PHYSCON 2009, Catania, Italy, September, 1–September, 4 2009

CONTROL PROBLEMS FOR A BODY MOVEMENT IN THE VISCOUS MEDIUM

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Abstract
The paper deals with mathematical model of movement of a body in a viscous medium. The problem of optimum control to the viscous medium from initial position in set is considered by moving of a body. Movement occurs at Reynolds's greater numbers that generates effects of failure of the laminar boundary layer, caused by return difference of a gradient of pressure. Thus behind a body the vortex path is formed. Frequency of failure of whirlwinds is expressed in the form of the dimensionless parameter. Asymmetrical formation of whirlwinds leads to occurrence periodic cross-section to speed of power influences on a body. Oscillatory movements, especially as a result develop if frequency of formation of whirlwinds comes nearer to own frequency of fluctuations of a body.

Key words
Optimal power flow, optimal control, Kármán Trail, fluid dynamics.

1 Introduction

Autonomous vehicles and robots intended for work in atypical environment has proved to form a great body of knowledge interesting from the viewpoint of challenging applications and being the source of new theoretical research. Particular emphasis is placed on mobile manipulation robots. Just this term is preferred in [Chernous'ko, Bolotnik and Gradetskii, 1989] intended for work in a viscous medium. It is caused, for example, by a need in robots to inspect and assimilate water tanks, and to do various technological works in those places.

Design of such vehicles is a complicated problem. The situation when one has to deal with rather limited energy supply of vehicles is natural and, sometimes, inevitable. Then, the following control problem is topical: to find the laws of the control forces and momentums behavior so as to move it from the initial position to a given one for minimum energy consumption. Such a problem is close to the ones of dynamic optimization considered by [Chernous'ko, Bolotnik and Gradetskii, 1989; Beletskii, 1973; Avetisyan, Akulenko and Bolotnik, 1987].

The problem has a number of special features. First, it is irregular (see [Krasovskii, 1968]), because the Euler–Lagrange equations do not contain controls in an explicit form, and, hence, the optimal controls cannot be determined in terms of the state and adjoint variables. Second, as it was found out, there are impulse components in the control forces and momentums optimum programs. Therefore, the classical variational techniques cannot be directly applied to find these programs. The third feature follows from the second one and consists of calculating the energy consumption. The point is that one has to define the well-posed procedure of multiplying impulse controls by discontinuous velocities.

So, the speech goes about a new set of problems being topical from the viewpoint of the theory of singular [Bryson and Ho, 1969; Gabasov and Kirillova, 1973; Gurman, 1985] solutions of dynamic optimization problems.

The totality of the problems solved in the present paper can be used in both the applied theory of singular dynamic optimization problems and design of perspective samples of new machines.

2 Problem Statement

Throughout this paper we regard the term "medium" as fluid or gas. However, for sake of being intuitive we use the term "fluid". Hydrodynamic constraints listed below, being satisfied, give a possibility to analyze necessary conditions for optimality for the problems men-

*Researches was supported by the Program no. P-04 of Presidium of the Russian Academy of Sciences "Fundamental Problems of Nonlinear Dynamics"

tioned in the Introduction. It is assumed that an inertial system and, inside it, a right Cartesian coordinate system $Ox_1x_2x_3$ are chosen.

Let $\mathbf{v}(t,x) = \mathbf{v}(t,x_1,x_2,x_3)$ be the velocity vector of fluid particle at the point $M(x_1,x_2,x_3)$ at the instant t, and v_1, v_2, and v_3 be its projections in the coordinate axes. The first two constraints are reduced to the following.

Constraint 1. *Fluid is incompressible.*

With account of the equation of continuity, this constraint is equivalent to zero velocity of the volume strain

$$\operatorname{div} \mathbf{v} = 0. \qquad (1)$$

Constraint 2. *The generalized Newton hypothesis (see* [Slezkin, 1955]*) is fulfilled*

$$P = -pE + \mu\Big(\frac{\partial \mathbf{v}}{\partial x} + \Big(\frac{\partial \mathbf{v}}{\partial x}\Big)^*\Big), \qquad (2)$$

where P is the linear operator defined by the stress tensor, $p = p(t,x)$ denotes the scalar field of pressure, μ is the dynamic viscosity coefficient, E is the identity mapping, $\frac{\partial \mathbf{v}}{\partial x}$ is the Frechet derivative, and $\Big(\frac{\partial \mathbf{v}}{\partial x}\Big)^*$ is the conjugate operator.

