EVOLUTION OF CLUSTERS IN
LARGE-SCALE DYNAMICAL NETWORKS

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Abstract

Recent tremendous progress in electronics, complexity theory and network science provides new opportunities for intellectual control of complex large-scale systems operating in turbulent environment via networks of interconnected miniature devices, serving as actuators, sensors and data processors. Actual dynamics of the resulting control systems are too sophisticated to be examined controlled by traditional methods, which primarily deal with ordinary differential equations. However, their complexity can be dramatically reduced by fast processes, organizing the elementary units of the system (called *agents*) into relatively small number of *clusters*. The clusters emerge and deteriorate in response to changes in the environment, and the processes of their formation and destruction are very short in time. During the periods of the clusters' existence, the system's dynamics is essentially *low-dimensional* due to synchronization between the agents in each cluster. An enormously complicated system is thus reduced to a finite-dimensional model with time-varying structure of the state vector. The low-dimensionality of the reduced model allows to control it by using classical methods, e.g. model-predictive or adaptive control.

This philosophy of complex systems control is illustrated on an experimental setup, called the "airplane with feathers". The wings of this airplane are equipped with arrays of microsensors, microcomputers, and microactuators ("feathers"). The feathers self-organize into clusters by using a multi-agent consensus protocol; the aim of this coordination is to reduce the perturbing forces, affecting the airplane in a turbulent flow.

1 Introduction

Whereas interconnected systems have been studied for many decades, recently it has been realized that many of them are governed by similar principles and obey similar mathematical models and thus can be examined by the same methods. This has given rise to new paradigms in study of complex structures and their dynamics, manifested by *Network Science* [Barabasi, 2002, Newman et al., 2006, Easley and Kleinberg, 2010], Multi-Agent Systems (MAS) theory [Shoham and Leyton-Brown, 2008, Mesbahi and Egerstedt, 2010] and the theory of complex systems (or "complexity") [Bar-Yam, 1997]. Attracting enormous attention of researchers from various fields, these new emerging areas *"break down barriers between physics, chemistry and biology ... and the soft sciences of psychology, sociology, economics,and anthropology"* [Bar-Yam, 1997, Wooldridge, 2002]. Interest to system-theoretic properties of complex networks has been motivated by necessity to develop robust and resilient algorithms to control large-scale and spatially distributed systems such as e.g. continental power grids or smart infrastructures, functioning in the information rich world [Murray, 2003, Samad and Annaswamy, 2011, Annaswamy et al., 2017]. To describe complexity effects, two types of models can be used: *agent-based* (networked, microscopic) and *statistical* (known under numerous alternative names, see below).

1.1 Statistical (Macroscopic) Modeling

Historically, statistical models of complex dynamics had appeared and been examined long before the emer-

gence of network science. Since elementary parts of a complex system (e.g. individuals in a society, cars in dense urban traffic or molecules of some gas in a vessel) are numerous and their trajectories can be chaotic, the direct measurement of each part's characteristics is practically impossible. A natural idea is thus to describe the *distribution* of some variable of interest (e.g. the voters' preferences before an election or referendum, cars' or molecules' velocities etc.) or even some special metrics of such a distribution (e.g. temperature or average kinetic energy of the molecules). For example, models of fluid dynamics and continuum mechanics [Batchelor, 1967, Mase, 1967] describe the evolution of density and velocity of *flows*, where individual particles are replaced by indistinguishable *infinitesimal elements* of the flow. Similar statistical ("macroscopic") models naturally describe flows in transportation networks [Helbing, 1995].

In biology and social sciences, statistical (compartmental) models date back to early works on mathematical epidemics [Kermack and McKendrick, 1927] and mathematical bioscience [Rashevsky, 1935, Rashevsky, 1938]. Behaviors between social or biological agents in such models are portrayed by interactions of multiple "species" or *compartments* [Jacquez, 1985]. Compartments are indecomposable entities that can grow or decline. For instance, in SIR models of epidemic spread the three compartments stand for the proportions of susceptible (S), infectious (I) and recovered (R) individuals; in the classical Lotka-Volterra model compartments are populations of predators and preys. Similar statistical models, arising in evolutionary game theory [Maynard Smith, 1982] and sociodynamics [Weidlich, 1971], are used to describe behaviors of social and economic agents. Such models can e.g. describe how the *distribution* of opinions evolves (paying no attention to the opinion of a particular social actor) and are sometimes also referred as Eulerian or "continuum-agent" [Como and Fagnani, 2011, Canuto et al., 2012, Mirtabatabaei and Bullo, 2012, Hendrickx and Olshevsky, 2016].

1.2 Agent-Based (Microscopic) Models

The *networked* or *agent-based* approach considers a complex system as a union of simple autonomous units, referred to as *agents*.[1] This bottom-up approach is broadly used in sociology since the pioneering works on consensus in social groups and evolution of social power [French Jr., 1956, Harary, 1959, Abelson, 1967, DeGroot, 1974], modeling large-scale biological populations (e.g. herds,

flocks and schools [Reynolds, 1987]), ensembles of particles [Vicsek et al., 1995] and coupled oscillators [Kuramoto, 1991, Mirollo and Strogatz, 1990]. Statistical models can be thought as limits of agent-based models as the number of agents tends to infinity (in practice, the system becomes very large) and distinctions between individual agents are neglected.

Usually the agents are mapped to nodes of some *network*, or *interaction graph*. The graph's arcs represent connections, or *couplings* between the agents. The coupling can stand for a physical interaction or presence of information flow between the two agents. In most interesting situations, the connections are sparse: each agent interacts with only a few number of neighbors, adjacent to its node in the graph, and has no information about the remaining part of the network. The topology of the graph can be static or time-varying, in the latter case, it can evolve independently of the agents or *co-evolve* with them [Belykh et al., 2014a]. It should be noted that structural and statistical properties of complex graphs, representing e.g. large-scale power grids or human relations in social groups, lead to a number of difficult problems on the borderline between statistical physics and graph theory [Watts and Strogatz, 1998, Barabási and Albert, 1999, Barabási et al., 2000, Albert and Barabási, 2002, Newman, 2003]. Many problems in this area has, in fact, been anticipated by extensive studies in social network analysis [Wasserman and Faust, 1994, Freeman, 2004]. However, the most intriguing problems are concerned with *dynamics* over complex networks.

1.3 Synchronization and Consensus

Perhaps, the most studied phenomena in dynamic complex networks are various effects of "self-organization" and "spontaneous order" [Néda et al., 2000, Strogatz, 2003] emerging from local interactions among the agents. The term "spontaneous order" has appeared in social philosophy and economics [Barry, 1982] and means self-organization of social or economic self-interested actors, established as a result of their independent actions rather than governmental regulations. In mathematical theory of dynamic networks, this term is typically understood as a kind of *synchronization* [Blekhman, 1988a, Blekhman et al., 1997, Strogatz, 2003, Fradkov, 2007, Pikovsky et al., 2001, Arenas et al., 2008, Wu, 2007, Fradkov, 2017, Boccaletti et al., 2018] between the agents' trajectories. Studies on self-synchronization among dynamical systems date back to the seminal experiments by Huygens in 17th century; the efforts to reproduce his experiments has led to a number of new results and open problems [Bennett et al., 2002, Czolczynski et al., 2011, Pena Ramirez et al., 2016]. Several decades after Huygens' experiments, de Mairan [de Mairan, 1729] discovered the effect of

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[1] In computer and software engineering, an agent is a computer system or software module that is capable of independent action on behalf of its owner or user [Wooldridge, 2002]. In control theory, "agent" is as autonomous dynamical subsystem of a complex system, having some interoperability capacity.

forced synchronization (or *entrainment*) of plants' circadian clocks with an environmental periodic signals. Studies on entrainment of biological rhythms gave birth to the field of chronobiology and opened up the new perspectives in control of limit-cycle oscillators [Johnson, 1999, Roenneberg et al., 2003, Efimov, 2011, Sacre and Sepulchre, 2014, Proskurnikov and Cao, 2017c, Medvedev et al., 2018]. The seminal work [Pecora and Carroll, 1998b] on master-slave synchronization of chaotic system has attracted enormous attention of physicists and engineers. The idea of chaotic synchronization has found numerous applications in telecommunications; this and other aspects of chaos "taming" can be found in surveys [Andrievsky and Fradkov, 2004, Fradkov and Evans, 2005]. Special cases of synchronization are asymptotic coordinate synchronization and consensus [Olfati-Saber et al., 2007a, Ren and Beard, 2008, Ren and Cao, 2011] that take their origin in sociological studies [Proskurnikov and Tempo, 2017, Proskurnikov and Tempo, 2018]. Synchronization and consensus lie in the heart of many algorithms, used by living and artificial agents to reach some kind of coordinated behavior, e.g. motion of large-scale biological formations [Reynolds, 1987], distributed optimization [Tsitsiklis et al., 1986, Nedic et al., 2010] and filtering [Kar and Moura, 2013], flocking and rendezvous of mobile robots [Olfati-Saber, 2006, Bullo et al., 2009], load balancing [Amelina et al., 2015] and synchronization of computer clocks [Schenato and Fiorentin, 2011].

1.4 From Synchronous to Clustered Behaviors

Synchronization is, however, not the only possible form of a complex system's organization. Much more complicated (and less studied) are various forms of *clustering*[2], also called partial or cluster synchronization. Special case of clustering includes the situation, where the states or outputs of all agents should be separated by a positive distance, as in the problems of collision avoidance in swarms and mobile sensor deployment [Ny et al., 2012, Schwager et al., 2011]. Much more interesting are situations where the number of clusters is much smaller than the number of agents; as will be discussed, such a clustering greatly enables efficient control of the system's dynamics using a few control inputs. In terms of statistical models with infinite number of agents, synchronization typically corresponds to a unimodal distribution with a "weak" tail, where clustering corresponds to a bimodal (polarization) or multimodal distribution.

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[2]The term "clustering" has many different meanings; cluster synchronization of dynamical systems should not be confused with problems of clustering (cluster analysis) in artificial intelligence and problems of graph clustering (community detection) in algorithmic graph theory. The latter problem is closely related to *percolation theory*, examining the behavior of connected clusters in random graphs.

Paradoxically, models providing agents' splitting into a few clusters are much less studied than synchronization and consensus algorithms; the structural and dynamical properties of the networks leading to formation and deterioration of stable clusters still remain, to a great extent, unexplored. The models explaining cluster synchronization can be divided into several classes.

One can expect that clustering may be caused by the agents' *heterogeneity*. One can expect, for instance, that in a network constituted by agents of $K \geq 2$ different types, agents of the same type should reach synchrony, constituting thus K different clusters. Such models of clustering have been examined e.g. in [Smet and Aeyels, 2009, Lu et al., 2010], see also recent works [Lu et al., 2017, Du et al., 2018] and references therein. Heterogeneity does not necessarily implies different dynamics of agents. Heterogeneous dynamics of agents can be caused by presence of non-identical exogenous signals, that can be considered as control inputs, disturbances or exclusive information [Wu et al., 2009, Aeyels and Smet, 2011, Xia and Cao, 2011]. In social networks, agents differ in their susceptibility to external influence, being fully or partially closed to assimilation of the peers' opinions [Acemoglu et al., 2013, Friedkin, 2015, Parsegov et al., 2017]. Besides agents, heterogeneity may occur in couplings, e.g. agents can split into communities or clusters such that couplings within a single community are stronger, in some sense, than couplings between different communities; different clusters may even *repulse* [Xia and Cao, 2011, Fläche and Macy, 2011, Altafini, 2012, Altafini, 2013, Proskurnikov and Cao, 2014, Proskurnikov et al., 2014, Ruiz-Silva and Gonzalo Barajas-Ramírez, 2018].

Homogeneous networks, however, also exhibit cluster synchronization. One of the reasons for that can be the *symmetry* of couplings, enabling stable invariant manifolds of partial synchrony [Belykh et al., 2000, Pogromsky et al., 2002, Pogromsky, 2008]; similar effects have been earlier revealed in oscillator networks [Okuda, 1993]. Desynchronization and cluster formation in oscillator networks may be caused by presence of common noise [Gil et al., 2009]. Recently, it has been demonstrated that clustering is possible even if the agents and couplings are identical during special structure of the coupling functions that can e.g. vanish as the distance between agents grows (the principle of *bounded confidence* models [Hegselmann and Krause, 2002, Deffuant et al., 2001, Proskurnikov and Tempo, 2018]) or the agents approach some *stationary points* [Proskurnikov and Cao, 2016]. The distribution of agents between clusters can be uniform or highly unequal, e.g. one huge cluster including all opinions and few clusters of 1-2 elements each. In oscillator networks, such dispropor-

tional structures of clusters are known as "chimera states" [Abrams and Strogatz, 2004].

