ENOC-2008, Saint Petersburg, Russia, June, 30–July, 4 2008

REGULAR AND CHAOTIC DYNAMICS OF THE SWING

Anton O. Belyakov
Institute of Mechanics
Moscow State Lomonosov University
Russia
a_belyakov@inbox.ru

Alexander P. Seyranian
Institute of Mechanics
Moscow State Lomonosov University
Russia
seyran@imec.msu.ru

Angelo Luongo
DISAT
Universita di L'Aquila
Italy
luongo@ing.univaq.it

Abstract
Dynamic behavior of weightless rod with a point mass sliding along the rod axis according to periodic law is studied. This is the simplest model of child's swing. Asymptotic boundaries of stability domains are derived near resonance frequencies. Regular and chaotic motions of the swing under change of problem parameters are found and investigated both analytically and numerically.

Key words
Swing, parametric resonance, rotational orbits, chaos

1 Introduction

Oscillations of the swing is one of the classical problems in mechanics. As probably everyone can remember, to swing a swing one must crouch when passing through the middle vertical position and straighten up at the extreme positions, i.e. perform oscillations with a frequency which is approximately twice the natural frequency of the swing. Despite the popularity of the swing, in the literature on oscillations and stability where this problem is referred to [Kauderer, 1958; Bogolyubov and Mitropol'skii, 1974; Magnus, 1976; Panovko and Gubanova, 1987; Arnold, 1989; Bolotin, 1999] there are not many analytical and numerical results on the swing behavior dependent on parameters. Among recent papers we cite [Seyranian, 2004] on the stability analysis of the swing.

The present paper is devoted to study of regular and chaotic motions of the swing. Stability conditions of vertical position and limit cycles are obtained, and existence conditions for regular rotations are derived. These conditions are justified by numerical simulations. Domains for chaotic motions are found and analyzed in parameter space.

2 Main relations

Equation for motion of the swing is derived with the use of angular momentum alteration theorem and taking into account linear damping forces

$$(ml^2\dot{\theta})^{\cdot} + \gamma l^2\dot{\theta} + mgl\sin(\theta) = 0, \tag{1}$$

where m is the mass, l is the length, θ is the angle of the pendulum deviation from the vertical position, g is the acceleration due to gravity. The upper dot indicates the time derivative.

It is assumed that the length of the pendulum changes according to the periodic law

$$l = l_0 + a\varphi(\Omega t), \tag{2}$$

where l_0 is the mean pendulum length, a and Ω are the amplitude and frequency of the excitation, $\varphi(\tau)$ is the smooth periodic function with period 2π and zero mean value.

We introduce the following dimensionless parameters and variables

$$\tau = \Omega t,\ \varepsilon = \frac{a}{l_0},\ \Omega_0 = \sqrt{\frac{g}{l_0}},\ \omega = \frac{\Omega_0}{\Omega},\ \beta = \frac{\gamma}{m\Omega_0}. \tag{3}$$

Then, equation (1) can be written in the following form

$$\ddot{\theta} + \left(\frac{2\varepsilon\dot{\varphi}(\tau)}{1+\varepsilon\varphi(\tau)} + \beta\omega\right)\dot{\theta} + \frac{\omega^2\sin(\theta)}{1+\varepsilon\varphi(\tau)} = 0. \tag{4}$$

Here the upper dot denotes differentiation with respect to new time τ. This equation will be studied in the following sections via asymptotic and numerical techniques depending on the three dimensionless problem parameters: the excitation amplitude ε, the damping β, and the frequency ω.

3 Motion at small excitation amplitude

When the excitation amplitude ε is small, we can expect the amplitude of oscillations also to be small. So, we can expand the sine into a Taylor's series around zero in equation (4). Changing the variable by

$$q = \theta(1 + \varepsilon\varphi(\tau)) \quad (5)$$

in equation (4) and multiplying it by $1 + \varepsilon\varphi(\tau)$ we obtain the equation for q as

$$\ddot{q} + \beta\omega\dot{q} - \frac{\varepsilon\left(\ddot{\varphi}(\tau) + \beta\omega\dot{\varphi}(\tau)\right)}{1 + \varepsilon\varphi(\tau)} q + \omega^2 \sin\left(\frac{q}{1 + \varepsilon\varphi(\tau)}\right) = 0. \quad (6)$$

Let us suppose that ε and β are small parameters as well as the variable q. Then, neglecting terms of higher order equation (6) takes the following form

$$\ddot{q} + \beta\omega\dot{q} + \left[\omega^2 - \varepsilon(\ddot{\varphi}(\tau) + \omega^2\varphi(\tau))\right] q - \frac{\omega^2}{6} q^3 = 0. \quad (7)$$

3.1 Stability of the vertical position

Let us analyze the stability of the trivial solution $q = 0$ for the nonlinear equation (7). Its stability with respect to the variable q is equivalent to that for the equation (4) with respect to θ due to relation (5). According to the Lyapunov's theorem about stability based on the linear approximation the stability and instability of the solution $q = 0$ of the equation with periodic coefficients (7) is determined by those of the linearized equation

$$\ddot{q} + \beta\omega\dot{q} + \left[\omega^2 - \varepsilon(\ddot{\varphi}(\tau) + \omega^2\varphi(\tau))\right] q = 0. \quad (8)$$

