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CONSTRUCTING LYAPUNOV–KRASOVSKII FUNCTIONALS FOR SOME CLASSES OF SWITCHED POSITIVE SYSTEMS WITH INFINITE DELAY

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Abstract

The problem of absolute stability is investigated for a switched positive Persidskii system with infinite delay. With the aid of a special construction of Lyapunov–Krasovskii functional, sufficient conditions of absolute stability are derived. These conditions are formulated in terms of the existence of a positive solution for an auxiliary system of linear algebraic inequalities. It is shown that a similar construction of Lyapunov–Krasovskii functional can be used for the permanence analysis for a Lotka–Volterra model of popolation dynamics.

Key words

Positive system, switching, infinite delay, Lyapunov–Krasovskii functional, Lotka–Volterra model, permanence.

1 Introduction

Stability analysis of time-delay systems is a fundamental and challenging research problem. In the past decades, this problem has attracted considerable attention of researchers in the systems and control community (see, e.g., [Fridman, 2014; Khac, Vlasov and Pyrkin, 2022; Mei, Efimov, Ushirobira, and Fridman, 2023; Ngoc and Anh, 2019; Xu, Liu, Krstic and Feng, 2022] and references therein). However, most of the existing results are obtained for the case of bounded delays. At the same time, in various applications such as population dynamics, mechanics, social science, networked control, etc., models with infinity delays are used [Fridman, 2014; Hino, Murakami and Naito, 1993; Kolmanovskii and Myshkis, 1999; Xu, Liu, Krstic and Feng, 2022].

Some stability conditions for systems with infinite delays were derived in [Kolmanovskii and Myshkis, 1999; Xu, Liu, and Feng, 2020; Xu, Liu, Krstic and Feng, 2022] on the basis of development of the Lyapunov direct method. In [Fridman, 2014], stability analysis of linear time-invariant systems with unbounded delay with application to a problem of traffic control are presented. Stability of certain types of Lotka–Volterra models of population dynamics with infinite delays was studied, e.g., in [Hino, Murakami and Naito, 1993; Ma, Huo and Liu, 2009; Montes de Oca and Vivas, 2006; Xu, Chaplain and Davidson, 2002].

Moreover, it is worth noting that mathematical models of numerous practical systems should possess the positivity property (solutions with nonnegative initial conditions remain nonnegative as time increases) [Kazkurewicz and Bhaya, 1999; Ngoc and Anh, 2019]. Therefore, an important subclass of time-delay systems is that of positive time-delay systems. In particular, in [Naito, Murakami, Shin and Ngoc, 2007; Ngoc and Anh, 2019], positive linear Volterra integro-differential systems were investigated, and conditions of positivity and exponential stability were derived. On the other hand, in various applications, realistic models should take into account such effects as nonlinearities, parametric uncertainties, switching of operation modes [Fridman, 2014; Kolmanovskii and Myshkis, 1999; Liberzon, 2003].

In the present contribution, a switched positive Persidskii type system with infinite delay is considered. Right-hand sides of this system are represented as linear combinations of separable sector nonlinearities with switched coefficients. The problem of the absolute stability of the system is studied. To solve this problem, a special construction of Lyapunov–Krasovskii functional is used. It should be noted that in [Aleksandrov, 2024] absolute stability conditions for positive Persidskii-type system with constant delay were obtained. The objective of the present paper is to extend the approach developed in [Aleksandrov, 2024] to the case of infinite delay.

In addition, we will show that a counterpart of the proposed construction of Lyapunov–Krasovskii functional can be used to derive permanence conditions for a switched Lotka–Volterra model of population dynamics.

