Quantum phase transitions in an effective Hamiltonian

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We study the effects of quantum phase transitions in a generic Hamiltonian describing interaction between a *fast* and a *slow* systems. We obtain and analyze the effective potential for the *slow system*. The effective Hamiltonians in the thermodynamic limit describes a bifurcation of the ground state leading to the effect of the Quantum Phase Transition in the *slow system*. We treat specific examples, atom-field and atom-atom interactions.

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I. INTRODUCTION

Frequently, in the process of interaction between two quantum systems, only one of them can be detected experimentally. In this case, a variety of physical effects appear in the process of such interaction which can be described in terms of an effective Hamiltonian corresponding to the observed system. The simplest example of such a situation arises when a *fast system* interacts with a *slow system*. Then, the *fast system* can be adiabatically eliminated and the *slow system* is described by an effective Hamiltonian. These considerations were assumed in the famous Born-Oppenheimer approximation. A regular approach to the quantum dynamics of the observed system is provided by the Lie transformation method [1, 2]. The advantage of this method consists in the possibility of varying the system's parameters, changing relations between them, which allows us to describe different physical regimes using the same mathematical tool. In particular, such an important example as expansion on the resonances in quantum systems not preserving the number of excitations can be obtained [3]. In this case a generic Hamiltonian governing interaction of two subsystems beyond the Rotating Wave Approximation (RWA) can be represented as a series in operators describing all possible transitions in the system.

Several interesting features appearing in the process of interaction of quantum systems can be realized by studying evolution of only two generic quantum system with one quantum channel. Even in such a simple case we may discriminate at least three interesting limits: a) when the interaction constant g is much higher than the characteristic frequencies of both interacting systems; b) when g is smaller than the frequencies of the systems and c) when g is higher than the frequency of one system but smaller that the frequency of the other one.

The a) case of very strong coupling should be studied carefully, because using the expansion parameter like an interaction constant over a characteristic frequency could be quite tricky. For instance, the type of the spectrum corresponding to the non-perturbed and to the perturbed systems can be different: either continuous or discrete.

The b) case corresponds to a situation where the resonance expansion is applicable. This particular case leads to dispersive-like interactions [4]. As it was shown in Ref. [3], the evolution is governed by an effective Hamiltonian describing a certain resonant interaction and the representation space of the total system can be always divided into (almost) invariant subspaces.

The last case c) possesses a peculiar property: besides of finding a corresponding effective Hamiltonian, we can also project it out to the lower energy state of the fast system, which would never get excited under given relations between the system's parameters, and thus, describe an effective dynamics of the slow system in the limit of strong interaction. It is well known that in this regime such an interesting effect as Quantum Phase Transitions may occur.

The quantum phase transitions (QPT) are a common feature of non-linear quantum systems. Such transitions occur at zero temperature and are associated with an abrupt change in the ground state structure. QPT are related to singularities in the energy spectrum and, at the critical points defining QPT, the ground state energy is a non-analytic function of the system's parameters [6]. Qualitatively, for a wide class of quantum systems, several important properties of QPT can be studied in the so-called thermodynamic (semiclassical) limit [7, 8]. Then, QPT can be analyzed in terms of a classical effective potential energy surface [9]. In this language QPT are related to the appearance of a new classical separatrix when the coupling parameters acquire certain values. According to the standard semiclassical quantization scheme and the correspondence principle, the energy density is proportional to the classical period of motion, diverging on the separatrix, which explains a high density of quantum states at the critical points.

In this article we study effective Hamiltonians describing evolution of a generic quantum system X interacting with a quantum system Y in the case where the

characteristic frequency of the system X is essentially lower than the corresponding frequency of the system Y, $\omega_X \ll \omega_Y$, and the interaction constant g satisfies the strong coupling condition: $\omega_X \lesssim g \ll \omega_Y$. We show that, depending on the type of interaction and the nature of quantum systems different physical situations take place, but generically such effective Hamiltonians describe Quantum Phase Transitions in the *slow system*.

II. EFFECTIVE HAMILTONIAN

Let us consider the following generic Hamiltonian describing an interaction between two quantum systems:

$$H = \omega_1 X_0 + \omega_2 Y_0 + g\left(X_+ + X_-\right)\left(Y_+ + Y_-\right), \quad (1)$$

where X_0 and Y_0 are the free Hamiltonians of the X and Y systems respectively, and such that $\omega_1 \ll \omega_2$. The above Hamiltonian does not preserve the *total excitation number operator* $N = X_0 + Y_0$ and, in the limit $\omega_1, \omega_2 \gg g$, leads to the appearance of *multiphoton-type* interactions of the form $X_+^n Y_-^m$ which, under certain physical conditions on the frequencies $\omega_{1,2}$, describe *resonant* transitions between energy levels of the whole system (see [3] and references therein).