1.5 Clusters and Control of Networks

Besides portraying important phenomena, observed in natural, social and robotic networks, models of synchronization and clustering play an important role in *control* of complex networks. It has been shown in [Zubarev, 1974] that no finite number of variables (degrees of freedom) can be sufficient to describe a non-equilibrium process with a variable number of degrees of freedom. At the same time, it is known that complex phenomena such as turbulence and spatiotemporal chaos [Xiao et al., 1998, Grigoriev et al., 1997] can be very controlled in a very "parsimonious" way. For instance, to suppress turbulence efficiently[3], one needs a finite number of control inputs that is much less than the number of positive Lyapunov exponents [Xiao et al., 1998], provided that these controls are injected in proper points of the system.

This seminal idea is known as *pinning control* [Grigoriev et al., 1997, Wu et al., 2009, Yu et al., 2014, Fu et al., 2017a]. In multi-agent control, this idea is illustrated by algorithms of *leader-following* synchronization [Ren and Beard, 2008, Lewis et al., 2014]: a group of agents applied a usual consensus algorithm in order to follows a single *leader* agent (influencing the remaining systems directly or indirectly), which can be externally controlled. Similar in spirit algorithms of *containment control* [Ren and Cao, 2011] gather a group of agents inside a convex polytope, spanned by the states of several leaders; controlling the leaders' trajectories, one then can control the motion of the whole swarm without precise synchronization of their trajectories with the leaders. A similar role in social networks is played by stubborn agents [Yildiz et al., 2013, Proskurnikov and Tempo, 2017, Parsegov et al., 2017].

The clustered structure of a network substantially simplifies its structure as a control system and enables relatively simple (low-dimensional) controllers, which can operate in uncertain conditions, as exemplified by modern methods of robust and stochastic model predictive control (MPC) [Mayne et al., 2000, Mesbah, 2018, Seron et al., 2018], adaptive control [Fradkov et al., 1999, Fradkov, 2007], data-driven and learning-based control [Benosman, 2018].

1.6 Forced Clustering and Slow-Fast Dynamics

Summarizing the discussion from the previous subsections, it can be said that "spontaneous order" effects such as synchronization and clustering in complex networks are not only crucial for understanding natural and social phenomena, but also important from

control-theoretic prospective and enable efficient control of very complex dynamics. In spite of enormous progress in understanding these effects, as well as controllability properties of complex systems, most of the existing results are limited in several aspects.

First, the existing results are primarily focused on understanding the roles of *couplings* between the agents and *connectivity* properties of the interconnection topology in reaching full or partial synchronization. Much less understood is the role of *environment*, which can exert heterogeneous forces on one or several agents (one can imagine viscous friction forces, arising when agents move through some medium, or beams of particles that are bombarding agents). In turn, the environment can depend on the agents' states, e.g. neural cells can secrete neurotransmitter peptides, changing conductivity of the intercellular medium [Gonze et al., 2005, Vasalou and Henson, 2011], moving charged particles create a time-varying electromagnetic field etc. The arising clusters can be *forced* by the environmental "cues" and thus adapt to the changing environment. Vortex-wave turbulence structures in fluid, gas, multiphase and plastic streams [Meshcheryakov and Khantuleva, 2015, Khantuleva and Meshcheryakov, 2016] can also be described in terms of birth and dissolution of clusters. Although cluster synchronization dramatically reduces the dimension of a control system, this dimension (and structure of the state vector) is time-varying. One paradigm to cope with such systems is *hybrid systems* theory [Goebel et al., 2009], imposing however a number of restrictions on the system's dynamics. In statistical physics, studies on cluster dynamics in fact date to Boltzmann's works [Gabrielov et al., 2008].

Second, most of the existing results consider systems with a single time scale, whereas in reality some processes within an agent or unfolding in the whole network can be much faster than others, as exemplified by spiking (very fast) and rest (relatively slow) dynamics in neuron ensembles [Pereira et al., 2007]. One way to describe fast processes is to consider them as *pulses*; such networks can be studied by methods of impulsive systems theory [Gelig and Churilov, 1998] or hybrid systems [Goebel et al., 2009]. Most studied systems of this type are *pulse-coupled oscillator* (PCO) networks that portray behavior of biological neurons and have found many applications in electronics [Mirollo and Strogatz, 1990, Kuramoto, 1991, Proskurnikov and Cao, 2017d, Núñez et al., 2015]. Another approach is the method of singular perturbations, broadly used in physics and mechanics [Cole, 1968, Dyke, 1964]. Networks with singularly perturbed agents have not been thoroughly studied, most of them stipulate linear dynamics of the agents [Yang et al., 2017, Rejeb et al., 2018] or cope with very special type of networks, e.g. Kuramoto-like

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[3]Notice that, unlike the case of linear equations in semigroups [Nefedov and Sholokhovich, 1986], most of control problems for nonlinear equations remain, to many extent, open.

oscillator ensembles [Dörfler and Bullo, 2012]. In practice, external control can be rather slow compared to the internal processes of cluster formation and deterioration. Understanding of interrelation between fast processes of clustering and slow-input controllability of the network remains an unsolved problem.

1.7 The Paper's Contribution and Organization

In this paper, we propose a new general "philosophy" of complex processes modeling, exploiting multi-agent dynamical models with variable structures[4] of clusters. The relevant mathematical concepts and brief literature survey are given in Section 2. This philosophy has been, to a great extent, inspired by a practical problem of reducing an aircraft's vibration in a turbulent flow. In Section 3, we present a novel approach to this problem, employing a special equipment ("feathers", consisting of miniature sensors, computers and actuators) installed on the wing surface and used to equalize the air pressure. Section 4 concludes the paper, discussing some open problems and directions for future research.

2 Clusters Formation and Related Problems

As has been already mentioned, many complex phenomena studied in natural, engineering and social sciences can be described as emergence and deterioration of clusters in complex dynamical networks. The clusters can emerge due to some endogenous structural and/or dynamical properties of the system (that is, the topology and structure of coupling), but can be also influenced by external factors (including e.g. control inputs). As a result of fast external processes, the clusters deteriorate, and the agents reassemble into a different configuration of clusters. Supposing that the processes of clusters' formation and destruction are sufficiently fast, so that the configuration of clusters remains stable for sufficiently long time, one can replace each cluster by a single "averaged" system and thus dramatically reduce the state space of the overall complex systems. This general program decomposes into a number of difficult problems, in particular:

1. why and how do the agents split into clusters?
2. how to predict the structure and number of clusters, given their previous configuration?
3. how fast is the cluster formation process?
4. how to control a cluster in a non-destructive way, preserving consensus of constituent agents?
5. which properties of the original complex system are inherited by its reduced-order model, obtained by "gluing together" agents from the same cluster and treating them as a single "macro-agent"?

The existing literature on cluster synchronization mainly focuses on the first two problems, which, how-

ever, are also far from being thoroughly studied. Problems 3 and 4 are mainly considered for special situations (e.g. linear dynamics of the agents). Problem 5 relates to the advanced theory of *singularly perturbed* systems, studying coupled dynamics on different time scales (dynamics of the cluster restructuring are very fast, compared to periods of the clusters' co-existence).

2.1 Multi-Agent Networks

Henceforth we confine ourselves to the special situation, where the system is constituted by a finite[5] (yet large) number of agents $N \gg 1$, indexed 1 through N; we use $\mathcal{N} = \{1, \ldots, N\}$ to denote the set of agents. Agent i is described by the differential equations

$$\dot{x}_i(t) = f_i(x_i(t), \theta_i(t), u_i(t), U_i(t)), \quad t \geq 0,$$
$$y_i(t) = g_i(x_i(t), \theta_i(t)), \; z_i(t) = h_i(x_i(t), \theta_i(t)). \tag{1}$$

Here $x_i(t) \in \mathbb{R}^{n_i}$ stands for the state vector of agent $i \in \mathcal{N}$. The joint state vector of the system $X(t) = [x_1^\top, \ldots, x_N^\top]^\top \in \mathcal{X} = \mathbb{R}^{\bar{n}}$ thus belongs to the space of very high dimension $\bar{n} = \sum_{i \in \mathcal{N}} n_i$. Each agent can depend on the (finite- or infinite-dimensional) uncertain vector $\theta_i \in \Theta_i$, generally, time-varying (in some situations, it is convenient to consider θ_i as a stochastic variable). The components of this vector can represent parameters of the agent or external signals, which can stand for e.g. reference trajectories or disturbances. The functions f_i and θ_i should, of course, be "good" enough to guarantee a solution's existence; both of them, however, may become discontinuous due to rapid changes in the structure of the agent and its environment. The relevant existence theorems can be found in classical monographs on functional equations and differential inclusions [Coddington and Levinson, 1955, Hale, 1977, Gelig et al., 2004].

Along with the continuous-time model (1), discrete-time agents may be considered whose states evolve as

$$x_i(t+1) = f_i(x_i(t), \theta_i(t), u_i(t), U_i(t)). \tag{2}$$

Continuous-time dynamics naturally describe physical processes, and discrete-time equations arise in models in digital devices and software components. Many "cyberphysical" systems arising in engineering include both physical and digital components, and hence are described by *impulsive* [Gelig and Churilov, 1998] or *hybrid* models [Goebel et al., 2009], where some state components evolve in continuous-time as in (1), and

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[4]The theory of cluster dynamics should not be confused with variable-structure systems (VSS) theory, focusing on sliding-mode control of uncertain systems [Emelyanov, 1967, Utkin, 1992].

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[5]The concepts related to synchrony and clustering can be introduced to continuum agents, however, the dynamics of such agents are infinite-dimensional, since the relevant models deal with evolution of probability measures or densities. To keep things simple, we confine ourselves to finite-dimensional dynamics.

the others obey recurrent equations as in (2). Unless otherwise stated, we consider continuous-time agents (1).

Each agent has two control inputs, which we refer to as "fast" and "slow" controls respectively. The first of them, $u_i(t)$ is used for *coupling* between the agents that provides global consensus or cluster synchronization (such algorithms will be considered below in detail). The second input $U_i(t)$ is used to control the agent. It can be e.g. identical for each agent in a cluster, treating thus clusters as "macro-agents", and depend on some averaged characteristics such as the average of agents in a cluster (as the number of agents $N \to \infty$, such an average can be approximated by an integral over some domain). Such an example will be considered below in Subsection 2.5. Another paradigm (known as "pinning control") is to influence a single agent or few agents in each cluster, in this situation, $U_i(t) = 0$ for most agents except for a few "pinned" leaders.

Notice that in general the agents may be not only heterogeneous, but even have state vectors of different dimensions (more generally, belonging to different vector spaces). To describe clustering of agents, some "distance" between the agents has to be introduced. For this reason, the output $z_i(t)$ is introduced, containing some characteristics of interest. We always assume that z_i have same dimension $\dim z_i(t) = \dim z_j(t) \, \forall i, j$, which allows to measure distances between these outputs. In examples considered below, usually $z_i = x_i$.

Besides z_i, the agent has another output y_i that is used for the interactions, or coupling, among the agents. At each time t, the set $\mathcal{N}^i(t) \subseteq \mathcal{N}$ is defined (possibly, empty), whose members we refer to as *neighbors* of agent i. Decision of agent i at time t is based on the outputs $y_j(t)$, "displayed" by the neighbors.[6] The set of outputs $\{y_j(t)\}_{j \in \mathcal{N}^i}$ can be measured by the agent (e.g. a mobile robot can detect positions and velocities of its neighbors) or communicated by the neighbors. As a result of interaction with neighbors, each agent takes a decision or computes the control command

$$u_i(t) = \mathcal{F}_i(Y^i(t)), \quad Y^i(t) = (y_j(t))_{j \in \mathcal{N}^i(t)}. \quad (3)$$

Here $\mathcal{F}_i$, $i \in N$ are functions, defined on appropriate spaces. The coupling algorithm (3) (also called *protocol*) is *distributed* in the sense that each agent uses only "local" information $Y^i(t)$ from its neighbors. Notice that $\mathcal{F}_i$ can be random functions; also, in the case of constant sets $\mathcal{N}^i$ one can consider more general functionals, depending on the whole trajectory $\{Y(s) : 0 \leq s \leq t\}$. Notice that the coupling (3) can stand for some physical connection between the agents. For instance, agents can represent mechanical

oscillators (pendulums or vibrating using), installed on the same platform [Blekhman, 1988b].