Let a body of bounded size with sufficiently smooth boundary S move in fluid. One of the fluid mechanics axioms is the sticking condition: at the body surface points the velocity vector of fluid particle is equal to the velocity vector of the corresponding body point. This condition and the constraint 1 imply that in the case of translational motion of the body the following equality is fulfilled at its surface (see [Slezkin, 1955])

$$\Big(\frac{\partial \mathbf{v}}{\partial x}\Big)^* \mathbf{n} = 0, \qquad (3)$$

where $\mathbf{n}$ is the unit vector of the outward normal to the surface S at the point x.

The stress on an element dS of the body surface is calculated by the formula $\mathbf{p}_n = P\mathbf{n}$, where $\mathbf{n}$ is the unit vector of the outward normal to dS. This equality and (2) yield the formula for the principal vector of the forces acting from fluid upon the body surface (hydrodynamic forces)

$$\mathbf{R} = \iint\limits_S \Big(-pE + \mu\Big(\frac{\partial \mathbf{v}}{\partial x} + \Big(\frac{\partial \mathbf{v}}{\partial x}\Big)^*\Big)\mathbf{n}\, dS. \qquad (4)$$

The formula for the principal momentum of hydrodynamic forces can be obtained similarly. According to (3), if the body moves translationally, then the formula (4) is reduced to

$$\mathbf{R} = \iint\limits_S \Big(-pE + \mu\frac{\partial \mathbf{v}}{\partial x}\Big)\mathbf{n}\, dS. \qquad (5)$$

We need further the so-called moving coordinate system $O_cy_1y_2y_3$ with the body inertia center as the origin and the axes rigidly connected with the body.

To find the principal vector and momentum, one has to calculate on the body surface the pressure and the Frechet derivative of the fluid velocity vector. To do this, one has to solve a certain boundary-value problem for the vector-valued Navier–Stokes equation. This equation is written out below in the moving system $O_cy_1y_2y_3$ with axes parallel to the corresponding axes of the system $Ox_1x_2x_3$ (the body is assumed to move translationally). Let $\mathbf{V}$ be the velocity vector of the body, and $x_c(t)$ be the radius vector of its inertia center. In the moving coordinate system, denote the absolute velocity vector of fluid and the pressure as follows:

$$\hat{\mathbf{v}}(t,y) = \mathbf{v}(t, x_c(t) + y),\ \hat{p}(t,y) = p(t, x_c(t) + y).$$

Then the Navier–Stokes equation is of the form

$$\frac{\partial \hat{\mathbf{v}}}{\partial t} = -\frac{\partial \hat{\mathbf{v}}}{\partial y}(\hat{\mathbf{v}} - \mathbf{V}) - \frac{1}{\rho}\Big(\frac{\partial \hat{p}}{\partial y}\Big)^* + \nu \operatorname{div} \frac{\partial \hat{\mathbf{v}}}{\partial y} + \mathbf{F}, \quad (6)$$

where $\mathbf{F}$ is the strength of the gravity field, ρ is the fluid density, $\nu = \mu/\rho$ is the kinematic viscosity coefficient.

Now, the above-mentioned boundary-value problem is reduced to finding the solution of a system of partial differential equations, namely, equation (6) plus the equation of continuity $\operatorname{div} \hat{\mathbf{v}} = 0$. This solution must satisfy the sticking condition $\hat{\mathbf{v}}(t,y)\Big|_S = \mathbf{V}$ and the natural condition $\lim\limits_{y\to\infty} \hat{\mathbf{v}}(t,y) = 0$.

A flow is accepted to call established or stationary if the field of its absolute velocity vectors in the moving coordinate system does not change in time. Obviously, if the body moves translationally, the necessary condition for the flow to be stationary is $\mathbf{V} = \mathbf{V}_0 = \mathrm{const}$.

The formulae for the power of the drag force acting upon a homogeneous solid sphere, in stationary cases considered by Stokes and Oseen, are presented below.

The Stokes procedure ignores in (6) the strength of the gravity field and the term $\frac{\partial \hat{\mathbf{v}}}{\partial y}(\hat{\mathbf{v}} - \mathbf{V})$. As a result, the expression for the drag force becomes $D = 6\pi\mu a V_0$, where V_0 is the magnitude of the velocity vector $\mathbf{V}_0$, and a is the radius of the solid sphere. For the purpose of forthcoming generalizations, it is convenient to rewrite this expression as follows:

$$D = C_D^{St}\rho S V_0^2/2, \qquad C_D^{St} = 24/\mathrm{Re}. \qquad (7)$$

Here $S = \pi a^2$, C_D^{St} is the drag coefficient, and $\mathrm{Re} = 2aV_0/\nu$ is the Reynolds number.