The raising-lowering operators $X_\pm$, $Y_\pm$ describe transitions between energy levels of the systems X and Y respectively and consequently obey the following commutation relations:

$$[X_0, X_\pm] = \pm X_\pm, \quad [Y_0, Y_\pm] = \pm Y_\pm. \quad (2)$$

We do not impose any condition on the commutators between transition operators, which are generally some functions of diagonal operators and of some integrals of motion $[N_1, X_0] = [N_2, Y_0] = 0$:

$$[X_+, X_-] = \nabla_{X_0}\phi_1(X_0, N_1), \quad (3a)$$
$$[Y_+, Y_-] = \nabla_{Y_0}\phi_2(Y_0, N_2), \quad (3b)$$

where $\phi_1(X_0, N_1) = X_+ X_-$ and $\phi_2(Y_0, N_2) = Y_+ Y_-$ are some polynomials of X_0 and Y_0 respectively (from now on we omit the dependence on integrals $N_{1,2}$ in the arguments) and $\nabla_z \phi(z) = \phi(z) - \phi(z+1)$. The

objects $(X_0, X_\pm)$ and $(Y_0, Y_\pm)$ are known as polynomial deformed algebras $sl_{pd}(2, R)$ [11].

Now, we will be interested in the limit where the *slow system* frequency is less than/or of the order of the coupling constant, $\omega_1 \lesssim g \ll \omega_2$. Following the method described in Ref. [2] we can adiabatically remove all the terms that contain the fast system's transition operators, $Y_\pm$. In particular, the counter-rotating term $X_+ Y_+ + X_- Y_-$ and the rotating term $X_+ Y_- + X_- Y_+$ can be eliminated from the Hamiltonian (1) by a subsequent application of the following Lie-type transformations:

$$U_1 = \exp\left[\varepsilon\left(X_+ Y_+ - X_- Y_-\right)\right], \quad (4a)$$
$$U_2 = \exp\left[\epsilon\left(X_+ Y_- - X_+ Y_-\right)\right], \quad (4b)$$

where the small parameters, ε and ϵ, are defined by

$$\varepsilon = \frac{g}{\omega_2 + \omega_1} \ll 1 \quad \text{and} \quad \epsilon = \frac{g}{\omega_2 - \omega_1} \ll 1. \quad (5)$$

The transformations (4a) and (4b) generate different kinds of terms: such as $X_\pm^n Y_\pm^k + h.c.$, $X_\pm^n Y_\mp^k + h.c.$, $Y_\pm^n + h.c.$, and $X_\pm^n + h.c.$ with coefficients depending on X_0 and Y_0. Under the condition $\omega_1, g \ll \omega_2$ all the rapidly oscillating terms, i.e. those containing powers of $Y_\pm$, can be removed by applying transformations similar to (4), with properly chosen parameters. Then, the effective Hamiltonian is diagonal for the operators of the Y system. The result can be expressed as a power series of the single parameter $\delta = g/\omega_2 \ll 1$.

It is worth noting that it is not enough that δ be a small parameter for the formal expansion in (4) (and the subsequent transformations). A balance is necessary between the *effective dimensions* of the subsystems and δ. The *effective dimensions* of the system depend on the order of the polynomials $\phi_{1,2}$, and on the powers of the elements $X_{\pm,0}$ and $Y_{\pm,0}$ involved in each transformation. It was shown before [3], that the powers of the small parameters are increasing faster than the powers of $X_{\pm,0}$ and $Y_{\pm,0}$, which implies that we can focus on the *effective dimensions* introduced with (4).

Taking into account the above mentioned considerations, keeping only terms up to third order in δ and disregarding small corrections to the effective transition frequencies, we arrive at the following effective Hamiltonian:

$$\begin{aligned}
H_{eff} &= \omega_1 X_0 + \omega_2 Y_0 - 2\omega_1 \delta^2 \nabla_{x,-y}\Phi(X_0, Y_0 + 1) + g\delta\nabla_y \phi_y(Y_0)\left(X_+ + X_-\right)^2 \\
&\quad + \frac{1}{2}g\delta^3 \nabla_y\left(\phi_y(Y_0)\nabla_y^2\phi_y(Y_0 - 1)\right)\left(X_+ + X_-\right)^4,
\end{aligned} \quad (6)$$

where

$$\Phi(X_0, Y_0 + 1) = \nabla_{X_0, Y_0}\left[\phi_1(X_0)\phi_2(Y_0)\right], \quad (7)$$

and the generalized displacement operators are defined

as

$$\nabla_{mX_0, nY_0}f(X_0, Y_0) = f(X_0, Y_0) - f(X_0 + m, Y_0 + n),$$

for m and n integers.