In many situations, it is convenient to depict the relations $j \in N^i(t)$ (agent j influences agent i at time t) by directed *arcs* $i \to j$. A collection of such arcs $\mathcal{E}(t)$ along with the set of nodes $\mathcal{N}$ constitutes the directed interaction graph $\mathcal{G}(t) = (\mathcal{N}, \mathcal{E}(t))$, referred also to as the *topology* of the network (1),(3).

Definition 1. *The triple consisting of 1) family of agents* (1)*; 2) interaction graph* $\mathcal{G}(t)$*; and 3) coupling protocol* (3) *is called **multi-agent network**.*

For the subsequent definition, it is convenient to introduce the deviations between the outputs

$$\Delta_{ij}(t_*) \triangleq \|z_i(t_*) - z_j(t_*)\|, \; \Delta_{ij}(\infty) = \varlimsup_{t_* \to \infty} \Delta_{ij}(t_*).$$

Here $\| \cdot \|$ stands for some predefined norm in the space of outputs. On finite-dimensional spaces, all norms are equivalent, however, from technical viewpoint computation of some norm may appear simpler. Most typically, $\|z\|$ stands for one of the norms $|z|_1 = \sum_i |z_i|$, $|z|_\infty = \max_i |z_i|$ or $|z| = |z|_2 = \sqrt{z^\top z}$.

Definition 2. *We say that agents i and j are (output) synchronized, or reach (output) consensus at time t_* if $\Delta_{ij}(t_*) = 0$. Similarly, agents i and j are asymptotically (output) synchronized if $\Delta_{ij}(\infty) = 0$. More generally, the agents are ε-synchronized at time t_* (respectively, asymptotically ε-synchronized) if $\Delta_{ij}(t_*) \leq \varepsilon$, (respectively, if $\Delta_{ij}(\infty) \leq \varepsilon$).*

A subset of agents $M \subseteq \mathcal{N}$ is synchronized *(at time t_* or asymptotically) if any pair $i, j \in M$ is synchronized in this sense. If this holds for all agents ($M = \mathcal{N}$), the system is said to be globally synchronized. The same definitions apply to ε-synchronization.*

It should be noticed that synchronization from Definition 2 is a special case of *coordinate* synchronization. In this paper, we do not consider general concept of synchronization in dynamic systems; the interested readers are referred to monographs and surveys [Blekhman, 1988b, Blekhman et al., 1997, Fradkov, 2017, Proskurnikov and Fradkov, 2016]. Notice that synchronization of two agents at time t_*, in general, does not imply synchronization at any $t \geq t_*$: as the system evolve, synchronization can be destroyed. However, usually algorithms used to reach consensus provide the invariance of *global* synchronization: if $z_i(t_*) = z_j(t_*)$ holds *for any* i, j, this relation retains its validity for $t \geq t_*$ (more formally, it holds until the system is "singularly" disturbed, see below).

Definition 3. *Assume that $\mathcal{N} = M_1 \cup \ldots \cup M_k$, where $M_l \neq \emptyset$ are disjoint. We say that $\{M_s\}_{s=1}^k$ is (ε, δ)-*

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[6]For the reasons of simplicity, we do not consider numerous communication issues such as e.g. delays, dropouts and asynchronous messaging between the agents.

clustering of the agents at time t_* if

$$\Delta_{ij}(t_*) \le \varepsilon \ \forall i,j \in M_s, s = 1, \dots, k$$
$$\Delta_{ij}(t_*) > \delta \ \forall i \in M_l \ \forall j \in M_m, \ l \ne m. \quad (4)$$

Similarly, $\{M_s\}_{s=1}^k$ is the asymptotical (ε, δ)-clustering of the agents if

$$\Delta_{ij}(\infty) \le \varepsilon \ \ \forall i,j \in M_s, s = 1, \dots, k$$
$$\varliminf_{t_* \to \infty} \|z_i(t_*) - z_j(t_*)\| > \delta \quad (5)$$
$$\forall i \in M_l \ \forall j \in M_m, \ l \ne m.$$

Some clusters M_j may consist of only one agent. A $(0,0)$-clustering is henceforth referred to simply as clustering (or cluster synchronization) of the agents.

Notice that in the second inequality of (5) we use the *lower* limit rather than the upper one, thus, for t_* being large, agents from different clusters are δ-separated *for any $t \ge t_*$*. Obviously, ε-synchronization is a special case of (ε, δ)-clustering with the only cluster $M_1 = \mathcal{N}$ (in this case, $\delta \ge 0$ can be arbitrary). The most interesting situation of (ε, δ)-clustering is the case where, first, $\delta \ge \varepsilon$ (the inter-cluster distance is greater than the clusters' diameters) and, second, the number of clusters k is relatively small compared to $N = |\mathcal{N}|$.

Effects of full or cluster synchronization (as well as other "spontaneous order" phenomena that are beyond the scope of this paper) among the agents are explained by presence of *couplings* among them. A natural question arises how a network synchronizes or splits into clusters. This problem has attracted enormous attention of different research communities in the recent decades. In spite of a substantial progress, the behaviors of quite simple models of dynamic networks (such as e.g. Kuramoto oscillators [Dörfler and Bullo, 2014a] or models of opinion dynamics [Proskurnikov and Tempo, 2018]) remain, in many extent, mysterious. In the next subsections, we consider several most studied situations.

2.2 Consensus via Iterative Averaging

We start with consensus algorithms, based on the principle of iterative averaging (similar dynamics appear in the literature under different names, e.g. *local voting* [Amelina et al., 2015, Vergados et al., 2018], *Laplacian flows* [Bullo, 2018, Proskurnikov and Cao, 2017a] or *rendezvous* algorithm [Lin et al., 2007a]). Historically, such algorithms originate from mathematical sociology and rational choice theory [French Jr., 1956, French and Raven, 1959, Friedkin, 1986, DeGroot, 1974, Abelson, 1964, Abelson, 1967].

2.2.1 The French-DeGroot and Abelson Models

One of the first models, explaining how consensus among the agents can arise, was suggested by French [French Jr., 1956]. Consider a group of N social actors, whose opinions at the stage $t = 0, 1, 2, \dots$ are expressed by some scalar values $x_1(t), \dots, x_N(t)$. The pattern of interpersonal relations is described by a graph $G = (V, E)$, whose nodes are in one-to-one correspondence with the actors and each arc corresponds to direct communication between the actor at the head of the arc and that at its tail. At each step agents simultaneously calculate the means of their own opinions with opinions, communicated by their neighbors. Introducing the adjacency matrix $A = (a_{ij})$ of the graph, the dynamics of the ith opinion is given by

$$x_i(t+1) = \frac{\sum_{j=1}^N a_{ij} x_j(t)}{\sum_{j=1}^N a_{ij}}, \quad i \in 1:N; \ t = 0, 1, \dots$$
$$(6)$$

This model was investigated by F. Harary [Harary, 1959], who established the first graph-theoretic criterion of asymptotic consensus ("unanimity") which means that all of the opinions $x_i(t)$ converge to the same limit[7]: $x_i(t) \to x^0 = x^0(x(0))$ as $t \to \infty$ for all $i \in 1:N$ or, equivalently,

$$x(t) \xrightarrow[t\to\infty]{} x_* = \mathbb{1}_N x^0, \quad \mathbb{1}_N \triangleq \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^N. \quad (7)$$

The iteration (6) is a special case of more general dynamics, referred to as linear *opinion pooling* [DeGroot, 1974]. DeGroot's model involves a *row-stochastic*[8] matrix $W = (w_{ij})$ and is given by

$$x_i(t+1) = \sum_{j=1}^N w_{ij} x_j(t) \Leftrightarrow X(t+1) = WX(t). \quad (8)$$

Here $X(t) = [x_1(t), \dots, x_N(t)]^\top$ stands for the stack vector of opinions. Establishing of consensus among the opinions $x_j(t)$ under any initial conditions $X(0)$ means that the limit $W_* = \lim_{t\to\infty} W^t$ exists, and $W_* =$

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[7]Consensus in this sense may seem a more restrictive condition than asymptotic synchrony from Definition 2, but in fact the convergence of the opinions to the same limit (7) and vanishing of their deviations appear to be equivalent in the case of linear algorithms. The corresponding fact remains valid for time-varying matrices $W(t)$ and is known as the equivalence of weak and strong ergodicity of Markov chains (or infinite matrix products) [Seneta, 1981].

[8]A square matrix $W = (w_{ij})$ is row-stochastic, if all of its entries are non-negative $w_{ij} \ge 0$ and the sum in each row $\sum_{j=1} w_{ij} = 1$. Any such matrix is the transition matrix of a stationary Markov chain [Norris, 1998], where w_{ij} is a probability of transition from the state i to state j.

$(1, 1, \ldots, 1)^\top \pi_*^\top$, where $\pi_*^\top = (\pi_1, \ldots, \pi_N)$ is a non-negative row vector. Obviously, $W_* W = W_*$, and hence $\pi_*^\top W = \pi_*$. Thus π_* is the Perron-Frobenius left eigenvector of W, corresponding to the maximal eigenvalue 1. Since W_* is row-stochastic, one also has $\sum_i \pi_i = 1$. It can be easily shown that π_* is the only eigenvector with such properties. Notice also that the consensus opinion, on which the agents converge, is

$$x^0 = \pi_*^\top x(0). \tag{9}$$

Conditions ensuring the existence of vector π_* have been thoroughly investigated in the theory of Markov chains; the aforementioned consensus property is equivalent to the *ergodicity* of the Markov chain with the transition matrix W, whose unique stationary distribution is π_*. Starting at any initial distribution $\pi^0 = (\pi_1^0, \ldots, \pi_N^0)$, the chain converges to the *stationary distribution* $\pi_* = \pi^0 W_* = \lim_{t \to \infty} \pi^0 W^t$. A necessary and sufficient condition on W, guaranteeing the ergodicity, is being a *SIA* matrix (stochastic, indecomposable, aperiodic) [Ren and Beard, 2008]. From the algebraic viewpoint, this means that all eigenvalues of W except for $\lambda_1 = 1$ are strictly stable ($|\lambda_j| < 1$, $j = 2, \ldots, N$), and λ_1 has multiplicity 1 (such matrices are also called *fully regular* [Gantmacher, 2000].) In terms of the Markov chain, this means that all recurrent states [Norris, 1998] of the Markov chain communicate to each other and aperiodic. A more convenient graphical condition can be obtained, introducing the interaction graph of the multi-agent system. The set of the ith agent's neighbors is $\mathcal{N}^i = \{j : w_{ji} > 0\}$ (obviously, the ith agent is affected by the jth one if and only if $w_{ij} \neq 0$). Communicating recurrent classes correspond to the only *source* strongly connected component [Harary et al., 1965], from which all other components can be reached by paths. The existence of such a component is equivalent to the existence of a directed (out-branching) *spanning tree* in a graph [Harary et al., 1965, Ren and Beard, 2008, Blondel et al., 2005, Cao et al., 2008]. Consensus is reached if and only if this source component is *aperiodic*, that is, greatest common divisor of all cycles' lengths is 1. In the simplest situation where $w_{ii} > 0 \forall i$, aperiodicity automatically holds, and the system reaches consensus if and only if the graph has a directed spanning tree [Ren and Beard, 2008, Blondel et al., 2005, Cao et al., 2008].

The mulit-agent system (8) can be easily rewritten in the general form (2), (3), for instance,

$$x_i(t+1) - x_i(t) = u_i(t), \quad y_i(t) = x_i(t)$$
$$u_i(t) = \sum_{j \neq i}^N w_{ij}(y_j(t) - y_i(t)). \tag{10}$$

Hence, the evolution of an agent's opinion is driven

by its *deviations* $y_j - y_i$ from the neighbors' outputs[9]. This coupling mechanism is often called *diffusive* [Pogromsky and Nijmeijer, 2001].

A continuous-time counterpart of (10), replacing the opinion's increment by the instantaneous "velocity", is

$$\dot{x}_i(t) = -\sum_{j=1}^N \varepsilon_{ij}(x_i(t) - x_j(t)) \Leftrightarrow$$
$$\Leftrightarrow \dot{X}(t) = -LX(t). \tag{11}$$

Here $L = (l_{ij})$ is a so-called *Laplacian* matrix [Ren and Beard, 2008, Mesbahi and Egerstedt, 2010, Olfati-Saber and Murray, 2004a], its off-diagonal entries $l_{ij} = -\varepsilon_{ij}$ are non-positive (i.e. $\varepsilon_{ij} \geq 0$) and the rows sum to zero: $l_{ii} = \sum_{j=1}^N \varepsilon_{ij}$. It is remarkable that the algorithm (11) first appeared and was studied in [Abelson, 1964], long before the recent "boom" in multi-agent and networked control. Consensus in (11) holds if and only of the interaction graph, corresponding to the adjacency matrix (ε_{ij}), has a directed spanning tree. This criterion has been formulated in [Abelson, 1964], however, the proof was incomplete. The first correct proof appeared in [Agaev and Chebotarev, 2000]. Similarly to the discrete-time case, the consensus value can be represented as (9), where $\pi_* \geq 0$ is the (unique) left eigenvector of the Laplacian matrix L, such that $\pi_*^\top L = 0$, $\sum_j \pi_j = 1$. The detailed history and analysis of properties of the algorithms (8),(11) is available in [Proskurnikov and Tempo, 2017].