2 Statement of the Problem

We use the following notation. $\mathbb{R}$ is the field of real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with the associated norm $\|\cdot\|$ of a vector, $\mathbb{R}^{n\times n}$ is the vector space of $n \times n$ matrices. Let $\mathbb{R}^n_+$ be the nonnegative cone of $\mathbb{R}^n$: $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x \geq 0\}$ and $\operatorname{int}\mathbb{R}^n_+$ be the interior of $\mathbb{R}^n_+$. A matrix $A \in \mathbb{R}^{n\times n}$ is nonnegative if all of its entries are nonnegative. A matrix $A \in \mathbb{R}^{n\times n}$ is called Metzler if all its off-diagonal entries are nonnegative. For a vector $\eta = (\eta_1, \ldots, \eta_n)^\top$, the notation $\eta \gg 0$ ($\eta \ll 0$) means that $\eta_i > 0$ ($\eta_i < 0$) for $i = 1, \ldots, n$. Let $BC((-\infty, 0], \mathbb{R}^n)$ be the space of continuous and bounded functions $\varphi(\xi) : (-\infty, 0] \mapsto \mathbb{R}^n$ with the uniform norm $\|\varphi\|_{BC} = \sup_{\xi\in(-\infty,0]} \|\varphi(\xi)\|$.

Let the switched system

$$\dot{x}(t) = P_\sigma f(x(t)) + Q_\sigma \int_{-\infty}^{t} h(t-\xi) f(x(\xi)) d\xi \quad (1)$$

be given. Here $x(t) \in \mathbb{R}^n$, $f(x)$ is a separable vector function: $f(x) = (f_1(x_1), \ldots, f_n(x_n))^\top$, where the functions $f_i(x_i)$, which are said to be admissible nonlinearities, are continuous for $|x_i| < \Delta$ ($0 < \Delta \leq +\infty$) and satisfy the sector-like constraints $x_i f_i(x_i) > 0$ for $x_i \neq 0$, $i = 1, \ldots, n$, scalar kernel $h(u)$ is nonnegative and continuous for $u \geq 0$, $\sigma = \sigma(t)$ is a piecewise constant function defining the switching law, $\sigma(t) : [0, +\infty) \mapsto \{1, \ldots, N\}$, P_s, Q_s are constant matrices, $s = 1, \ldots, N$.

According to the standard assumption [Liberzon, 2003], we will consider the case where the function $\sigma(t)$ admits only finitely many discontinuities on every finite interval. Such switching laws will be called admissible.

The system (1) belongs to well known class of Persidskii type systems [Kazkurewicz and Bhaya, 1999]. These systems are widely used for modeling automatic control systems, population dynamics, neural networks, opinion dynamics, etc. [Kazkurewicz and Bhaya, 1999; Mei, Efimov, Ushirobira, and Fridman, 2023].

Every solution $x(t) = x(t, t_0, \varphi)$ of (1) is defined by the initial conditions $t_0 \geq 0$ and $\varphi(\xi) \in BC((-\infty, 0], \mathbb{R}^n)$. For a solution $x(t)$, x_t denotes the restriction of this solution to the interval $(-\infty, t]$, i.e., $x_t : \xi \mapsto x(t + \xi)$ for $\xi \in (-\infty, 0]$.

Definition 1. [Ngoc and Anh, 2019] *The system (1) is called positive if, for any $t_0 \geq 0$ and any nonnegative for $\xi \in (-\infty, 0]$ initial function $\varphi(\xi)$, the corresponding solution satisfies $x(t, t_0, \varphi) \geq 0$ for $t \geq t_0$.*

Assumption 1. *The matrices P_s are Metzler and the matrices Q_s are nonnegative, $s = 1, \ldots, N$.*

Remark 1. Under Assumption 1, the system (1) is positive (see [Ngoc and Anh, 2019]).

From the continuity of $f(x)$ and the sector restrictions it follows that $f(0) = 0$. Hence, there exists the zero solution of (1).

Definition 2. *The system (1) is called absolutely stable if its zero solution is asymptotically stable for any admissible nonlinearities and any admissible switching signal.*

The objective of the present paper is to obtain the absolute stability conditions for positive system (1). Our analysis will be based on the use of a special construction of diagonal Lyapunov–Krasovskii functional. To derive conditions under which the absolute stability can be proven with the aid of such a functional, we will apply an approach that was first proposed in [Pastravanu and Matcovschi, 2014] and subsequently developed in [Aleksandrov, 2021; Aleksandrov, 2024]. In addition, we will show that a similar functional can be used for the permanence analysis of a switched Lotka–Volterra model with infinite delay describing interaction of species in a biological community.

