fig. 1. Deterministic Lyapunov exponent (a) and stochastic sensitivity factor (b).

2.2 Sensitivity of 3D-cycles

For 3D-cycles, due to singularity, the matrix $W(t)$ has the following decomposition

$$W(t) = \lambda_1(t)v_1(t)v_1^\top(t) + \lambda_2(t)v_2(t)v_2^\top(t).$$

Here $\lambda_1(t) \geq \lambda_2(t) \geq \lambda_3(t) \equiv 0$ are eigenvalues, and $v_1(t), v_2(t), v_3(t)$ are eigenvectors of the matrix $W(t)$. The constructive method for computation of this decomposition is presented in (Bashkirtseva and Ryashko (2004)). Consider a stochastic Lorenz system

$$\dot{x} = \sigma(-x+y) + \varepsilon\dot{w}_1 \qquad \sigma = 10,\ b = \frac{8}{3}$$

$$\dot{y} = rx - y - xz + \varepsilon\dot{w}_2$$

$$\dot{z} = -bz + xy + \varepsilon\dot{w}_3$$

received by the addition of small additive stochastic disturbances to classical deterministic Lorenz model. Here $w_i(t)$ $(i = 1,2,3)$ are independent standard Wiener processes, ε is a parameter of noise intensity.

For deterministic Lorenz model ($\varepsilon = 0$) an interval $99.524 < r < 100.795$ is well-known as a period doubling bifurcations zone with infinite chain of limit cycles. This r-interval is divided into subintervals $I_1, I_2, I_4, ..., I_{2^n}, ...$ with limit cycles $\Gamma_1,\ \Gamma_2,\ \Gamma_4..., \Gamma_{2^n}, ...$. Here Γ_k is a non-symmetric stable $(y^2x)^k$ periodic orbit ($k-$ cycle) observed on subinterval I_k.

For Lorenz model $I_1 = (99.98, 100.795)$, $I_2 = (99.629, 99.98)$, $I_4 = (99.547, 99.629)$ etc.

The appearance of noise results in stochastic deformation of the deterministic unforced cycles. Under the random disturbances the trajectories of a stochastically forced system leave the deterministic cycle and form some bundle around it.

For the description of stochastic cycle as a whole it is possible to use stochastic sensitivity factor $m = \max\lambda_1(t),\ t \in [0,T]$. Function $m = m(r)$ describes a variation of cycles stochastic sensitivity as parameter r changes.

Consider this function on the intervals $I_1, I_2, I_4, ...$ (see Fig.2). The branches of $m(r)$ are qualitatively similar on these intervals.

The minimum $m(r)$ on each interval corresponds to values $r_1, r_2, r_4, ...$ (supercycles).

As parameter r decreases or increases from r_k, function $m(r)$ monotonically grows. As parameter approaches bifurcation points function $m(r)$ tends to infinity. However, minimum values $m_k = m(r_k)$ for different intervals are unequal essentially. Here we have some self-similarity: $m_2 \approx 5m_1, m_4 \approx 5m_2, ...$ Thus transition to the next bifurcation

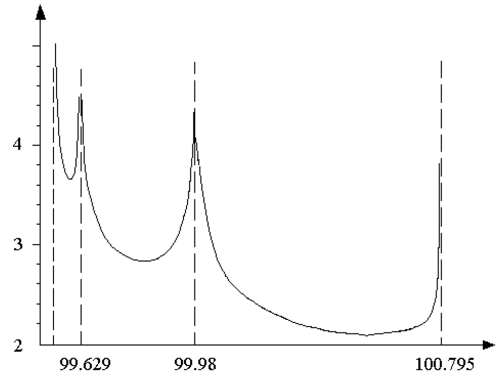

Fig. 2. Stochastic sensitivity factor

interval is accompanied by increase of cycle sensitivity five times. It is possible to resume that sensitivity function predicts chaos signaling about increase of cycle sensitivity by fast rising of values $m_1, m_2, m_4, ...$

3. SENSITIVITY CONTROL

Consider a stochastic system with a control of the form

$$\dot{x} = f(x,u) + \varepsilon\sigma(x,u)\dot{w}, \qquad (7)$$

where x is $n-$dimensional state variable, u is r-dimensional vector of control functions, $f(x,u)$, $\sigma(x,u)$ are vector functions, $w(t)$ is $n-$dimensional Wiener process, ε is scalar parameter of disturbances intensity. It is supposed that for $\varepsilon = 0$ and $u = 0$ the system (7) has $T-$periodic solution $x = \xi(t)$ with a phase trajectory γ (cycle).