The consensus algorithm (11) is in fact a special case of *speed gradient* (or gradient descent) dynamics [Fradkov et al., 1999, Fradkov, 2005, Khantuleva and Shalymov, 2017]. Consider the "cost function" that penalizes disagreement between the agents

$$Q(x) = \frac{1}{2} \sum_{i,j=1}^N \varepsilon_{ij} |x_i - x_j|^2. \tag{12}$$

A straightforward computation shows [Olfati-Saber and Murray, 2004b, Olfati-Saber et al., 2007b] that (41) is nothing else than the conventional speed-gradient algorithm

$$\dot{x}_i = -\frac{\partial Q}{\partial x_i}, \quad i = 1, \ldots, N.$$

In other words, each agent is constantly moving in the

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[9]The agent's equation could also be written as $x_i(t+1) = u_i(t)$, from control viewpoint, (10) is more natural (the state usually cannot be controlled directly) and stands for the discretized integrator.

direction of fastest descent of the cost function. Since

$$\frac{d}{dt} Q(x(t)) = -\sum_{i=1}^{N} \left(\frac{\partial Q}{\partial x_i}\right)^2, \qquad (13)$$

and the graph is connected, the set where $\dot{Q} = 0$ (equivalently, $\dot{x}_i = 0$) coincides with the synchronous manifold, i.e. the set of all $\mathbf{X}$ such that $x_i = x_j \, \forall i, j$.

An important property of the algorithms (8) and (11) is the *robustness* of synchronization against external disturbances. For instance, the algorithm

$$\dot{x}(t) = -Lx(t) + \theta(t), \qquad (14)$$

where $\theta(t) \in \mathbb{R}^N$ is the vector of *bounded* disturbances, provides ε-synchronization with the accuracy ε proportional to $\sup_{t \geq 0} \max_{i,j} |\theta_i(t) - \theta_j(t)|$. In the case of vanishing at infinity $\theta(\cdot)$, asymptotic synchronization is preserved. Mathematical results on consensus robustness for continuous-time and discrete-time cases are available in [Shi and Johansson, 2013a]. Besides this, consensus algorithms are robust against *delays* in measurements [Münz et al., 2010, Münz et al., 2011, Tian and Liu, 2008, Blondel et al., 2005, Proskurnikov, 2012, Shi and Johansson, 2013a, Amelina et al., 2015, Seuret et al., 2008, Proskurnikov and Matveev, 2018].

Another important property of the consensus protocol has been established in [Fu et al., 2017b]. Assuming that the initial vector $x(0)$ is a continuous random variable with a finite entropy $h(x(0))$, the same holds for any $x(t)$ (also being random). Furthermore, the entropy is non-increasing along the trajectories, in particular, $h(x(t)) \leq h(x(0))$. Another relation between the entropy and consensus will be discussed in Section 2.3.

2.2.2 Convergence Rate and Time

A natural question arises how to measure the convergence speed of algorithms (8),(11). Algebraically, this speed depends on the eigenvalues of matrix W (respectively, Laplacian L). Denoting the vector of consensus opinions $x_* = W_* x(0) = (\pi_*^\top x(0)) \mathbb{1}_N$ and recalling that $W_* \mathbb{1}_N = W \mathbb{1}_N = \mathbb{1}_N$ (the matrices W and $W_* = \lim_{t\to\infty} W^t$ are row-stochastic), one has $(W - W_*) x_* = 0$. Furthermore, $\pi_*^\top x(t+1) = \pi_*^\top W x(t) = \pi_*^\top x(t)$, that is, $\pi_*^\top x(t) \equiv \pi_*^\top x_*$ and $W_* x(t) \equiv W_* x_* = x_*$. Hence,

$$x(t+1) - x_* = (W - W_*)(x(t) - x_*), \qquad (15)$$

entailing that $|x(t) - x_*| \leq c\rho^t x(0)$, where $c > 0$ is a constant and $\rho \triangleq \rho(W - W_*)$ stands for the spectral radius of the matrix $W - W_*$. Let $\lambda_1 = 1, \lambda_2, \ldots, \lambda_N$

be the eigenvalues of W, sorted in a way that $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_N|$. Then, as can be shown, the eigenvalues of $W - W_*$ are $0, \lambda_2, \ldots, \lambda_N$, and hence $\rho = |\lambda_2|$ [Xiao and Boyd, 2004, Olshevsky and Tsitsiklis, 2011]. In other words, the *asymptotic* convergence rate depends on the second largest eigenvalue of W. Similarly, in the case of continuous-time dynamics (11), one has [Olfati-Saber et al., 2007a, Ren and Beard, 2008] $|x(t) - x_*| \leq ce^{-Re\,\lambda_2(L)t}$, where λ_2 stands for the eigenvalue of L with the second minimal real part[10].

For practical reasons, it is often more important to know the *convergence time* of the algorithm, namely, how long will it take the agents to reach ε-synchrony. Instead of the maximal deviation $\max_{i,j} |x_i - x_j|$, it is often more convenient to evaluate the deviation $\|x(t) - x_*\|$; the choice of norm here can be arbitrary, however, for technical reasons the norm $\|\cdot\| = |\cdot|_\infty$ is usually considered [Olshevsky and Tsitsiklis, 2011, Nedić and Ozdaglar, 2010]. Using (15), one has $\|x(t) - x_*\| \leq \|(W - W_*)^t\| \|x(0) - x_*\|$, and the problem is to estimate the number

$$T(r) \triangleq \min\{t : \|(W - W_*)^t\| = \|W^t - W_*\| \leq r\}.$$

Obviously, the agents reach ε-consensus in no longer than $T(\varepsilon/\|x(0) - x_*\|)$ iterations.

A question arises how to estimate ρ and $T(r)$ in a general situation[11]. The most studied is the case of bidirectional graphs ($i \in \mathcal{N}^j \Leftrightarrow j \in \mathcal{N}^i$) with self-loops ($i \in \mathcal{N}^i$), where each agent assigns equal influence weights to its neighbors. $w_{ij} = \frac{1}{|\mathcal{N}^i|} \forall j \in \mathcal{N}^i$. The result by [Olshevsky and Tsitsiklis, 2011] states that

$$\rho \leq 1 - \gamma N^{-3},$$

in such a situation, where $\gamma > 0$ is a constant, independent of N or any specific graph. Moreover, this asymptotical estimate cannot be improved: there exist graphs, on which $\rho \geq 1 - \tilde{\gamma} N^{-3}$. Using this result, it can be shown [Olshevsky and Tsitsiklis, 2011] that

$$T(r) = O\left(N^3 \log(N/r)\right),$$

however, the tightness of this estimate remains an open problem (it is known, however, that $T(r) = \Omega(N^3 \log r)$ for an appropriate graph with N nodes.

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[10]In can be shown that if the graph has a directed spanning tree, then $\lambda_1 = 0$ is the only eigenvalue on the imaginary axis and $Re\,\lambda_j > 0$ for $j = 2, \ldots, N$. Without loss of generality, one may assume that $0 = Re\,\lambda_1 < Re\,\lambda_2 \leq \ldots \leq Re\,\lambda_N$.

[11]It can be noticed that for a general Laplacian matrix L and sufficiently small $\alpha > 0$, the matrix $I - \alpha L$ is row-stochastic, which reduces the estimation of the continuous-time system's convergence rate of to the discrete-time case [Xiao and Boyd, 2004].

The problem of convergence rate's estimation becomes even more challenging for randomized algorithms, where either matrix $W(t)$ is random (as e.g. in gossiping algorithms, surveyed in [Proskurnikov and Tempo, 2018]) or the system is disturbed by random noises. In such situations, synchronization is often understood in mean-square sense; however, one may also try to estimate (random) time of reaching ε-consensus, provided that disturbances are sufficiently small compared to ε. The relevant results can be found in [Amelina et al., 2015, Vergados et al., 2018, Granichin and Khantuleva, 2017].

2.2.3 Extensions and Applications

The properties of SIA matrices [Cao et al., 2008, Jadbabaie et al., 2003, Ren and Beard, 2008] and Lyapunov-based arguments [Moreau, 2005, Blondel et al., 2005, Münz et al., 2011] allow to obtain consensus criteria under *non-stationary* interactions among the agents, where the row-stochastic matrix $W = W(t)$ in (8) or the Laplacian matrix $L = L(t)$ in (11) evolves; the relevant results can be found in [Ren and Beard, 2008, Proskurnikov and Tempo, 2018]. These criteria require the existence of spanning tree in the union of interaction graphs over sufficiently long period and in general are only *sufficient* for reaching agreement. Necessary and sufficient conditions are known only for special types of interaction graphs [Matveev et al., 2013, Hendrickx and Tsitsiklis, 2013, Shi and Johansson, 2013b].

It should be noticed that dynamics similar to (8) and (11) naturally arise in many applications. Consensus algorithms (called also *protocols*) similar to (8) were independently introduced by J. Tsitsiklis [D.Bertsekas and Tsitsiklis, 1989, Tsitsiklis, 1984] in some distributed algorithms for multi-processor clusters. Similar dynamics were suggested for modeling of alignment in ensembles of moving particles [Vicsek et al., 1995, Helbing, 2001], that can be considered as a deterministic analogue of phase transitions in thermodynamics. First-order protocols (11) (with delayed couplings) arise in *microscopic* traffic flow models [Helbing, 2001, Chandler et al., 1958, Michiels et al., 2009a, Sipahi et al., 2007, Proskurnikov, 2013], where the graph has a structure of chain or cycle.

2.3 From Consensus to Clustering

While the phenomenon of consensus has been studied up to certain exhaustiveness, opinions of social actors often do not reach any agreement but rather form highly irregular fractions or *clusters* of different sizes. Along with the consensus problem, the problem of describing clustering mechanisms has arisen in mathematical sociology [Abelson, 1967], where

it is now referred to as the *community cleavage* problem [Friedkin, 2015] or Abelson's *diversity puzzle* [Kurahashi-Nakamura et al., 2016]. Since convex-combination mechanisms (8),(11) have been widely adopted in opinion dynamics modeling, a question has arisen which factors lead to clustering of opinions under such a protocol. The proposed explanations can be (roughly) divided into several major classes.

2.3.1 Clustering Due to Lack of Connectivity

The most obvious explanation for absence of consensus is the absence of connectivity in the network, e.g. if the graph is composed of two or more disjoint components (equivalently, W is a block-diagonal matrix), then the opinions in these components may become disparate. Less obvious is the situation where the graph is weakly connected yet does not have a directed spanning tree, that is, its condensation [Harary et al., 1965] has several source components, and any other component is reachable from them (equivalently, the graph is covered by a *spanning forest* of several trees).

As a simple example, one may consider the system (8) with several *stubborn* agents. Agent i is stubborn if $w_{ii} = 1$, whereas $w_{ij} = 0 \, \forall j \neq i$. Such an agent is anchored at its initial opinion $x_i(t) \equiv x_i(0)$, being insusceptible to social influence; at the same time, stubborn agents can influence the others.

In presence of several stubborn agents with heterogeneous opinions, the group does not reach consensus, but rather demonstrates clustering behavior. Typically, the opinions of normal ("open-minded") agents get into the convex hull, spanned by the stubborn agents' opinions. This *containment* property of consensus dynamics is explored in problems of *containment control* with multiple static or dynamic leaders [Ren and Cao, 2011, Proskurnikov and Tempo, 2017]. Notice, however, that there is no simple relation between the number of stubborn agents and number of clusters. The structure of clusters, formed by consensus-like algorithms over weakly connected graphs, depends on the matrix of spanning forests, whose computation for large-scale graph is challenging [Chebotarev and Agaev, 2002, Chebotarev and Agaev, 2014].