3 Absolute Stability Conditions

Denote $H(t) = \int_t^{+\infty} h(u)du$ for $t \geq 0$. We will impose the following additional constraints on the kernel in (1).

Assumption 2. *Let $H(0) < +\infty$.*

Assumption 3. *Let $\int_0^{+\infty} H(t)dt < +\infty$.*

Remark 2. For instance, the exponential kernel $h(u) = \exp(\alpha u)$ with $\alpha < 0$ satisfies Assumptions 2 and 3. Such kernels are widely used in models of control systems [Fridman, 2014].

Theorem 1. *Let Assumptions 1–3 be fulfilled. If there exist a positive rational number ν with odd numerator and denominator, vectors $\theta \gg 0$, $\eta \gg 0$ and numbers ω_1, ω_2 such that*

$$\nu\omega_1 + \omega_2 < 0, \quad (2)$$

$$(P_s + H(0)Q_s)\theta \leq \omega_1\theta, \quad s = 1, \ldots, N, \quad (3)$$

$$(P_s + H(0)Q_r)^\top \eta \leq \omega_2\eta, \quad s, r = 1, \ldots, N, \quad (4)$$

then the system (1) is absolutely stable.

Proof. For a given value of ν, choose a diagonal Lyapunov–Krasovskii functional candidate as follows

$$V(x_t) = \sum_{i=1}^{n} a_i \int_0^{x_i(t)} f_i^\nu(u)du$$

$$+\sum_{i=1}^{n} b_i \int_{-\infty}^{t} H(t-\xi) f_i^{\nu+1}(x_i(\xi)) d\xi, \qquad (5)$$

where $a_i > 0$, $b_i \geq 0$ are tuning parameters. Consider the derivative of this functional along the solutions of (1). We obtain

$$\dot{V} = \sum_{i,j=1}^{n} a_i p_{ij}^{(\sigma)} f_i^{\nu}(x_i(t)) f_j(x_j(t))$$

$$+\sum_{i,j=1}^{n} a_i q_{ij}^{(\sigma)} f_i^{\nu}(x_i(t)) \int_{-\infty}^{t} h(t-\xi) f_j(x_j(\xi)) d\xi$$

$$-\sum_{i=1}^{n} b_i \int_{-\infty}^{t} h(t-\xi) f_i^{\nu+1}(x_i(\xi)) d\xi$$

$$+\sum_{i=1}^{n} b_i H(0) f_i^{\nu+1}(x_i(t)).$$

Here $p_{ij}^{(\sigma)}$ and $q_{ij}^{(\sigma)}$ are entries of the matrices P_σ and Q_σ, respectively.

Let θ_i and η_i be components of the vectors θ and η, $a_i = \eta_i/\theta_i^{\nu}$, $z_i(t) = f_i(x_i(t))/\theta_i$, $i = 1, \ldots, n$. Then

$$\dot{V} = \sum_{i,j=1}^{n} \eta_i \theta_j p_{ij}^{(\sigma)} z_i^{\nu}(t) z_j(t)$$

$$+\sum_{i,j=1}^{n} \eta_i \theta_j q_{ij}^{(\sigma)} z_i^{\nu}(t) \int_{-\infty}^{t} h(t-\xi) z_j(\xi) d\xi$$

$$-\sum_{i=1}^{n} b_i \theta_i^{\nu+1} \int_{-\infty}^{t} h(t-\xi) z_i^{\nu+1}(\xi) d\xi$$

$$+\sum_{i=1}^{n} b_i \theta_i^{\nu+1} H(0) z_i^{\nu+1}(t)$$

$$\leq \sum_{i=1}^{n} \eta_i \theta_i p_{ii}^{(\sigma)} z_i^{\nu+1}(t)$$

$$+\sum_{i,j=1,\ i\neq j}^{n} \eta_i \theta_j p_{ij}^{(\sigma)} |z_i(t)|^{\nu} |z_j(t)|$$