The stabilizing regulator we shall select from the class U of admissible feedbacks $u = u(x)$ satisfying conditions:
(a) $u(x)$ is sufficiently smooth and $u|_\gamma = 0$;
(b) for the deterministic system

$$\dot{x} = f(x,u)$$

the solution $x = \xi(t)$ is exponentially stable in the neighbourhood Γ of cycle γ.

3.1 Control and SSF

Consider in detail the case $n = 2$. Sensitivity function μ for $u \in U$ is a solution of boundary-value problem

$$\dot{\mu} = a(t)\mu + b(t),\ \ \mu(0) = \mu(T). \qquad (8)$$

Here

$$a(t) = p^\top(t)(F^\top(t) + F(t))p(t),$$

$$b(t) = p^\top(t)S(t)p(t) \tag{9}$$

$$F(t) = \frac{\partial f}{\partial x}(\xi(t),0) + \frac{\partial f}{\partial u}(\xi(t),0)\frac{\partial u}{\partial x}(\xi(t))$$

$$S(t) = \sigma(\xi(t),0)\sigma^\top(\xi(t),0),$$

$p(t)$ is normalized vector orthogonal to cycle γ at a point $\xi(t)$.

As we see, the variation of control u allows to change the only coefficient $a(t)$ in the equations (9). Note that outcome of control depends only on values of the derivative $\frac{\partial u}{\partial x}$. It gives us possibility to simplify the structure of used regulator.

3.2 Choice of regulator structure

Consider Taylor's expansion of control function $u(x)$ at a point γ

$$u(x) = u(\gamma) + \frac{\partial u}{\partial x}(\gamma)(x-\gamma) + O(\|x-\gamma\|^3).$$

For $\gamma = \gamma(x)$ taking into account condition (a), we get

$$u(x) = \frac{\partial u}{\partial x}(\gamma(x))\Delta(x) + O(\|\Delta(x)\|^3).$$

As we see, a first approximation $u_1(x)$ for arbitrary smooth control function $u \in U$ for small deviations $\Delta(x) = x - \gamma(x)$ is the feedback

$$u_1(x) = \Phi(\gamma(x))\Delta(x). \tag{10}$$

Here $\Phi(\gamma(x))$ is the feedback matrix coefficient. Appropriate t-parametric representation for this matrix looks like $K(t) = \Phi(\xi(t))$.

As mentioned above (see (8), (9)), capabilities of control by sensitivity function $\mu(t)$ are completely determined by linear approximation of a function $u(x)$ and independent on higher order terms. It allows to restrict our consideration without loss of generality by more simple regulators(10) in the following form

$$u = K(t(x))\Delta(x). \tag{11}$$

Thus the feedback coefficient matrix $K(t)$ completely determines capabilities of the regulator (11) to synthesize SSF $\mu(t)$.

Connect controlled coefficient $a(t)$ in the equation (8) with feedback matrix $K(t)$ directly. Really, it follows from (9) that

$$a(t) = a_0(t) + a_1(t)$$

$$a_0(t) = 2q^\top(t)p(t) \tag{12}$$

$$a_1(t) = 2\beta^\top(t)k(t),$$

where

$$q(t) = A(t)p(t),\ \beta(t) = B(t)p(t)$$

$$A(t) = \left[\frac{\partial f}{\partial x}(\xi(t),0)\right]^\top$$

$$B(t) = \left[\frac{\partial f}{\partial u}(\xi(t),0)\right]^\top \tag{13}$$

$$k(t) = \frac{\partial u}{\partial x}(\xi(t))p(t) \ = \ K(t)p(t)$$

Note that the vector k is a derivative of a control function u from (11) in the direction of normal vector p.