Because the effective Hamiltonian (6) is diagonal for the operators of the Y *fast system*, we may project it out onto a minimal energy eigenstate of the Y system, $|\psi_0\rangle_Y$, substituting Y_0 by its eigenvalue y_0: $Y_0|\psi_0\rangle_Y = y_0|\psi_0\rangle_Y$.

The first order effect then comes from the term $\sim (X_+ + X_-)^2$, while the term $\sim (X_+ + X_-)^4$ defines a *fine structure* of the *effective potential*, obtained after projecting the effective Hamiltonian (6) onto the state $|\psi_0\rangle_Y$.

It is important to stress that, although δ is a small parameter, the effect of the terms $\sim \delta^n$, $n \geq 1$, could be in principle comparable with the main diagonal term $\omega_1 X_0$, especially if the algebra of X operators describe a *big subsystem*, i.e., large spin or big photon number. In this case non-trivial effects such as QPT may occur.

Now, we may proceed with analysis of the effective Hamiltonian (6) in the thermodynamic limit, focusing on the possible bifurcation of the ground state.

III. EXAMPLES

A. Atom-field interaction (Dicke model)

The Hamiltonian governing the evolution of A symmetrically prepared two-level atoms interacting with a single mode of quantized field has the form

$$H = \omega_1 \hat{n} + \omega_2 S_z + g(S_+ + S_-)(a^\dagger + a), \quad (8)$$

where $\hat{n} = a^\dagger a$ and $S_{z,\pm}$ are generators of the $(A+1)$-dimensional representation of the $su(2)$ algebra.

1. Effective field dynamics

First let us suppose that the atoms form a *fast subsystem* so that,

$$X_0 = \hat{n}, \quad X_+ = a^\dagger, \quad X_- = a, \quad Y_0 = S_z, \quad Y_\pm = S_\pm,$$

and thus, $\phi_y(Y_0) = C_2 - S_z^2 + S_z$ and $\phi_x(X_0) = \hat{n}$, where $C_2 = A/2(A/2+1)$ is the eigenvalue of the Casimir operator of the $su(2)$ algebra (integral of motion corresponding to the atomic subsystem).

Projecting the effective Hamiltonian onto the minimum energy state of the atomic system $|0\rangle_{at}$, so that $y_0 = -A/2$, we obtain the following effective Hamiltonian for the field mode:

$$H_{eff} = \tilde{\omega}_1 \hat{n} - Ag\delta \left(a + a^\dagger\right)^2 + gA\delta^3 \left(a + a^\dagger\right)^4, \quad (9)$$

where $\tilde{\omega}_1 = \omega_1(1 - 2A\delta^2)$.

Rewriting (9) in terms of position and momentum operators,

$$H_{eff} = \frac{\tilde{\omega}_1}{2}(p^2 + x^2) - 2Ag\delta x^2 + 4gA\delta^3 x^4,$$

we immediately detect that QPT in this case is related to the bifurcation of the effective potential $U(x) = (\tilde{\omega}_1/2 - 2Ag\delta)x^2 + 4gA\delta^3 x^4$ from a single minimum at $A < \tilde{\omega}_1/(4g\delta)$ to a double well structure at $A > \tilde{\omega}_1/(4g\delta)$. The physical effect associated with this QPT consists of a spontaneous generation of photons in the field mode. In some sense, the virtual photons, always presented in the Dicke model (8), are *condensed* into the real photons after crossing the critical point $A = \tilde{\omega}_1/(4g\delta)$. It is worth noting that this does not happen if the RWA is applied to (8).

2. Effective atomic dynamics

In the opposite case, when the atoms form a *slow subsystem* we have

$$X_0 = S_z, \quad X_\pm = S_\pm, \quad Y_0 = \hat{n}, \quad Y_+ = a^\dagger, \quad Y_- = a.$$

Projecting the effective Hamiltonian onto the minimum energy state of the field mode $|0\rangle_f$, so that $y_0 = 0$, the effective Hamiltonian acquires the form

$$H_{eff} = \tilde{\omega}_1 S_z - 4g\delta S_x^2 + 2\omega_2\delta^2 S_z^2 + 16g\delta^3 \{S_x^2, S_z\}, \quad (10)$$

where $\tilde{\omega}_1 = \omega_1 - 2\omega_2\delta^2 - 16g\delta^3$.