$$+\sum_{i,j=1}^{n} \eta_i \theta_j q_{ij}^{(\sigma)} |z_i(t)|^{\nu} \int_{-\infty}^{t} h(t-\xi) |z_j(\xi)| d\xi$$

$$-\sum_{i=1}^{n} b_i \theta_i^{\nu+1} \int_{-\infty}^{t} h(t-\xi) z_i^{\nu+1}(\xi) d\xi$$

$$+\sum_{i=1}^{n} b_i \theta_i^{\nu+1} H(0) z_i^{\nu+1}(t).$$

Using the Young inequality [Fridman, 2014], we obtain

$$\dot{V} \leq \sum_{i=1}^{n} \eta_i \theta_i p_{ii}^{(\sigma)} z_i^{\nu+1}(t)$$

$$+\frac{\nu}{\nu+1} \sum_{i,j=1,\ i\neq j}^{n} \eta_i \theta_j p_{ij}^{(\sigma)} z_i^{\nu+1}(t)$$

$$+\frac{1}{\nu+1} \sum_{i,j=1,\ i\neq j}^{n} \eta_i \theta_j p_{ij}^{(\sigma)} z_j^{\nu+1}(t)$$

$$+\frac{\nu}{\nu+1} H(0) \sum_{i,j=1}^{n} \eta_i \theta_j q_{ij}^{(\sigma)} z_i^{\nu+1}(t)$$

$$+\frac{1}{\nu+1} \sum_{i,j=1}^{n} \eta_i \theta_j q_{ij}^{(\sigma)} \int_{-\infty}^{t} h(t-\xi) z_j^{\nu+1}(\xi) d\xi$$

$$-\sum_{i=1}^{n} b_i \theta_i^{\nu+1} \int_{-\infty}^{t} h(t-\xi) z_i^{\nu+1}(\xi) d\xi$$

$$+\sum_{i=1}^{n} b_i \theta_i^{\nu+1} H(0) z_i^{\nu+1}(t)$$

$$= \frac{\nu}{\nu+1} \sum_{i=1}^{n} \eta_i z_i^{\nu+1}(t) \sum_{j=1}^{n} \left(p_{ij}^{(\sigma)} + H(0) q_{ij}^{(\sigma)} \right) \theta_j$$

$$+\sum_{i=1}^{n} \theta_i z_i^{\nu+1}(t) \left(\frac{1}{\nu+1} \sum_{j=1}^{n} \eta_j p_{ji}^{(\sigma)} + b_i \theta_i^{\nu} H(0) \right)$$

$$+\frac{1}{\nu+1} \sum_{i=1}^{n} \theta_i \int_{-\infty}^{t} h(t-\xi) z_i^{\nu+1}(\xi) d\xi \sum_{j=1}^{n} \eta_j q_{ji}^{(\sigma)}$$

$$-\sum_{i=1}^{n} b_i\theta_i^{\nu+1}\int_{-\infty}^{t} h(t-\xi)z_i^{\nu+1}(\xi)d\xi$$

$$\leq \frac{\nu\omega_1}{\nu+1}\sum_{i=1}^{n}\eta_i\theta_i z_i^{\nu+1}(t)$$

$$+\sum_{i=1}^{n}\theta_i z_i^{\nu+1}(t)\left(\frac{1}{\nu+1}\sum_{j=1}^{n}\eta_j p_{ji}^{(\sigma)} + b_i\theta_i^{\nu}H(0)\right)$$

$$+\frac{1}{\nu+1}\sum_{i=1}^{n}\theta_i\int_{-\infty}^{t} h(t-\xi)z_i^{\nu+1}(\xi)d\xi\sum_{j=1}^{n}\eta_j q_{ji}^{(\sigma)}$$

$$-\sum_{i=1}^{n} b_i\theta_i^{\nu+1}\int_{-\infty}^{t} h(t-\xi)z_i^{\nu+1}(\xi)d\xi.$$

If

$$b_i\theta_i^{\nu} = \frac{1}{\nu+1}\max_{r=1,\ldots,N}\sum_{j=1}^{n}\eta_j q_{ji}^{(r)}, \quad i=1,\ldots,n,$$

then

$$\dot{V} \leq \frac{\nu\omega_1+\omega_2}{\nu+1}\sum_{i=1}^{n}\eta_i\theta_i z_i^{\nu+1}(t).$$