3.3 Control goal and choice of regulator parameters

The aim of control is the synthesis of desired SSF for cycle γ of stochastic system (7). Let $\bar{\mu}(t) \in M$ be a some assigned SSF. Here

$$M = \{\mu \in C^1_{[0,T]} \mid \mu(t) > 0, \quad \mu(0) = \mu(T)\}.$$

Denote by μ_u a SSF of cycle γ for stochastic system with control $u \in U$.

Definition 2. A cycle γ is called completely stochastic controllable if for all $\bar{\mu} \in M$ there exists $u \in U$ such that $\mu_u = \bar{\mu}$.

Theorem. A cycle γ is completely stochastic controllable if and only if

$$\beta(t) \neq 0 \quad \forall t \in [0,T]. \tag{14}$$

The function $\bar{\mu}(t)$ is connected with control parameter $k(t)$ by the following equation

$$\beta^\top(t)k(t) = \frac{\bar{a}_1(t)}{2}, \tag{15}$$

where

$$\bar{a_1}(t) \ = \ (\dot{\bar{\mu}}(t) - a_0(t)\bar{\mu}(t) - b(t))/\bar{\mu}(t).$$

The equation (15) for (14) has infinite set of the solutions (control is not unique). Consider additional optimal criterion

$$\|k(t)\|^2 \longrightarrow \min. \tag{16}$$

The problem (15), (16) has the unique solution

$$\bar{k}(t) = \frac{\bar{a}_1(t)\beta(t)}{2\beta^\top(t)\beta(t)}$$

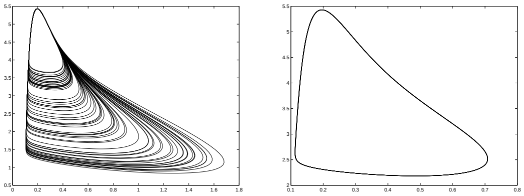

Fig. 3. Stochastic forced Brusselator

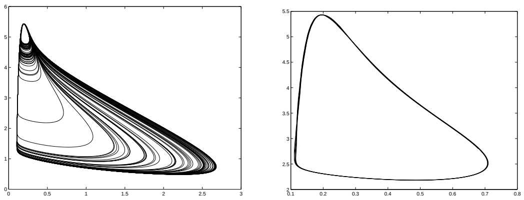

Fig. 4. Periodic forced Brusselator

Here feedback matrix in (11) can be found as follows

$$\bar{K}(t) \;=\; \frac{\bar{a}_1(t)}{2\beta^{\top}(t)\beta(t)}B(t).$$

Consider forced Brusselator (6) with control

$$\dot{x} = a - (b+1)x + x^2 y + u_1 + \varepsilon\theta_1$$

$$\dot{y} = bx - x^2 y + u_2 + \varepsilon\theta_2, \qquad a = 0.2$$

where ε is an external force intensity, u_1 and u_2 are control functions.

For $b_* = 1.064082$ in absence of control this system for stochastic and periodic disturbances $\theta_i(t)$ is generator of chaos (see subsection 2.1). An extraordinary sensitivity of Brusselator is connected with huge values of stochastic sensitivity function.

Now we demonstrate the controlling chaos for this model. Let us take $\bar{\mu}(t) \equiv 5$. Such choice of SSF is dictated by the desire to have oscillations with small sensitivity.

The results of a direct numerical simulation of the forced trajectories for controlled Brusselator are presented in Fig.3 (for stochastic $\theta_i(t) = \dot{w}_i$) and Fig.4 (for periodic $\theta_1 = cos(\omega t)$, $\theta_2 = 0$. Here left plots demonstrate dynamics of system without control and right plots with control. As we see, constructed regulator gives us the solution of chaos control problem.

4. CONCLUSION

Suggested stochastic sensitivity function technique is a useful tool for a quantitative description for a system response on the random external disturbances. Using SSF, we can predict some singularities in dynamics of stochastically and periodically forced oscillators. The critical (chaotic) values of Brusselator parameter were found and the thin effects observed in stochastic Lorenz model near chaos in a period-doubling bifurcations zone were investigated. The new method of stochastic cycles control based on SSF allows to form the stochastic attractor with desired features by feedback regulator. This constructed regulator really provides the solution of a controlling chaos problem.

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