For our analysis it is convenient to perform a $\pi/2$ rotation in (10) around axis y, transforming the (10) Hamiltonian into

$$\tilde{H}_{eff} = -\tilde{\omega}_1 S_x - 4g\delta S_z^2 + 2\omega_2\delta^2 S_x^2 - 16g\delta^3 \{S_z^2, S_x\}. \quad (11)$$

In the thermodynamical limit we may replace the atomic operators by the corresponding classical vectors over the two-dimensional sphere, i.e.,

$$S_z \to \frac{A}{2}\cos\theta, \quad S_x \to \frac{A}{2}\sin\theta\cos\phi, \quad S_y \to \frac{A}{2}\sin\phi\sin\theta,$$

and thus rewrite the effective Hamiltonian (11) as a classical Hamiltonian function,

$$\begin{aligned} H_{cl} = &-\frac{A}{2}(\tilde{\omega}_1\cos\phi\sin\theta + 2Ag\delta\cos^2\theta \\ &-A\omega_2\delta^2\cos^2\phi\sin^2\theta \\ &+4A^2g\delta^3\cos\theta\cos\phi\sin^2\theta). \end{aligned} \quad (12)$$

The first two terms in the above expression describe the thermodynamical limit of the Lipkin-Meshkov model [8] and determine the critical point of QPT, $\xi = 4Ag\delta/\tilde{\omega}_1 = 1$, which again is related to the bifurcation of the ground state: a single minimum at $\sin\phi = 0$, $\cos\theta_* = 0$ splits into two minima at $\sin\phi = 0$, $\cos\theta_{**} = \pm\sqrt{1 - \xi^{-2}}$ for $\xi > 1$. It is worth noting that the global minimum of H_{cl} at $\xi < 1$ converts into a local maximum for $\xi > 1$, so that $H_{cl}(\theta_{**}) < H_{cl}(\theta_*)$. This means that the atoms, initially prepared at the minimum of the Hamiltonian function, spontaneously change their ground state energy at some value of the system's parameters. Classically,

this implies appearance of a separatrix, which leads to the discontinuity on the energy density spectrum in the thermodynamic limit. It is also worth noting that there is a loss of the rotational symmetry in this process: the new ground state is obviously not invariant under rotations around axis x, while the initial ground state is clearly invariant under x-rotations.

It is easy to see that the last two terms in (12) are of lower order in the parameter $A\delta$ and can be neglected in the first approximation for description of QPT at $\xi = 1$.

B. Spin-spin interaction

As a second example let us consider a dipole-dipole like interaction, that is,

$$X_0 = S_{z1},\ X_\pm = S_{\pm 1}, \quad Y_0 = S_{z2},\ Y_\pm = S_{\pm 2}.$$

The effective Hamiltonian for the *slow spin system* (after projecting onto the lowest state of the *fast spin system* with eigenvalue $-A_2/2$) takes the form similar to (10), with

$$H_{eff} = \tilde\omega_1 S_{z1} - 2A_2 g\delta S_{x1}^2 + 2A_2\omega_1\delta^2 S_{z1}^2$$
$$+16gA_2\delta^3 S_{x1}^4 + 24g\delta^3 A_2^2\{S_x^2, S_z\}, \quad (13)$$

where $\tilde\omega_1 = \omega_1 - 2A_1\omega_1\delta^2 - 20g\delta^3 A_1^2$. The first two terms are dominant for $\delta \ll 1$ and describe the Lipkin-Meshkov model, so that the critical point is reached at $\xi = 4A_2A_1^2 g\delta/\tilde\omega_1 = 1$ in the thermodynamical limit. The effect of the rest of the terms in (13) is negligible in the vicinity of $\xi = 1$.

IV. CONCLUSIONS

We deduce the effective Hamiltonian of a generic slow quantum system interacting with another fast oscillating system when the total excitation number is not preserved. Analyzing those effective Hamiltonians in the thermodynamic limit we have observed a bifurcation of the ground state leading to the effect of the Quantum Phase Transitions.

It is interesting to note that, for multidimensional systems, when algebraically the X system is a direct sum of several non-interacting subsystems, an interesting effect of generation of entangled states (in the non-preserving excitation case) can be observed. Really, let us suppose that in (1) $X_{0,\pm} = X_{0,\pm 1} + X_{0,\pm 2}$, $[X_{j,1}, X_{j,2}] = 0, j = 0, \pm$; then the corresponding effective Hamiltonian (up to a first non-trivial order in δ) takes the form

$$H_{eff} \approx \omega_1(X_{0,1} + X_{0,2}) + \omega_2 Y_0 +$$
$$g\delta[(X_{+,1} + X_{-,1})^2 + (X_{+,2} + X_{-,2})^2$$
$$+2(X_{+,1} + X_{-,1})(X_{+,2} + X_{-,2})]\nabla_y \phi_y(Y_0),$$

where we can clearly see that the last term contains the operator product $\sim X_{+,1}X_{+,2}$ which, together with quadratic terms in $X_{\pm,1(2)}$, implies a spontaneous generation of entangled states of X_1 and X_2 starting from the minimum energy state. This can be corroborated by the entangling power measure by considering a uniform distribution of the initial factorized states [12]. Thus we can say that, in the regime studied here, entanglement can be generated in a bipartite system in the vicinity of a Phase Transition.

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