2.3.2 External Inputs and Information

The idea of stubborn agents can be further extended, replacing the (static) radical opinions by additional external inputs, static or dynamic, that influence some of the agents. Consider, for instance, the system (14), where $\theta_i(t) = 0$ for all agents except for $i \in I_0$, where I_0 is a set of several "informed" leaders, whose inputs $\{\theta_j(t) : j \in I_0\}$, are mutually different. One can expect that such a group also exhibits cluster synchronization; the relevant conditions have been found in [Smet and Aeyels, 2009, Xia and Cao, 2011, Aeyels and Smet, 2011]. A similar idea lies in the heart of the so-called Friedkin-Johnsen

(FJ) model [Friedkin and Johnsen, 1999], squarely based on the DeGroot dynamics (8),

$$x(t+1) = AWx(t) + (I - A)u. \quad (16)$$

Similar to (8), here W is a row-stochastic matrix, A is a non-negative *diagonal* matrix, $0 \leq a_{ii} \leq 1$, and u is a constant vector. In the original model from [Friedkin and Johnsen, 1999], $u_i = x_i(0)$, and the diagonal entry a_{ii} is treated as an agent's *susceptibility* to opinion assimilation. If $a_{ii} = 0$, the agent is totally stubborn $x_i(t) \equiv x_i(0)$. A maximally susceptible agent with $a_{ii} = 1$ assimilates opinions of the other agents in the same way as in (8)

$$x_i(t+1) = \sum_j w_{ij} x_j(t).$$

An agent with $a_{ii} \in (0, 1)$ is partially "anchored" at its initial opinion u_i and factors it into every iteration of the opinion. As discussed in [Proskurnikov and Tempo, 2017], in general u_i may be considered as a "prejudice" of agent i, influencing dynamics of its opinion. In spite of its simplicity, many properties of the model (16), including graph-theoretic conditions for its stability, convergence rate and the structure of final opinions have been studied only recently [Parsegov et al., 2017, Proskurnikov and Tempo, 2017, Proskurnikov et al., 2017]. Recently, extensions of the Friedkin-Johnsen models have been proposed, allowing to describe dynamics of multidimensional opinions, representing individual positions on several logically related topics [Parsegov et al., 2017].

2.3.3 Balance and Negative Weights

An elegant extension of the consensus algorithms (8) and (11), allowing the agents' splitting into two clusters with opposite opinions ("polarization") is known as the Altafini model [Altafini, 2012, Altafini, 2013, Liu et al., 2017, Xia et al., 2016]. The discrete-time Altafini model is coincident to (8) with the only difference is that W may have negative off-diagonal entries w_{ij}, $i \neq j$, however, the diagonal weights are non-negative and the matrix of *absolute values* $(|w_{ij}|)_{i,j}$ is row-stochastic. The continuous-time Altafini model is coincident to (11), however, L is a *signed Laplacian* matrix, which means that l_{ij}, $i \neq j$, may be positive or negative and $l_{ii} = \sum_j |l_{ij}|$. Note that the Altafini model steps away from the convex combination mechanism, in particular, the convex hull of the opinions can expand.

In the case of non-negative coefficients (respectively, w_{ij} or $\varepsilon_{ij} = (-l_{ij})$), the Altafini model reduces to the usual consensus algorithm. Otherwise, the positive sign of a coupling gains can be considered as "trust" or "friendship" between the corresponding

agents, whereas the negative gain means "distrust" or "enmity". If the graph has a directed spanning tree and is *structurally balanced* [Harary, 1953], i.e. agents split into two "hostile camps" such that members of the same camp are "friends" ($w_{ij} \geq 0$), and the members of different camps are "enemies" ($w_{ij} \leq 0$), then the Altafini model provides polarization of the opinions: members of the two camps agree on the opposite values x^0 and $-x^0$, where x^0 depends on the initial conditions and the graph's structure. If these conditions do not hold, the model can provide a more complicated type of clustering, called "interval bipartite consensus" [Meng et al., 2016]. A degenerate situation is also possible, where the system is exponentially stable and all opinions converge to 0.

For the detailed analysis of Altafini models over static and time-varying graphs the reader is referred to [Proskurnikov et al., 2014, Proskurnikov and Cao, 2014, Meng et al., 2016, Xia et al., 2016, Proskurnikov and Cao, 2017a, Proskurnikov and Cao, 2017b, Liu et al., 2017].

2.3.4 Nonlinear Couplings

It can be noticed that the aforementioned models of clustering stipulate some *heterogeneity* of the agents (e.g. presence of heterogeneous disturbances, different levels of attachment to the initial opinions) or their couplings (e.g. positive/negative). Also, the number of clusters in such models is usually very large (in general, each agent can form its own cluster), except for the Altafini model, which can lead to polarized opinions (with only two clusters). It appears, however, that the cluster synchronization (with arbitrary number of clusters, which can be greater than two but much smaller than the number of agents) can be provided by special structure of *couplings* among the agents. Unlike (8) and (11), the relevant models of clustering are essentially *nonlinear*. The nonlinearity of couplings is, however, is not sufficient for clustering: consensus is guaranteed by many nonlinear algorithms, preserving the convex hull [Moreau, 2005, Lin et al., 2007b, Matveev et al., 2013, Proskurnikov, 2013].

A number of clustering models can be written as

$$\dot{x}_i = \sum_{j=1}^{N} \varepsilon_{ij} \varphi(x_j, x_i), \quad (17)$$

where $\varphi(\cdot)$ is a (nonlinear) coupling maps and $\varepsilon_{ij} \geq 0$ are coupling gains, encoding the interaction graph. In the case where $\varphi(\xi, \zeta)(\xi - \zeta) > 0$ for any $\xi \neq \zeta$ and φ is continuous, this protocol guarantees consensus [Lin et al., 2007b, Papachristodoulou et al., 2010, Matveev et al., 2013]. It is remarkable that algorithm (17) naturally arises in some models of statistical mechanics, e.g. considering x as a discrete probability distribution ($\sum_i x_i = 1$) and as-

suming that $\varphi(x_i, x_j) = \log(x_j/x_i) = \log x_j - \log x_i$ and $\varepsilon_{ij} = \varepsilon_{ji}$, one obtains the speed-gradient dynamics maximizing the conventional Shannon entropy $H(x) = \sum_i x_i \log x_i$ [Fradkov et al., 2016]. Consensus at the uniform distribution corresponds to the maximum of the entropy. Remarkably, similar entropy-maximizing (MAXENT) algorithms for other types of entropies arising in statistical physics lead to non-uniform distributions, corresponding to clustering behavior [Khantuleva and Shalymov, 2018, Shalymov and Fradkov, 2018].

One way to provide clustering is to require clustering is to assume that $\varphi(x) = 0$ for $|x| \geq R$, which is in harmony with effects of *homophily* and *biased assimilation* [Dandekar et al., 2013] in social dynamics: a social actor readily accepts opinions of like-minded individuals (agents j such that $|x_j(t) - x_i(t)| < R$), examining the deviant opinions with discretion or rejecting them. The discrete-time counterpart of this model was first proposed in [Krause, 2000] and is now referred to as the Hegselmann-Krause model [Hegselmann and Krause, 2002]

$$x_i(t+1) = \frac{1}{|\mathcal{N}^i(x(t))|} \sum_{j \in \mathcal{N}^i(x(t))} x_j(t),$$

$$\mathcal{N}^i(x) \triangleq \{j : |x_j - x_i| < R\}. \tag{18}$$

The model (18) is a modification of DeGroot's dynamics (8), where the graph is *state-dependent* and evolves together with the agents' opinions (such networks are called *co-evolutionary* or *coevolving*).

Bounded confidence models (in continuous and discrete time) have been extensively studied in the literature [Hegselmann and Krause, 2002, Deffuant et al., 2001, Lorenz, 2007, Blondel et al., 2009, Blondel et al., 2010, Ceragioli and Frasca, 2012, Motsch and Tadmor, 2013, Jabin and Motsch, 2014, Frasca et al., 2016], however, their nonlinear structure makes them very difficult for analysis; in particular, the results. Some results on stability analysis of fixed points in the state space (standing for possible configurations of opinion clusters) are given in [Blondel et al., 2009, Blondel et al., 2010, Ceragioli and Frasca, 2012]; it was noticed, in particular, that some trajectories can converge to unstable equilibria. A novel model of opinion dynamics with bounded confidence has been proposed in [Pilyugin and Campi, 2018]. In the latter model, agents' attitudes belong to a symmetric

interval, e.g. $[-1, 1]$, and evolve as follows

$$x_i(t+1) = \begin{cases} \tilde{x}_i(t+1), & \tilde{x}_i(t+1) \in [-1, 1] \\ -1, & \tilde{x}_i(t+1) < -1 \\ +1, & \tilde{x}_i(t+1) > 1. \end{cases},$$

$$\tilde{x}_i(t+1) = x_i(t) + \frac{h}{|\mathcal{N}^i(x(t))|} \sum_{j \in \mathcal{N}^i(x(t))} x_j.$$

Here $h > 0$ is a constant, and $\mathcal{N}^i(x)$ is the same as in (18). Two main differences with the model (18) are the opinion saturation at the extremal values and the effect of opinion *reinforcement*: if $\mathcal{N}^i(x(t)) = \{i\}$, agent i gets no counter-argument against its attitude and tends to "strengthen" it: $x_i(t) \mapsto (1 + h)x_i(t)$. It has been shown in [Pilyugin and Campi, 2018] that this model leads to polarization of opinions at the extremal values ± 1; the structure of arising clusters can be determined. Unlike the model from [Ceragioli et al., 2016] with similar properties, the model from [Pilyugin and Campi, 2018] does not employ negative couplings between the agents.

There are other models with nonlinear couplings, admitting clusters. For instance, a model proposed in [Fläche and Macy, 2011] allows *repulsion* between distant opinions, whereas close opinions attract[12]. The structure of clusters in this model still remains a challenging question; in the experiments reported in [Fläche and Macy, 2011], no more than 5 clusters are reported. Another model of cluster dynamics has been proposed in [Proskurnikov and Cao, 2016]

$$\dot{x}_i = (h'(x_i))^{[\alpha]} \sum_{j=1}^{N} a_{ij}(h(x_j) - h(x_i)),$$

$$y^{[\alpha]} \triangleq |y|^\alpha \, sgn \, y. \tag{19}$$

Here $\alpha > 0$ is a positive constant and $h(x)$ is a coupling function, controlling the structure of clusters. In the simplest situation of *linear* coupling maps $h(x) = kx$, $k \neq 0$, the model (19) becomes a special case of the consensus algorithm (11), where $l_{ij} = -|k|a_{ij}$. Consensus is the most typical outcome (under the assumption of the network's connectivity) whenever h is strictly monotone. Otherwise, the dynamics (19) allows *clustered* equilibria that can be classified [Proskurnikov and Cao, 2016] and are determined, to a great extent, by the stationary points of h (i.e. the points where $h'(x_i) = 0$). Notice that an agent starting at such a point is usually[13] stubborn: $x_i(t) \equiv x_i(0)$ in

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[12]This type of repulsion should not be confused with the aforementioned Altafini's model, where the relation between the agents depends on their relations, rather than on the distance of opinions.

[13]Formally, this requires the uniqueness of the Cauchy problem's solution; we do not consider the relevant conditions now.

view of (19). Under some conditions (e.g. the all-to-all communication case where $a_{ij} = 1\,\forall i, j$), it has been proved [Proskurnikov and Cao, 2016] that any of the solutions converges to one of the equilibria; the general case is a topic of ongoing research.

2.4 Synchronization of Agents with High-Order and Nonlinear Dynamics

In the previous subsections, we primarily considered agents with the simplest first-order dynamics $\Delta x_i(k) = u_i(k)$ or $\dot{x}_i(t) = u_i(t)$. In the recent decades, synchronization problems have been considered for agents of some other types, e.g. ensembles of phase-coupled [Kuramoto, 1984, Kuramoto, 1991, Strogatz, 2000, Arenas et al., 2008, Dörfler and Bullo, 2014b] and pulse-coupled [Mirollo and Strogatz, 1990, Kuramoto, 1991, Izhikevich, 2007, Proskurnikov and Cao, 2017d] oscillators, circadian clocks [Gonze et al., 2005, Proskurnikov and Cao, 2017c] and chaotic systems [Wu, 2007, Pecora and Carroll, 1998b, Fradkov, 2007].

In this section, we consider only a few general classes of agents, for which synchronization problems have been most studied.