This is a Hill's equation with damping. It is known that instability (i.e. parametric resonance) occurs near the frequencies $\omega = k/2$, where $k = 1, 2, \ldots$. The instability domains in the vicinity of these frequencies were obtained in [Seyranian, 2001] analytically. In three-dimensional space of the parameters ω, ε, and β these domains are described by the half-cones

$$(\beta/2)^2 + (2\omega/k - 1)^2 < r_k^2\varepsilon^2, \quad \beta \geq 0, \quad k = 1, 2, \ldots, \quad (9)$$

where $r_k = \frac{3}{4}\sqrt{a_k^2 + b_k^2}$ is expressed through the Fourier coefficients of the periodic function $\varphi(\tau)$

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(\tau)\cos(k\tau)d\tau, \quad (10)$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(\tau)\sin(k\tau)d\tau. \quad (11)$$

Inequalities (9) give us the first approximation to the instability domains. Hence, in the first approximation each k-th resonant domain depends only on k-th Fourier coefficients of the periodic excitation function. Particularly, for $\varphi(\tau) = \cos(\tau)$, $k = 1$ we obtain $a_1 = 1$, $b_1 = 0$, and $r_1 = 3/4$ after which the first instability domain takes the form

$$\beta^2/4 + (2\omega - 1)^2 < 9\varepsilon^2/16, \quad \beta \geq 0. \quad (12)$$

It follows from relation (3) that in physical time the swing is pumping with the excitation frequency Ω close to the critical frequencies $\Omega_k = 2\Omega_0/k$, where $k = 1, 2, \ldots$.

Inside the instability domains (9) the vertical position $\theta = 0$ becomes unstable and motion of the system can be either regular (limit cycle, regular rotation) or even chaotic.

3.2 Limit cycle

Limit cycle is a kind of regular motion which can also be studied with the assumption of small amplitudes of the system motion. So, we study the parametric excitation $\varphi(\tau) = \cos(\tau)$ of the nonlinear system (6) (hence, of the system (4)) at the first resonance frequency $\omega \approx 1/2$. We are looking for an approximate solution of system (6) in the form $q(\tau) = Q(\tau)\cos(\tau/2 + \Psi(\tau))$ by using the averaging method for resonant case described in the book [Bogolyubov and Mitropol'skii, 1974], where $Q(\tau)$ and $\Psi(\tau)$ are the slow amplitude and phase. As a result, we get a system of averaged first order differential equations for the slow amplitude and phase

$$\dot{Q} = -\frac{Q\beta\omega}{2} + \frac{Q\varepsilon(1 - \omega^2)}{2}\sin(2\Psi), \quad (13)$$

$$\dot{\Psi} = \omega - \frac{1}{2} - \frac{Q^2\omega^2}{8} + \frac{\varepsilon(1 - \omega^2)}{2}\cos(2\Psi). \quad (14)$$

This system gives steady solutions for $\dot{Q} = 0$, $\dot{\Psi} = 0$. Besides the trivial one with $Q = 0$ we obtain expressions for the amplitude and phase as

$$Q^2 = \frac{4}{\omega^2}\left(2\omega - 1 \mp \sqrt{\varepsilon^2(1 - \omega^2)^2 - \beta^2\omega^2}\right), \quad (15)$$

$$\Psi = \frac{1}{2}\arctan\left(\frac{\mp 4\beta\omega}{\sqrt{\varepsilon^2(1 - \omega^2)^2 - \beta^2\omega^2}}\right) + \pi j, \quad (16)$$

where $j = \ldots, -1, 0, 1, 2, \ldots$ and "$\arctan$" gives the major function value lying between zero and π.

In order to find boundaries of the resonance domain one should put $Q = 0$ in expression (15). These boundaries coincide with the boundaries of inequality (12) which is not a surprise because inequality (12) determines the instability region for the trivial solution $Q = 0$. There is an example for $\varepsilon = 0.04$ and$\beta = 0.05$ presented in Fig. 1 of amplitude–frequency response function (15) in comparison with numerical results (circles). Fig. 1 shows that we have a good coincidence with the numerical simulations up to the amplitude equals 1.

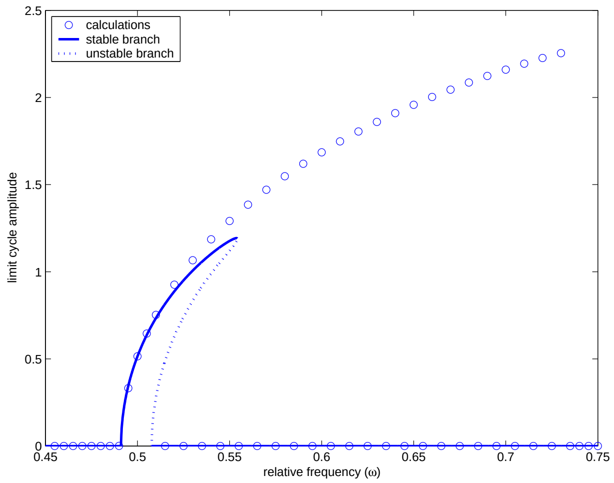

Figure 1. The amplitude–frequency characteristics for the parameters $\varepsilon = 0.04$ and $\beta = 0.05$.