The Oseen approach also ignores the gravitational forces action and the quadratic inertial terms, however, takes completely into account the velocity of the solid sphere in the Navier–Stokes equation. The following approximate result comes out:

$$\begin{gathered} D = C_D^{Os} \rho S V_0^2 / 2, \\ C_D^{Os} = 24(16 - \mathrm{Re}^2)(16\mathrm{Re} - 3\mathrm{Re}^2)^{-1}. \end{gathered} \tag{8}$$

Suppose that the body has a symmetry axis. If the body moves in such a manner that this axis remains in a given plane (for example, in the plane Oxy), then, according to the statics theorems for an absolutely solid body, the totality of forces acting from fluid upon the body can be reduced to the resultant one called the hydrodynamic force. As usual (see, for instance, [Appazov, Lavrov and Mishin, 1966]), the point of intersection of the symmetry axis and the line of the hydrodynamic force action is referred to as center of pressure. The hydrodynamic force is resolved into components parallel to the velocity vector $\mathbf{V}$ of the body inertia center and perpendicular to $\mathbf{V}$. The first component $\mathbf{D}$ is known as the drag force, and the second one $\mathbf{D}^l$ is called the lift force.

Let $\mathbf{i}, \mathbf{j}$ be the unit vectors in the directions Ox and Oy respectively. We need further a mapping that puts a vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ into correspondence to $\mathbf{a}^\perp = -a_2\mathbf{i} + a_1\mathbf{j}$. Let V be the magnitude of $\mathbf{V}$, D be that of the drag force, and D^l be that of the lift force. For needs of forthcoming references, it is convenient to formulate the following assertion as lemma.

Lemma 1. *The drag and lift forces are calculated by the formulae*

$$\begin{aligned} &\mathbf{D} = \mathrm{sgn}(\mathbf{V}, \mathbf{D}) D \frac{1}{V} \mathbf{V}, \\ &\mathbf{D}^\perp = \mathrm{sgn}(\mathbf{V}, \mathbf{D}) s D^\perp \frac{1}{V} \mathbf{V}^\perp, \\ &s = \mathrm{sgn}((\mathbf{V}, \mathbf{e})(\mathbf{V}, \mathbf{e}^\perp)), \end{aligned} \tag{9}$$

where $\mathbf{e}$ *is the directing vector of the body symmetry axis.*

The magnitude of the drag force acting upon the solid sphere is presented as (7) (or (8)) just to make the coefficient C_D a dimensionless quantity. In the considered case, such a presentation can be maintained for the magnitude of the stationary drag force, i.e.,

$$D = C_D \rho S V^2 / 2. \tag{10}$$

Analogously, the magnitude of the stationary lift force can be presented as

$$D^\perp = C_D^\perp \rho S V^2 / 2. \tag{11}$$

Here S is the area of the body projection onto the plane perpendicular to the velocity vector of the body inertia center.

According to the theory of dynamic similitude, the coefficients C_D and $C_D^\perp$ depend on the body shape, Reynolds and Frud numbers only.

Further we deal with mechanical systems of axially symmetric bodies (referred to as links). Let us introduce the following constraint.

Constraint 3. *Systems move in a volume of fluid which is either very extended or is enclosed within rigid boundaries.*

In the framework of the listed constraints, the coefficient C_D is, following to [Sedov, 1973], a function of the body shape, Reynolds number and, probably, the angle of attack between the velocity vector of the body inertia center and the symmetry axis, i.e., $C_D = C_D(\text{shape}, \mathrm{Re}, \alpha)$. To determine the angle of attack, one can use the formula

$$\alpha = -s \arccos |(\mathbf{e}, \mathbf{V}/V)|. \tag{12}$$

As the case of nonstationary flow is concerned, it should be noted that Bussinesk has generalized the Stokes approach to the case of nonuniform translational motion of a solid sphere and has received the formula for the drag force (see [Oseen, 1927]), which in terms of the distributional derivative D_t and distributional convolution [Schwatz, 1950] (denoted below by the symbol "$*$") is of the form

$$\mathbf{D} = -k_1 D_t \mathbf{V} - k_{00} \mathbf{V} - k_{01} \Big(\frac{1}{\sqrt{t}} * D_t \mathbf{V} \Big). \tag{13}$$

Here we use the notation

$$k_1 = \frac{2}{3} \pi a^3 \rho, \quad k_{00} = 6\pi\mu a, \quad k_{01} = 6\sqrt{\pi\nu}\rho a^2.$$

The first term of the formula (13), which corresponds to the so-called apparent additional mass, is the inertial component of the drag, and the second one presents the stationary Stokes formula.