2.4.1 Linear Stationary Networks
Recall that system (11) may be considered as a group of agents

$$\dot{x}_j(t) = u_j(t) \in \mathbb{R}, \qquad (20)$$

coupled via the following protocol

$$u_j(t) = \sum_{k=1}^{N} \varepsilon_{jk}(x_k(t) - x_j(t)). \qquad (21)$$

A natural extension of the first-order integrator agent (20) is a general linear time-invariant agent

$$\dot{x}_j(t) = Ax_j(t) + Bu_j(t),\ y_j(t) = Cx_j(t). \qquad (22)$$

where $x_j(t)$ stands for the *state vector* of the agent, $u_j(t) \in \mathbb{R}^m$ is its *control* or *input*, and $y_j(t) \in \mathbb{R}^l$ is some *output*. The state vectors are generally unavailable for measurement, and the only information agents can use comes to their own and neighbors' outputs, or even deviation among the outputs. Introducing joint stack vectors $X(t) = [x_1(t)^\top, \ldots, x_N(t)^\top]$, $Y(t) = [y_1(t)^\top, \ldots, y_N(t)^\top]$ and $U(t) = [u_1(t)^\top, \ldots, u_N(t)^\top]$, a natural extension of the protocol (21) is as follows [Ren and Beard, 2008,

Olfati-Saber et al., 2007a]:

$$u_j(t) = \sum_{k=1}^{N} \varepsilon_{jk} K(y_k(t) - y_j(t)) \Leftrightarrow \qquad (23)$$
$$\Leftrightarrow U(t) = -(L \otimes K)Y(t).$$

Here K is a $m \times l$-matrix and $\otimes$ stands for the Kronecker product of two matrices [Laub, 2005]: if $A = (a_{ij})$ anf B are arbitrary matrices, their product is

$$A \otimes B \triangleq \begin{pmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & & \ddots & \vdots \\ a_{m1}B & a_{n2}B & \ldots & a_{mn}B \end{pmatrix}.$$

As we have discussed, the matrix L always has a zero eigenvalue $\lambda_1 = 0$, corresponding to the eigenvector $(1, 1, \ldots, 1)^\top \in \mathbb{R}^N$. This is the only eigenvector at 0 if and only if the interaction graph of the network has a directed spanning tree. In this case, all the remaining eigenvalues of $\lambda_2, \ldots, \lambda_N$ lie in the open right half-plane of $\mathbb{C}$: $Re\,\lambda_j > 0$ [Agaev and Chebotarev, 2005].

We are interested in asymptotic synchronization of the agents' states (the synchronization outputs are $z_i = x_i$); henceforth we refer to is as synchronization among the agents (22) for brevity. The following fundamental result [Fax and Murray, 2004, Olfati-Saber et al., 2007a, Li et al., 2010] establishes criterion for such a synchronization.

Theorem 1. *The protocol* (23) *establishes asymptotic synchronization among the agents* (22) *for any* $\{x_i(0)\}_{i=1}^N$ *if and only if two conditions hold:*

1. *the interaction graph has a directed spanning tree (and hence $\lambda_1 = 0$ is a simple eigenvalue of L);*
2. *for other eigenvalues $\lambda_2, \ldots, \lambda_N$ of L the matrices $A - \lambda_j BKC$ are Hurwitz.*[14]

Theorem 1 can be extended to more sophisticated models of networks, including those with time-delays [Olfati-Saber and Murray, 2004a, Michiels et al., 2009b]. Another approach to analysis of linear time-invariant networks is the *frequency-domain* technique [Tian and Liu, 2008, Tian and Zhang, 2012, Tian, 2012, Lestas and Vinnicombe, 2010, Münz, 2010, Münz et al., 2010], operating with Fourier or Laplace transforms of the solutions $x_j(t)$. Some extensions to heterogeneous agents and time-varying topologies can be found in [Wieland et al., 2011, Ren and Cao, 2011, Trentelman et al., 2013].

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[14]The matrix P is called Hurwitz if all of its eigenvalues have negative real parts or, equivalently, the system of differential equations $\dot{z}(t) = Pz(t)$ is exponentially stable.

2.4.2 Nonstationary Networks with Static Graphs and MSF Approach Whereas Theorem 1 appeared (in a modified form) only in [Fax and Murray, 2004] (see also [Wu, 2007]), its *non-stationary* extension, based on the *Master Stability Function* (MSF) approach has been widely known in physical literature since the seminal paper [Pecora and Carroll, 1998a], offering a synchronization criterion for the agents

$$\dot{x}_i(t) = A(t)x_i(t) + B(t)u_i(t),\ y_i(t) = C(t)x_i(t), \tag{24}$$

applying a non-stationary distributed algorithm

$$u_i(t) = \sum_{j=1}^{N} \varepsilon_{ij} K(t)(y_j(t) - y_i(t)). \tag{25}$$

It should emphasized that the topology of the network here remains fixed, whereas the agents and coupling gains may evolve. The system (24),(25) arises in [Pecora and Carroll, 1998a] from linearization[15] of a *time-invariant* nonlinear network

$$\dot{x}_i(t) = f(x_i(t)) + \sum_{k=1}^{N} \varepsilon_{ij}[g(x_j(t)) - g(x_i(t))] \tag{26}$$

around a non-equilibrium solution (e.g. a cycle) $x_1(t) = \ldots = x_N(t) = x_*(t)$.

For any $z \in \mathbb{C}$ consider a system of ordinary differential equations

$$\dot{\xi}(t) = [A(t) - zB(t)K(t)C(t)]\xi(t),$$

and let $\lambda_{max}(z)$ be its maximal Lyapunov exponent. The function λ_{max} is referred to as the *master stability function* (MSF) of the system. In the stationary case $\lambda_{max}(z)$ is simply the maximal real part of the eigenvalues of the matrix $A - zBKC$ and hence $\lambda_{max}(z) < 0$ if and only if this matrix is Hurwitz. Condition 2 in Theorem 1 can be now reformulated as follows: $\lambda_{max}(\lambda_j) < 0$ for $j = 2, \ldots, N$. In this form it appears to be valid in the non-stationary case [Pecora and Carroll, 1998a]:

Theorem 2. *The protocol* (25) *establishes asymptotic synchronization among the agents* (24) *if and only if two conditions hold:*

1. *the interaction graph has a directed spanning tree (thus $\lambda_1 = 0$ is a simple eigenvalue of L);*

2. *for other eigenvalues $\lambda_2, \ldots, \lambda_N$ of L one has $\lambda_{max}(\lambda_j) < 0$.*

In the case where the coefficients are periodic, the Lyapunov exponents are negative if and only if the Floquet multipliers are less than 1 in modulus; this condition can be validated by e.g. solving the periodic Riccati equation [Yakubovich et al., 2007]. Notice that Theorem 2, applied to (26), guarantees only *local* conditions for synchronization; the estimation of the relevant basin of attraction remains a non-trivial problem.

It is remarkable that a relaxation of the conditions from Theorems 1 and 2, in some situations, leads to cluster synchronization. This happens when the graph has some *symmetries*, which means that $SL = LS$ for some permutation matrix S. Existence of symmetries enables the existence of clusters, whose structure is determined by the graph's symmetry group [Pecora et al., 2014]. These clusters are stable (locally in the case of nonlinear agents) if the conditions of Theorem 2 hold for some spectral points of the Laplacian matrix. It is remarkable that results on *global* stability of such clusters are available only in special situations [Pogromsky et al., 2002, Pogromsky, 2008].

2.4.3 Synchronization among General Nonlinear Agents In spite of significant progress in recent years, see e.g. [Arenas et al., 2008, Belykh et al., 2014b], the problem of reaching synchronization or other type of regular behavior in a general complex network where both nodes and couplings may be nonlinear, and the topology may evolve, still remains a challenge. A wide class of such networks can be described by the following mathematical model. Let the dynamics of nodes be

$$\dot{x}_j(t) = f(x_j(t), u_j(t)),\ y_j(t) = h(x_j(t)), \tag{27}$$

coupled via the following non-stationary protocol

$$u_j(t) = \sum_{k=1}^{N} \varepsilon_{jk}(t)\varphi_{jk}(y_k(t) - y_j(t)). \tag{28}$$

Here $\varphi_{jk}(\cdot)$ are some mappings (in general, nonlinear), referred to as *coupling maps* or *couplings*, and $\varepsilon_{jk}(t)$ are *coupling gains* that characterize the couplings' "strengths" and define, in particular, the interaction topology (generally time-varying and disconnected). The goal is to disclose conditions, providing asymptotic state synchronization between the agents.

As was discussed in the foregoing, for *stationary* networks ($\varepsilon_{jk} = const$) synchronization can be proved *locally*, under some technical assumptions, via the MSF approach [Pecora and Carroll, 1998a]. More subtle techniques, applicable to time-varying topologies, were proposed in [Lü and Chen, 2005, Lu et al., 2007].

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[15]In general, the asymptotic stability of a time-varying linearization does not imply the stability of a solution, this holds only under additional *regularity* assumption [Leonov and Kuznetsov, 2007]. This assumption is valid e.g. for linearization at an equilibrium or a cycle [Leonov and Kuznetsov, 2007].

These methods, however, allow to estimate neither the basin of attraction for a synchronous solution nor the convergence speed.

The results, establishing *global* synchronization in a general network (27),(28), mostly employ either *contraction* principles [DeLellis et al., 2011] or Lyapunov functions. Many result explicitly or implicitly employ "incremental" Lyapunov function of the form $\bar{V}(X) = \sum_{j,k=1}^{N} V(x_j - x_k)$, where V stands for the positive definite quadratic form or another radially unbounded function. Often one can prove the protocol (28) to satisfy an "incremental" quadratic constraint:

$$\sum_{j,k=1}^{N} F(u_j - u_k, y_j - y_k) \geq 0. \qquad (29)$$

Suppose, for instance, that the interaction graph is undirected and the couplings are anti-symmetric

$$\varepsilon_{jk} = \varepsilon_{kj} \geq 0, \; \varphi_{jk}(y) = -\varphi_{kj}(y), \qquad (30)$$

assume also that $\varphi_{jk}(y)^\top y \geq 0$. A straightforward computation yields that $\sum_j u_j = 0$ and

$$0 \leq \sum_j \sum_k \varepsilon_{jk} \varphi_{jk}(y_k - y_j)^\top (y_j - y_k) =$$
$$= 2 \sum_j u_j^\top y_j = \sum_{j,k} (u_j - u_k)^\top (y_j - y_k),$$

which is a special case of (29). More subtle costraints (29) arise when the coupling obey general sector inequalities [Proskurnikov, 2014]. Assuming that (29) holds, to construct an incremental Lyapunov function $\bar{V}(X)$, it suffices to find such a function $V(x)$ that for any solutions x_1, u_1, y_1 and x_2, u_2, y_2 of (27) one has

$$\dot{V}(x_1(t) - x_2(t)) + F(u_1(t) - u_2(t), y_1(t) - y_2(t)) \leq 0. \qquad (31)$$

This principle is applicable to many networks with linear couplings $\varphi_{jk}(y) = Ky$ [Hamadeh et al., 2012, Stan and Sepulchre, 2007, Belykh et al., 2006b, Belykh et al., 2006a, Belykh et al., 2007, Belykh et al., 2005]. The condition (31) is a special case of *incremental dissipativity* [Stan and Sepulchre, 2007] which extends the conventional Willems dissipativity [Willems, 1972] has proved to be an important tool in network analysis [Liu et al., 2011, Proskurnikov et al., 2015].

In the case of *linear* agents (22) and nonlinear couplings (28), considered in [Proskurnikov, 2014, Proskurnikov and Matveev, 2015], incremental Lyapunov functions enable to derive counterparts of the seminal circle and Popov's crite-

ria [Gelig et al., 2004]; in the discrete-time, counterparts of the Tsypkin and Jury-Lee criteria are obtained [Proskurnikov and Matveev, 2018].

Notice that up to now we have dealt with *homogeneous* (identical) agents. In the case where agents are *passive* in Willems sense, some strong synchronization criteria, applicable to heterogeneous agents as well, were obtained in [Arcak, 2007, Chopra and Spong, 2006, Proskurnikov and Mazo, 2017]. In general, heterogeneity of the agents is recognized as a potential reason for *clustering*, e.g. K different types of systems can be clustered into K synchronous groups [Lu et al., 2010, Lu et al., 2017, Du et al., 2018].