Hence (see [Kolmanovskii and Myshkis, 1999]), the system (1) is absolutely stable. The proof is completed. □

Corollary 1. *Let Assumptions 1–3 be valid. If one of the following conditions is satisfied:*

(i) there exists a vector $\theta \gg 0$ such that

$$(P_s + H(0)Q_s)\theta \ll 0, \quad s=1,\ldots,N,$$

(ii) there exists a vector $\eta \gg 0$ such that

$$(P_s + H(0)Q_r)^{\top}\eta \ll 0, \quad s,r=1,\ldots,N,$$

then the system (1) is absolutely stable.

Indeed, in the case (i) the conditions of Theorem 1 will be fulfilled for sufficiently large values of ν, whereas in the case (ii) the conditions of Theorem 1 will be fulfilled for sufficiently small values of ν.

4 Permanence Analysis for a Lotka–Volterra Model

Next, consider the system

$$\dot{x}_i(t) = x_i(t)\left(c_i^{(\sigma)} + \sum_{j=1}^{n} p_{ij}^{(\sigma)} f_j(x_j(t)) + \sum_{j=1}^{n} q_{ij}^{(\sigma)}\int_{-\infty}^{t} h(t-\xi)f_j(x_j(\xi))d\xi\right), \quad (6)$$

$$i=1,\ldots,n.$$

This system belongs to well known class of Lotka–Volterra models describing interaction of species in biological communities (see [Hofbauer and Sigmund, 1998; Kazkurewicz and Bhaya, 1999; Pykh, 1983; Romero-Melendez and Castillo-Fernandez, 2022]). Here $x_i(t) \in \mathbb{R}$ is the population density of the i-th species, the functions $f_j(x_j)$ are defined for $x_j \in [0,+\infty)$, $h(u)$ is nonnegative and continuous for $u \geq 0$ scalar kernel, $\sigma = \sigma(t)$ is an admissible switching law, $\sigma(t) : [0,+\infty) \mapsto \{1,\ldots,N\}$, $p_{ij}^{(s)}$, $q_{ij}^{(s)}$, $c_i^{(s)}$ are constant coefficients. The coefficients $c_i^{(\sigma)}$ characterise the intrinsic growth rate of the i-th population, the terms $p_{ii}^{(\sigma)}x_i(t)f_i(x_i(t))$ and $q_{ii}^{(\sigma)}x_i(t)\int_{-\infty}^{t} h(t-\xi)f_i(x_i(\xi))d\xi$ reflect the self-interaction in the i-th population, whereas the terms $p_{ij}^{(\sigma)}x_i(t)f_j(x_j(t))$ and $q_{ij}^{(\sigma)}x_i(t)\int_{-\infty}^{t} h(t-\xi)f_j(x_j(\xi))d\xi$ for $j \neq i$ describe the influence of population j on population i.

According to the standard assumptions (see [Pykh, 1983]), we consider the case where functions $f_i(x_i)$ admit the following properties:

1) $f_i(x_i)$ are continuous and locally Lipschitz for $x_i \geq 0$;

2) $f_i(x_i) > 0$ for $x_i > 0$, and $f_i(0) = 0$;

3) $f_i(x_i) \to +\infty$ as $x_i \to +\infty$.

Taking into account the biological meaning of the model, we will consider the system (6) in $\operatorname{int}\mathbb{R}_+^n$. Let initial functions $\varphi(\xi)$ for solutions of (6) belong to the space $BC((-\infty,0],\operatorname{int}\mathbb{R}_+^n)$.

Remark 3. From the properties of functions $f_i(x_i)$ it follows that $\operatorname{int}\mathbb{R}_+^n$ is an invariant set for (6) [Pykh, 1983].

Actual problem of population dynamics is that of permanence of the investigated models [Hofbauer and Sigmund, 1998]. The permanence property means that the following two conditions are satisfied:

(i) Ultimate boundedness of solutions, i.e., the existence of a compact domain in the state space of a system such that each motion enters into it in a finite time and remains in this domain thereafter.