If the drag and lift forces can be described quite accurately by the formulae (9)–(11), then the flow is called quasistationary (see [Sedov, 1973]).

The work of the hydrodynamic forces is considered further as performance index. In [Zavalishchin, 2002] we show that solving the problem of optimal displacements of a solid sphere, if the flow is quasistationary, leads to the relative mistake in the optimal energy consumption about 3% only (Reynolds numbers are assumed to obey the restriction $\mathrm{Re} < 1$). The nonstationarity of the flow can be partially taken into account by means of introducing the apparent additional mass (see [Sedov, 1973; Daily and Harleman, 1966]).

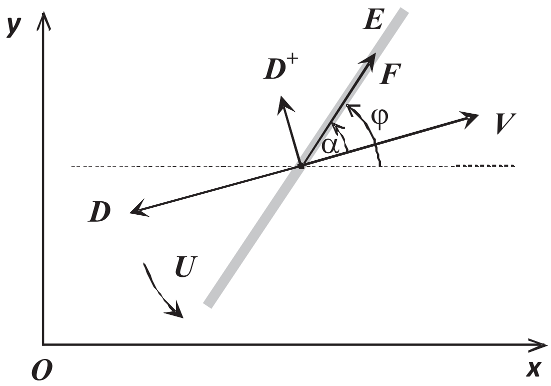

Figure 1. Forces and moments acting on a body

Hypothesis 1. *The optimal displacement of the system produces quasistationary flow of the system links.*

Lemma 2. *Let the constraints 1-3 be fulfilled, Hypothesis 1 hold, and the drag coefficient of each link be a homogeneous function of power m_c in Reynolds numbers corresponding to the optimal motion of the system. Then the magnitude of the drag force of each link is a homogeneous function of power $m = m_c + 2$ in the magnitude of its inertia center velocity. The same holds relative to the lift forces.*

For spherical or cylindrical bodies, the drag coefficient is an approximately homogeneous function of Reynolds numbers small enough, when it is approximately inversely proportional to them, and large enough, when it depends on them negligibly (in particular, such an interval is a fairly long left-hand half-neighborhood of the number $\mathrm{Re} = 5 \cdot 10^5$). In the first case $m = 1$, and in the second one $m = 2$.

Hypothesis 2. *The optimal displacements of the considered systems possess the following property: Reynolds numbers of each link provide that the drag and lift coefficients of the link are homogeneous functions of these numbers.*

In this article the model of moving in the viscous medium of solid body for Reynolds's greater numbers is investigated. The increase in speed generates a separation of a boundary layer and occurrence of turbulent effects. In turn the last is the reason of occurrence of cross-section fluctuations operating on the body. Further attempt of modelling of movement of a body in such conditions is undertaken.

3 Mathematical Model

In this section, we deal with a model of moving in a viscous medium of solid body (see Fig. 1) in plane Oxy. The state of the body is described by the generalized coordinates x, y and φ. Let V be the vector of centroid velocity $\mathbf{V} = (\dot{x}; \dot{y})^T$, F be the force acting along a body axis $\mathbf{F} = (F\cos\varphi; F\sin\varphi)^T$, E be the unit vector $\mathbf{E} = (\cos\varphi; \sin\varphi)^T$, D and $D^\perp$ are the drag force and lift force respectively

$$\begin{aligned}
&\mathbf{D} = (-D\cos(\varphi-\alpha); -D\sin(\varphi-\alpha))^T,\\
&\mathbf{D}^\perp = (-D^\perp\sin(\varphi-\alpha); D^\perp\cos(\varphi-\alpha))^T,
\end{aligned}$$

U be the angular moment. On a body mass force also actiones $\mathbf{F_m} = (0; -mg)^T$.

Kinetic energy is equal to

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}\frac{ml^2}{3}\dot{\varphi}^2. \tag{14}$$

Using the Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \tag{15}$$

one can obtain body moving equations

$$\begin{aligned}
m\ddot{x} &= Q_x\\
m\ddot{y} &= Q_y\\
\tfrac{1}{3}ml^2\ddot{\varphi} &= Q_\varphi
\end{aligned} \tag{16}$$

The generalized forces corresponding to the generalized coordinates will be the following

$$\begin{aligned}
Q_x &= -D\cos(\varphi-\alpha) - D^\perp\sin(\varphi-\alpha) +\\
&\quad + F\cos(\varphi)\\
Q_y &= -D\sin(\varphi-\alpha) + D^\perp\cos(\varphi-\alpha) +\\
&\quad + F\sin(\varphi) - mg\\
Q_\varphi &= U + M
\end{aligned} \tag{17}$$

The system of equations (16) and (17) describes body movement.