2.5 Consensus-preserving Control and Slow-Fast Dynamics in Networks

Up to now, we have considered a group of agents, reaching a consensus at some point; as follows from (9), this point depends on the initial condition and the structure of coupling matrix; the latter dependence is highly nonlinear. A natural question arises whether it is possible to control the multi-agent system (via additional inputs $U_i(t)$, as in (1)) in such a way that, first, synchrony among the agents is not destroyed and, second, the consensus value is driven towards the desired trajectory. The answer to this question is affirmative, and the construction used to solve this problem is known as the *mean-field control* [Kadanoff, 2009]. To keep things simpler, we confine ourselves to the case of *balanced* Laplacian matrix in (11), which means that $\mathbb{1}_N^\top L = 0$, and therefore (9) holds with $\pi_* = N^{-1} \mathbb{1}_N$ (average consensus) [Olfati-Saber et al., 2007a]. Notice first that an *identical* control input, added to all agents in (11), does not destroy consensus:

$$\dot{x}(t) = -Lx(t) + \mathbb{1}_N U(t) \Longrightarrow$$
$$x(t) = e^{-Lt}x(0) + \int_0^t e^{-L(t-s)} \mathbb{1}_N U(s)ds = \qquad (32)$$
$$= e^{-Lt}x(0) + \mathbb{1}_N \int_0^t U(s)ds.$$

Here we used the fact that $L\mathbb{1}_N = 0$ and therefore $e^{-Lt}\mathbb{1}_N = \mathbb{1}_N$. Since, by assumption, $e^{-Lt}x(0) \to x_*$ and $x_* = x^0 \mathbb{1}_N$, the asymptotic synchrony is preserved (furthermore, agents need the same time to reach ε-synchrony as in the uncontrolled system (11)).

Now it can be noticed that the consensus value $x^0 = N^{-1} \mathbb{1}_N x(t)$ obeys the integrator dynamics

$$\dot{x}^0(t) = U(t).$$

If one is capable to *measure* the average value $x^0(t)$ by some sensor (e.g. by organizing an opinion poll in the case of social network), it becomes possible to control

it by applying the proper control input $U(t)$ to all of the agents. For instance, one is able to drive it to a predefined value x_{des}^0 (constant or slowly changing) by using e.g. the conventional PI controller

$$U(t) = k_1(x_{des}^0 - x^0(t)) + k_2 \int_0^t (x_{des}^0 - x^0(s))ds.$$

Notice the properties of such a control algorithm:

– each agent directly interacts only with neighboring agents from $\mathcal{N}^i = \{j : l_{ij} \neq 0\}$;
– implicitly, each agent is also influenced by a (common) control input, preserving consensus;
– the external control input depends on an *averaged characteristics* of the cluster and may be considered as a "mean field", depending on all the agents;
– the resulting motion is a *superposition* of the two motions, caused respectively by the local couplings and the mean-field control;
– the "mean field" control depends only on some *averaged* characteristics;

A similar idea can be applied in a more general situation of multi-dimensional agents. Consider a group of linear agents, having a common input $U_i(t) = U(t)$

$$\dot{x}_j(t) = Ax_j(t) + Bu_j(t) + B_0U(t), \ y_j(t) = Cx_j(t). \tag{33}$$

Assume that the agents are coupled via the protocol (28), obeying symmetry conditions (30). Introducing the new variables $\tilde{x}_i(t) = x_i(t) - B_0 \int_0^t U(s)\,ds$, the system (33),(28) reduces to (22),(28), which means that if the algorithm (28) establishes consensus, this consensus is not destroyed by the mean-field control $U(t)$. Introducing the function $x^0(t) = N^{-1}\mathbb{1}_N x(t)$, which is no longer constant, and noticing that $\sum_i u_i(t) = 0$, one gets

$$\dot{x}^0(t) = Ax^0(t) + B_0U(t). \tag{34}$$

In other words, similar to the first-order case, the dynamics of the (single) cluster's *center* is *decoupled* from the "centripetal" consensus dynamics of individual agents. This separation principle opens up the perspective to control the large-scale formation of agents efficiently by controlling the formation's center $x^0(t)$, being an averaged characteristics of the multi-agent system. A similar principle can be applied to continuum agents ($N \to \infty$) under the assumption of coupling symmetry [Hendrickx and Olshevsky, 2016]; in the latter case, the average value has to be replaced by the integral over the cluster of agents. As the dynamics of an agent (and the cluster's center) become high-dimensional, partially uncertain and various constraints occur that have to be taken into account, for controlling

$U(t)$ efficient methods of advanced control theory have to be employed, such as e.g. MPC and adaptive control.

The idea of mean-field control thus appears to be a simple yet efficient technique to control the dynamics of clusters without destroying them; its practical implementation in the case of multiple clusters (discussed in the next sections), heterogeneous dynamics of agents or asymmetric coupling remains an open problem. A disadvantage of this approach is the necessity to create a "homogeneous" field, equally influencing all agents. An alternative approach is to control a single agent in the community; it can be expected that (under some conditions) the couplings between the agents make the whole group follow this dedicated "leader". The relevant algorithms are known as *leader-following* or *reference-tracking* consensus protocols. In the case of non-trivial desired trajectory of the leader one, however, either needs to estimate its derivative or use sliding-mode control. We do not consider such algorithms in this paper and refer the reader to [Ren and Beard, 2008, Lewis et al., 2014].

Another conjecture, inspired by the aforementioned decoupling property, is that general complex systems can be decoupled into low-dimensional dynamics and "slow" of a few cluster's centers (average values, or integrals of the states over the clusters) and high-dimensional and "fast" dynamics of individual agents, forming a cluster due to some coupling law. Whereas in general situation simple separation of these dynamics, similar to (34), does not seem to be possible, methods of averaging and singular perturbations could be employed, allowing to separate slow and fast processes in dynamical systems [Mitropolsky, 1967, Blekhman, 2000, Kokotovic et al., 1986, O'Malley, 1991, Cole, 1968]. It should be noticed, however, that the existing theory of singularly perturbed systems is mainly focused on two problems. The first problem, pioneered in [Tikhonov, 1948] is the convergence of the solutions of the "perturbed" system to "unperturbed" ones (such a convergence appears to be uniform only on compact intervals, which does not allow to investigate asymptotic behaviors of the perturbed systems) [Lizama and Prado, 2006, Parand and Rad, 2011]. The second problem developed in [Klimushev and Krasovskii, 1961, Khalil, 1981, Kokotovic et al., 1986] is the global asymptotic stability of singularly perturbed systems with a unique equilibrium. Behavior of other types of attractors under singular perturbations has been studied only in special situations [Fenichel, 1979, Krupa and Wechselberger, 2010, Jardon-Kojakhmetov et al., 2016]. Recently, a progress has been achieved in examining pendulumlike systems with multiple equilibria under singular perturbations; it has been shown, in particular, that "gradient-like" behavior (convergence of all solutions)

is usually preserved for small values of the parameter, as well as some characteristics of the solutions such as the number of slipped cycles [Smirnova et al., 2015b, Smirnova et al., 2015a, Smirnova et al., 2016, Smirnova and Proskurnikov, 2017]. The analysis of the aforementioned slow-fast dynamics effects in multi-agent networks remains, however, a difficult problem that is a subject of ongoing research.

3 Airplane with "Feathers" in a Turbulent Airflow

In this section, we give a detailed exposition of the problem formulated in [Granichin and Khantuleva, 2017].

3.1 Flight Stabilization in a Turbulence Region

The problem of reducing the aerodynamical forces influence force on the hull arises for an airplane that has fallen into a turbulence zone, where it is subject to rapidly changing forces and their moments that lead to potentially dangerous ride. Birds can partially control the influence of forces and moments with their feathers. Suppose we are able to construct an airplane with a large number of "feathers" that would smooth the influence of turbulent wind flows. From a traditional point of view we need a central fastest computer which performs very quickly complex computations collect based on collected data from all "feathers". In parallel with computation it feeds to all "feathers" computed control actions. This can be avoided if we take into account the possibility for self-organization between "feathers". For this purpose, at the first, we need to organize the local interaction between neighbor "feathers" by transmitting electric signals that by exchanging data regarding the forces acting on each "feather". At the second, we suppose that the "feathers" are able to turn in such a way that equalizes the forces between them induced by the turbulent flow. In the process of equating the induced forces, dynamic structures will form on the surface in the form of clusters with virtually identical values of induces forces. With time, these clusters will grow and merge with each other, reducing their number on the airplane's surface until they form a single cluster. By smoothing force fields acting on the airplane in a turbulence region, we can make its flow mode close to laminar.

Let an airplane with the total mass M is flying at a velocity $\mathbf{V}(t)$ due to the sum of forces acting on it

$$M\frac{d\mathbf{V}(t)}{dt} = \mathbf{F}_e - \mathbf{F}_d - \mathbf{F}_g + \mathbf{F}_l \qquad (35)$$

where the forces $\mathbf{F}_e, \mathbf{F}_d, \mathbf{F}_g, \mathbf{F}_l$ are engine thrust, air drag force, gravity and lift respectively. If the regime of the flight is stationary all the forces are counterbalanced $\mathbf{F}_e - \mathbf{F}_d - \mathbf{F}_g + \mathbf{F}_l = 0$, the plane is moving along the straight line at a constant velocity $\mathbf{V}_0 = col(V_0, 0, 0)$.

The gas flow over the plane considers being laminar and known for the given body's form. It means that the distribution of the forces over the body's surface is also known. When the airplane gets into the turbulent wind flow the forces distribution changes, the total balance of forces is broken, the flight becomes non-stationary and jolting

$$\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1, \; m\frac{d\mathbf{V}_1}{dt} = \mathbf{F}_1(t).$$

Besides deviations from the given trajectory due to the force $\mathbf{F}_1$ applied to the resultant force point $\mathbf{r}^1(t)$, moment of the force $\mathbf{M}_1(t) = [\mathbf{r}^1(t) \times \mathbf{F}_1(t)]$ causes irregular rotation of the plane around its center of mass changing angular momentum of the plane $\mathbf{L}_1(t) = [\mathbf{r}^1(t) \times m\mathbf{V}_1(t)]$ according to the law $\frac{d\mathbf{L}_1(t)}{dt} = \mathbf{M}_1(t)$. Hereinafter $[\cdot \times \cdot]$ denotes the vector product. This rotation leads not only to deviations from a given trajectory but also to a ride due to inertia and rapid turns around the center of mass in different directions. To a large extent the irregular rotation of the plane is connected to the effect of jolting. It is impossible to completely avoid the influence of a turbulent flow, but it is desirable to make it less chaotic.

In order to reduce jolting it would be desirable to compensate the forces acting on the plane from the turbulent flow by additional forces. To achieve this goal the special constructive elements like the bird's feathers on the body's surface may be used. Assume that the surface of an airplane wing consists of N similar elements ("feathers") $a_1, \ldots, a_N$ (see. Fig 1) Here $i \in \mathcal{N} = \{1, \ldots, N\}$ stands for the number of "feather". Denote

Figure 1. A scheme of an individual "feather".
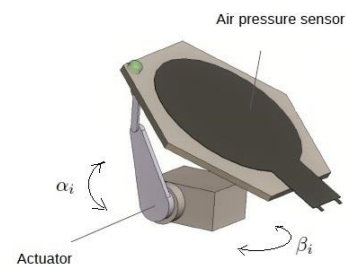
$\mathbf{r}_i$ is a vector from the center of mass of airplane to the center of "feather" a^i. Each element is equipped with a pressure sensor (force-sensitive resistor), allowing to measure pressure force y_i, and an actuator u_i with two servomechanisms, allowing to change the air pressure. Each feather can rotate in two directions, parameterized by *tilt angle* $\alpha_i \in [\alpha^-, \alpha^+]$ in the vertical plane and by *rotation angle* $\beta_i \in [\beta^-, \beta^+]$ in horizontal plane. The tilt and rotation angle determine, respectively, the vertical and cross components of the air drag force.

The integral air force acting on i-th element with area S has three components and three degrees of freedom (normal reaction of the plate and two angles α_i and rotation β_i) are needed to control the feathers

$$\mathbf{d}_i(t) = \mathbf{n}_i(t) C_d(\alpha_i(t), \beta_i(t)) S \rho (v_i(t))^2 / 2$$

where $\mathbf{n}_i(t)$ is the normal vector to i-th plate, $C_d(\alpha_i(t), \beta_i(t))$ is the aerodynamic coefficient of i-th plate defined by two angles $\alpha_i(t), \beta_i(t)$, $v_i(t)$ is the velocity magnitude of the flow over the i-th plate on the plane's surface at instant t. The force $\mathbf{d}_i(t)$ has three projections: one component along x-axis defines air resistance, vertical component determines the lift and the third is lateral yaw component. Summarizing the activities of all the feathers one gets a total force acting on the plane and changing its trajectory. For rotations of agent i we can use micro-control action $u_i(t)$

$$\begin{aligned}\dot{\alpha}_i(t) &= u_i^{(1)}(t), \\ \dot{\beta}_i(t) &= u_i^{(2)}(t).\end{aligned} \quad (36)$$

The macro control action $U_i(t)$ for agent i is defined by airplane engine power.