(ii) The condition that species do not extinct, i.e., no matter how small initial quantities of species are, starting from a certain time instant, their quantities will exceed some fixed positive values.

Some results on permanence of delay-free Lotka–Volterra systems were obtained in [Aleksandrov, Aleksandrova and Platonov, 2013; Chen, 2006; Hofbauer and Sigmund, 1998]. In [Aleksandrov, 2020; Ma, Huo and

Liu, 2009; Nakata and Muroya, 2010], the permanence conditions were studied for certain classes of biological models with constant delays. However, it should be noted that the permanence problem for systems with switching and infinite delay has not been sufficiently studied.

For the system (6), the corresponding definition is formulated in the following form.

Definition 3. *The system (6) is called permanent if there exist numbers* δ_1 *and* δ_2, $0 < \delta_1 < \delta_2$, *and, for any solution* $x(t, t_0, \varphi) = (x_1(t, t_0, \varphi), \dots, x_n(t, t_0, \varphi))^\top$ *with* $t_0 \geq 0$, $\varphi(\xi) \in BC((-\infty, 0], \operatorname{int} \mathbb{R}^n_+)$, *one can choose* $T \geq t_0$ *such that* $\delta_1 \leq x_i(t, t_0, \varphi) \leq \delta_2$, $i = 1, \dots, n$, *for* $t \geq T$.

Definition 4. *The system (6) is called absolutely permanent if it is permanent for any admissible functions* $f_i(x_i)$ *and any admissible switching law.*

To derive conditions of absolute permanence, we will impose some additional constraints on the right-hand sides of (6).

Assumption 4. *Let* $H(t) \leq \beta h(t)$ *for* $t \geq 0$, $\beta = \text{const} > 0$.

Remark 4. For instance, the exponential kernel considered in Remark 2 satisfies Assumptions 4.

Assumption 5. *Let* $c_i^{(s)} > 0$, $p_{ij}^{(s)} \geq 0$ *for* $i \neq j$, $q_{ij}^{(s)} \geq 0$, $i, j = 1, \dots, n$, $s = 1, \dots, N$.

Remark 5. From the biological point of view, Assumption 5 means that, for each species, birth rate is greater than mortality rate, and between any two species in a community there are interactions of the following types: "symbiosis", "compensalism" or "neutralism" [Hofbauer and Sigmund, 1998].

Denote $P_s = \left\{ p_{ij}^{(s)} \right\}_{i,j=1}^n$, $Q_s = \left\{ q_{ij}^{(s)} \right\}_{i,j=1}^n$, $s = 1, \dots, N$.

Theorem 2. *Let Assumptions 2, 4, 5 be fulfilled. If there exist a positive rational number* ν *with odd numerator and denominator, vectors* $\theta \gg 0$, $\eta \gg 0$ *and numbers* ω_1, ω_2 *such that the inequalities (2)–(4) hold, then the system (6) is absolutely permanent.*

Proof. For a given value of ν, construct the following counterpart of the functional (5):

$$V(x_t) = \sum_{i=1}^n a_i \int_1^{x_i(t)} \frac{f_i^\nu(u)}{u} du$$

$$+ \sum_{i=1}^n b_i \int_{-\infty}^t H(t-\xi) f_i^{\nu+1}(x_i(\xi)) d\xi,$$

where a_i, b_i are positive coefficients. Consider the derivative of this functional along the solutions of (6). In a similar way as in the proof of Theorem 1, it can be shown that, for appropriate choice of the coefficients a_i and b_i, the estimate

$$\dot{V} \leq -\gamma \sum_{i=1}^n \Big(f_i^{\nu+1}(x_i(t))$$

$$+ \int_{-\infty}^t H(t-\xi) f_i^{\nu+1}(x_i(\xi)) d\xi \Big) + M$$

is valid, where γ and M are positive constants. Hence, $\dot{V} < 0$ for

$$\sum_{i=1}^n \left(f_i^{\nu+1}(x_i(t)) + \int_{-\infty}^t H(t-\xi) f_i^{\nu+1}(x_i(\xi)) d\xi \right) > \frac{M}{\gamma}.$$

Taking into account Assumption 5, we obtain

$$\dot{x}_i(t) \geq x_i(t) \left(c_i^{(\sigma)} + p_{ii}^{(\sigma)} f_i(x_i(t)) \right), \quad i = 1, \dots, n.$$

This implies that $\dot{x}_i(t) > \bar{c}$ for $0 < x_i(t) < \delta_1$, $i = 1, \dots, n$, for some $\bar{c} > 0$, $\delta_1 > 0$.