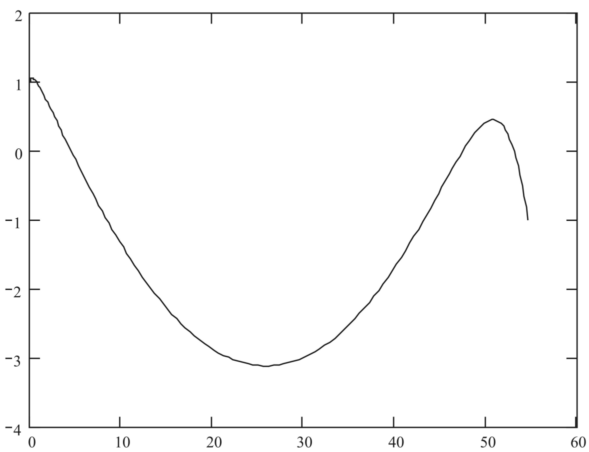

Figure 2. The phase trajectory $y(x)$

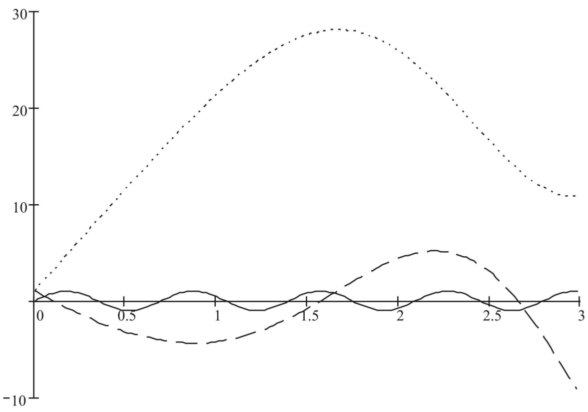

Figure 3. Control U, velocities x and y

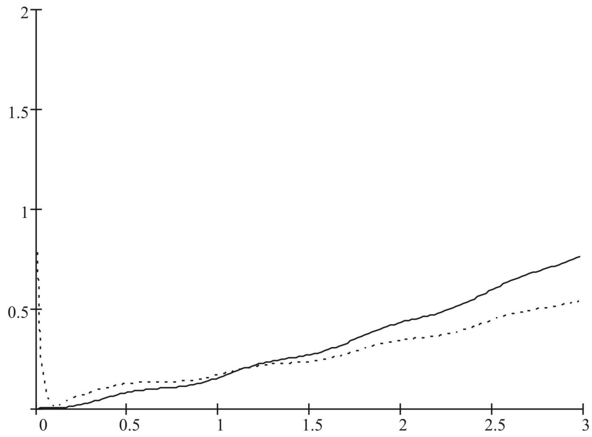

Figure 4. Angles φ and α

4 The case of Reynolds's greater numbers

The phase trajectory $y(x)$ of body movement is obtained by numerical integration of system (15) and (16) is represent on Fig. 2. It is visible that a body having overcome 60 metres deviates on a vertical axis on 3 metres. Control U– continuous line, velocities x– dot line, and y– dashed line, is represent on Fig. 3. At last angle φ– continuous line, and angle of attack α– dot line, (within 0,5 radians) is represent on Fig. 4. It should be noted that data for numerical experiment undertook from the book [Daily and Harleman, 1966].

5 Conclusion

One of the classical open-flow problems in fluid mechanics concerns the flow around a circular cylinder, or more generally, a bluff body. At very low Reynolds numbers the streamlines of the resulting flow is perfectly symmetric as expected from potential theory. However as the Reynolds number is increased the flow becomes asymmetric and the so called Strouhal number relates the frequency of shedding to the velocity of the flow and a characteristic dimension of the body. It is defined as $S_t = f_{st}S/V$. In the equation f_{st} is the vortex shedding frequency (or the Strouhal frequency) of a body at rest. The Strouhal number for a cylinder is 0.2 over a wide range of flow velocities. The phenomenon of lock-in happens when the vortex shedding frequency becomes close to a natural frequency of vibration of the structure. When this happens large and damaging vibrations can vortex street occurs.

Analyzing results of numerical experiment it is possible to draw following conclusions. Adaptive control allows to smooth influence of the Kármán trail. Thus the border of its occurrence is probably removed. It would be the small contribution to struggle against turbulence.

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