Suppose that airplane moves in a laminar stationary air flow from the initial instant t_0 till some instant t_1, during this time period $[t_0, t_1]$ the integral air force acting on i-th element ("feathers") is constant $\mathbf{d}_i^0$ and all "feathers" are lying on the surface: $\alpha^i = 0$ and $\beta^i = 0$ (see Fig. 2).
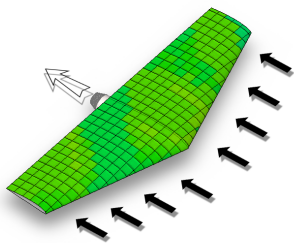

Figure 2.   The aircraft in a laminar wind flow.

Let $x_i(t)$ be the state vector of agent ("feathers") i. It consists of 11 components $x_i^{(j)}(t)$, $j = 1, 2, \ldots, 5$: 3 coordinates, 3 velocities, 3 rotations, and two angles. We assume that $y_i(t) = \|\mathbf{d}_i(t)\|$ without loss of generality. Let $z_i(t) = \mathbf{d}_i(t) - \mathbf{d}_i^0$ be the deviation of the integral air force acting on i-th element from initial vector $\mathbf{d}_i^0$ in laminar wind flow. For the forces $\mathbf{F}_e, \mathbf{F}_d, \mathbf{F}_g, \mathbf{F}_l$

we have in laminar wind flow

$$\begin{aligned}\mathbf{F}_e &= \sum_{i \in \mathcal{N}} \theta_i^{(e,0)} U_i^{(1)} + m_{N+1} U_{N+1}^{(1)}, \\ \mathbf{F}_d &= \sum_{i \in \mathcal{N}} \theta_i^{(d,0)} \mathbf{d}_i^0, \\ \mathbf{F}_g &= \sum_{i \in \mathcal{N}} \theta_i^{(g,0)} g + m_{N+1} g, \\ \mathbf{F}_l &= \sum_{i \in \mathcal{N}} \theta_i^{(l,0)} \mathbf{d}_i^0,\end{aligned} \quad (37)$$

where we introduce additional agent $N + 1$ located in the center of mass with $\mathbf{r}_{N+1} = 0$ and $m_{N+1} = M - \sum_{i \in \mathcal{N}} \theta_i^{(g)}$, and $\theta_i^{(e,0)} = \theta_i^{(g,0)} = m_i$ is an agent mass, $U_i^{(1)} = const$ is a constant engine power, coefficients $\theta_i^{(d,0)}$ and $\theta_i^{(l,0)}$ depend on the position and orientation of agent on the surface, g is gravity constant.

In the laminar wind flow, when the plane is moving along the straight line at a constant velocity V_0, dynamics of agent i is described by the differential equations

$$\begin{aligned}\dot{x}_i^{(1)}(t) &= V_0, \quad t \in [t_0, t_1], \\ \dot{x}_i^{(j)}(t) &= 0, \quad j = 2, 3, 6, \ldots, 11, \\ \dot{x}_i^{(4)}(t) &= \theta_i^{(e,0)} U_i^{(1)} - \theta_i^{(d,0)} \mathbf{d}_i^0 = 0, \\ \dot{x}_i^{(5)}(t) &= \theta_i^{(d,0)} \mathbf{d}_i^0 - \theta_i^{(g,0)} g = 0, \\ y_i(t) &= \|\mathbf{d}_i^0\|, \\ u_i(t) &= 0, \\ U_i^{(1)}(t) &= e, \\ U_i^{(j)}(t) &= 0, \quad j = 2, 3\end{aligned} \quad (38)$$

where constant e is determined by a constant engine power.

In turbulent flow "feathers" begin to rise and turn. Each feather is a plate on which the force is acting depending on its position relative to the flow direction. Fig. 3 shows the different forces (different colors) for different units of airplane in case of a turbulent wind flow when all feathers remain the initial (equal) orientations.

The forces deviations $z_i(t)$ are induced by the laminar flow small perturbations during the plates' turns and by the turbulent flow around them. We consider "intellectual feather" which are able to arrange a self-regulation in the system. We denote set $\mathcal{N}^i$ of feather i neighbors and assume that each feather i gets the information about $\mathbf{d}_j(t)$ and angles $\alpha_j(t), \beta_j(t)$ for $j \in \{i\} \cup \mathcal{N}^i$. This process should be much faster and can be repeated at any change of the turbulent flow. The system of such miniature intellectual agents has the ability to self-organization. As a result of the forces synchronization, groups of agents get approximately equal components
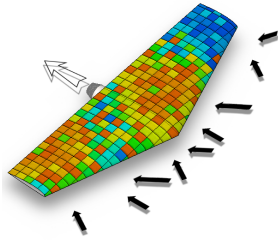
Figure 3.   The aircraft in a turbulence wind flow.

of the forces deviations acting on agents and form clusters on the surface of the plane (see Fig. 4). Such structuring is provided by the goal-directed collective behavior of the system.
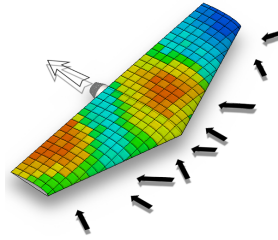

Figure 4.   The aircraft in a turbulence wind flow. Clustering of "feathers".

Suppose that with respect to the airplane's axis of motion the wings are large in the direction perpendicular to the motion and small lengthwise, while the size of the airplane hull in the perpendicular direction is also small. Under these assumptions, among disturbing forces $z_i(t)$ with amplitudes of approximately the same order of magnitude acting on the entire surface of the airplane the only vertical and lateral projections of the forces deviations are important for the problem of reducing the jolting, and the total effect of all other moments is small. For the mentioned above disturbing force $\mathbf{F}_1(t)$ we have

$$\mathbf{F}_1(t) = \sum_{i \in \mathcal{N}} z_i(t). \qquad (39)$$

We can use micro-control actions $u_i(t)$ to equalize deviation forces $z_i(t)$ if the changing of turbulent wind flow is sufficiently slow so that we have no any significant changing before instant t_2. Assume that we can do it during a small time interval τ: $t_1 + \tau < t_2$, and $z_i(t) = \mathbf{z}^1$. By virtue (39) we obtain that $\mathbf{F}_1(t) = N\mathbf{z}^1 = const$ during time interval $[t_1 + \tau, t_2]$. Hence, we have for the rotation moment that

$$\mathbf{M}_1(t) = \sum_{i \in \mathcal{N}} [\mathbf{r}_i \times z_i(t)] = N[\sum_{i \in \mathcal{N}} \mathbf{r}_i \times \mathbf{z}^1].$$

For homogeneous distribution of feathers on the symmetrical body $\sum_{i=1}^n \mathbf{r}^i = 0$ and we have $\mathbf{M}_1 = 0$. So, the most disturbing factor connected to jolting disappeared during the process of equalization of the forces acting on different feathers.

When the forces deviations induced by the turbulent wind flow are equalized the total force correction on account of the feather's work tends to $\mathbf{F}_1 = const$ and the flow over the body becomes almost laminar again until the turbulent wind will not change. Because of the small and constant force $\mathbf{F}_1$ the trajectory of the plane is slowly changing in one direction and can be corrected by the macro control U_i. More precisely, during time interval $[t_1 + \tau, t_2]$ we can use the following model of agent dynamics

$$\dot{x}_i(t) = f_i(x_i(t), \theta_i(t), u_i(t), U_i(t)),$$
$$y_i(t) = \|\mathbf{d}_i(t)\|. \qquad (40)$$

where model parameters $\theta_i(t)$ are determined by force $\mathbf{F}_1$, mass, position, orientation, and initial parameters of motion in the laminar wind flow.

3.2   Consensus Algorithm

We are interested in synchronization of the agents with respect to the outputs $z_i(t)$, $i = 1, \ldots, N$, introduced above. If it were possible to control directly the velocities $\dot{z}_i$, the vector-valued modification of the protocol (11) could be used

$$\dot{z}_i(t) = \gamma \sum_{j \in \mathcal{N}_i} b_{ij}(z_j - z_i), \quad i = 1, \ldots, N. \quad (41)$$

Here $B = (b_{ij})$ is a non-negative weighted adjacency matrix of the communication graph, $\mathcal{N}_i = \{j : b_{ij} \neq 0\}$ is the set of neighboring nodes for node i and $\gamma > 0$ is the control gain, introduced for convenience. Algorithm (41) coincides with (11), where $\varepsilon_{ij} = \gamma b_{ij}$. Henceforth we always assume that communication is bidirectional ($b_{ij} = b_{ji}$) and connected, which, as has been discussed, guarantees the exponential convergence of (41) to consensus. We have also seen that (41) in the case of undirected graph is nothing else than an algorithm of gradient descent, applied to the "energy" function

$$Q(\mathbf{Z}) = \frac{1}{2} \sum_{i,j=1}^{N} b_{ij} |z_j - z_i|^2. \qquad (42)$$

The algorithm (41), however, cannot be directly exploited in our situation since the actual control inputs are the rates $\dot{\alpha}_i$ and $\dot{\beta}_i$.

Notice that

$$\frac{\partial z_i^{(1)}}{\partial \alpha_i} = -y_i \sin(\alpha_i), \quad \frac{\partial z_i^{(1)}}{\partial \beta_i} = 0,$$

$$\frac{\partial z_i^{(2)}}{\partial \alpha_i} = y_i \cos(\alpha_i) \cos(\beta_i),$$

$$\frac{\partial z_i^{(2)}}{\partial \beta_i} = -y_i \sin(\alpha_i) \sin(\beta_i),$$

$$\frac{\partial z_i^{(1)}}{\partial y_i} = \cos(\alpha_i), \quad \frac{\partial z_i^{(2)}}{\partial y_i} = \sin(\alpha_i) \cos(\beta_i).$$

Consider now the control algorithm

$$\dot{\alpha}_i = -\left(\gamma \frac{\partial Q}{\partial z_i^{(1)}} + \frac{\partial z_i^{(1)}}{\partial y^i} \dot{y}^i \right) \left(\frac{\partial z_i^{(1)}}{\partial \alpha_i} \right)^{-1} = \quad (43)$$

$$= -\frac{\gamma}{y^i \sin(\alpha_i)} \sum_{j \in \mathcal{N}^i} b_{i,j}(z_j^{(1)} - z_i^{(1)}) + \frac{\dot{y}_i}{y_i} \cot(\alpha_i),$$

$$\dot{\beta}_i = -\left(\gamma \frac{\partial Q}{\partial z_i^{(2)}} + \frac{\partial z_i^{(2)}}{\partial \alpha_i} \dot{\alpha}_i + \frac{\partial z_i^{(2)}}{\partial y_i} \dot{y}_i \right) \left(\frac{\partial z_i^{(2)}}{\partial \beta_i} \right)^{-1} =$$
$$(44)$$

$$= -\frac{\gamma}{y_i \sin(\alpha_i) \sin(\beta_i)} \sum_{j \in \mathcal{N}^i} b_{i,j} \times$$

$$\times \left((z_j^{(2)} - z_i^{(2)}) - \cot \alpha_i \cos \beta_i (z_j^{(1)} - z_i^{(1)}) \right) +$$

$$+ \frac{\dot{y}_i}{y_i} \frac{\cot \beta_i}{(\sin \alpha_i)^2}.$$

Here $\gamma > 0$ is a control gain, and we assume that $\dot{\alpha}_i = \dot{\beta}_i = 0$ if $y_i = 0$.

A straightforward computation shows that the algorithm (43),(44) is equivalent to (41) and hence exponentially converges to consensus. More important, it is possible to estimate time needed to reach an approximate consensus in the sense that

$$|Q(\mathbf{Z}(t))| \leq \varepsilon. \quad (45)$$

The estimate involves ε, γ, initial conditions (that is, the values of $\mathbf{Z}(t_1+)$ and characteristics of the graph. The exact formulation is available in [Granichin and Khantuleva, 2017].

4 Conclusion and Future Work

The problem of determination of the constants $\theta_i(t)$) in equations (40) is not considered in this paper. This parameters remain constants during a finite time horizon. Traditional asymptotic methods cannot be used and we plan to apply for this case extended LSCR (Leave-out Sign-dominant Correlation Regions) approach [Amelin and Granichin, 2016], which was proposed earlier by M. Campi with co-authors to increase the effectiveness of adaptive control based on finite (not large) set of experimental data only [Campi et al., 2009, Campi and Weyer, 2010]. Another useful SPS (Sign-Perturbed Sums) method is proposed in [Care et al., 2017].

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