Denote

$$\Omega = \Big\{ x_t \in BC((-\infty, 0], \operatorname{int} \mathbb{R}^n_+) : \sum_{i=1}^n \Big(f_i^{\nu+1}(x_i(t))$$

$$+ \int_{-\infty}^t H(t-\xi) f_i^{\nu+1}(x_i(\xi)) d\xi \Big) \leq \frac{2M}{\gamma},$$

$$x_i(t) \geq \delta_1, \quad i = 1, \dots, n \Big\},$$

$$\widetilde{V} = \sup_{x_t \in \Omega} V(x_t).$$

It should be noted that $0 < \widetilde{V} < +\infty$. Let

$$\widetilde{\Omega} = \Big\{ x_t \in BC((-\infty, 0], \operatorname{int} \mathbb{R}^n_+) : V(x_t) \leq \widetilde{V},$$

$$x_i(t) \geq \delta_1, \quad i = 1, \dots, n \Big\}.$$

Then $\Omega \subset \widetilde{\Omega}$ and there exists $\delta_2 > 0$ such that $x_i(t) \leq \delta_2$, $i = 1, \dots, n$, for $x_t \in \widetilde{\Omega}$.

Consider a solution $x(t, t_0, \varphi) = (x_1(t, t_0, \varphi), \dots, x_n(t, t_0, \varphi))^\top$ of the system (6) with $t_0 \geq 0$, $\varphi(\xi) \in BC((-\infty, 0], \operatorname{int} \mathbb{R}^n_+)$. First, one can find $T_1 \geq t_0$ such that $x_i(t, t_0, \varphi) \geq \delta_1$, $i = 1, \dots, n$, for $t \geq T_1$. Next, there exists T_2 such that $T_2 \geq T_1$ and $x_{T_2}(t_0, \varphi) \in \Omega$. Hence, $x_t(t_0, \varphi) \in \widetilde{\Omega}$ and $x_i(t, t_0, \varphi) \leq \delta_2$, $i = 1, \dots, n$, for $t \geq T_2$. This completes the proof. □

5 Example

Consider the system (1) with $n = 2$, $N = 2$, $h(u) = \exp(-u)$ for $u \geq 0$,

$$P_1 = \begin{pmatrix} -5 & 0 \\ 0 & -4 \end{pmatrix}, \quad P_2 = \begin{pmatrix} -4 & 1 \\ 0 & -3 \end{pmatrix},$$

$$Q_1 = \begin{pmatrix} 0 & 5 \\ 3 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

It is easy to verify that in this case the inequalities (3) and (4) admit positive solutions if and only if

$$\omega_1 \geq \frac{\sqrt{71} - 9}{2}, \quad \omega_2 \geq \frac{\sqrt{73} - 7}{2}.$$

Applying Theorem 1, we obtain that, for any rational number ν with odd numerator and denominator satisfying the condition

$$\nu > \frac{\sqrt{73} - 7}{9 - \sqrt{71}},$$

one can construct a Lyapunov–Krasovskii functional of the form (5) guaranteeing the absolute stability of the considered system.

6 Conclusion

In the present contribution, a special construction of diagonal Lyapunov–Krasovskii functional for a switched positive Persidskii system with infinity delay is proposed. With the aid of such a functional, new conditions of absolute stability of the considered system are derived. A similar functional is used for the permanence analysis of a switched Lotka–Volterra model of population dynamics. The proofs of the presented theorems contain constructive procedures for the verification of the existence of required functionals and finding their parameters. An important direction for further research is an application of the developed approaches in problems of multiagent systems control.

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