3.3 Stability of the limit cycle

In order to study stability of the periodic solution $q_0(\tau) = Q\cos(\tau/2 + \Psi)$ we substitute it in equation (7) by the solution with small perturbation $q(\tau) = q_0(\tau) + u(\tau)$, where Q and Ψ are taken from expressions (15) and (16). Thus, in the first approximation we obtain a linear differential equation for $u(\tau)$

$$\ddot{u}+\beta\omega\dot{u}+\left[\omega^2+\varepsilon(1-\omega^2)\cos(\tau)-\frac{\omega^2}{2}q_0^2(\tau)\right]u=0. \tag{17}$$

Solution is stable or unstable simultaneously with that of nonlinear equation (6) according to the Lyapunov's theorem about stability based on the linear approximation. We have got again the Hill's equation with damping (17) depending on three parameters $\omega \approx 1/2$, $\beta \ll 1$, and $\varepsilon \ll 1$ with 2π-periodic excitation function

$$\Phi(\tau) = (1-\omega^2)\cos(\tau) - \frac{Q^2\omega^2}{2\varepsilon}\left(\cos(\tau/2+\Psi)\right)^2. \tag{18}$$

The domain of instability for equation (17) in the vicinity of the point $\omega = 1/2$, $\beta = \varepsilon = 0$ has the following form [Seyranian, 2001]

$$\beta^2/4 + (2\omega - 1 + a_0\varepsilon)^2 < \varepsilon^2(a_1^2 + b_1^2), \tag{19}$$

where the first Fourier coefficients of the function $\Phi(\tau)$ are the following

$$a_0 = -\frac{Q^2\omega^2}{2\varepsilon}, \tag{20}$$

$$a_1 = 1 - \omega^2 - \frac{Q^2\omega^2}{4\varepsilon}\cos(2\Psi), \tag{21}$$

$$b_1 = \frac{Q^2\omega^2}{4\varepsilon}\sin(2\Psi). \tag{22}$$

After some transformations we obtain the instability condition as

$$\mp Q^2\omega^2\sqrt{\varepsilon^2(1-\omega^2)^2 - \beta^2\omega^2} < 0 \tag{23}$$

which tells us that periodic solution (15), (16) with the sign plus is stable and that with minus is unstable.

4 Regular rotations

In this section we study regular rotations of the swing. During the rotation of the swing we obviously can not suppose θ to be small in equation (4). Let us consider ω^2 as a small parameter having the same order with the small parameters ε and β, which makes the system quasi-linear. Then, general equation (4) can be rewritten as a system with small excitation

$$\ddot{\theta} = -\varepsilon\left(\frac{2\dot{\varphi}(\tau)\dot{\theta}}{1+\varepsilon\varphi(\tau)} + \frac{\beta\omega}{\varepsilon}\dot{\theta} + \frac{\omega^2}{\varepsilon}\frac{\sin(\theta)}{1+\varepsilon\varphi(\tau)}\right). \tag{24}$$

Variable θ is neither small nor even limited, therefore we can introduce a new limited variable ψ via substitution $\theta = b\tau + \psi$, which is the solution of degenerate equation (24) $\ddot{\theta} = 0$. Constant b is the mean angular velocity of the swing rotation.

In order to use the general averaging method [Bogolyubov and Mitropol'skii, 1974] we have to write equation (24) in the form of the first order equation system with a small right side. For that reason we introduce a new variable v so that $\dot{\psi} = \sqrt{\varepsilon}v$, where $\sqrt{\varepsilon}$ is considered as a new small parameter. Thus, for the periodical excitation function $\varphi(\tau) = \cos(\tau)$ we obtain

$$\begin{pmatrix}\dot{\psi}\\ \dot{v}\end{pmatrix} = \sqrt{\varepsilon}X(\psi,v,\tau) + (\sqrt{\varepsilon})^2Y(\psi,v,\tau) + (\sqrt{\varepsilon})^3Z(\psi,v,\tau) + o((\sqrt{\varepsilon})^3), \tag{25}$$

where X, Y, and Z are the vectors with the following components

$$X_1 = v,\quad X_2 = 2b\sin(\tau) - \frac{\omega^2}{\varepsilon}\sin(b\tau+\psi), \tag{26}$$

$$Y_1 = 0,\quad Y_2 = 2v\sin(\tau) - \frac{\beta\omega}{\varepsilon\sqrt{\varepsilon}}b, \tag{27}$$

$$Z_1 = 0,\quad Z_2 = -b\sin(2\tau) + \frac{\omega^2}{\varepsilon}\sin(b\tau+\psi)\cos(\tau) - \frac{\beta\omega}{\varepsilon\sqrt{\varepsilon}}v. \tag{28}$$

With the averaging method we find the first, second and third order approximations of the system. It is the third approximation of averaged equation where regular rotations with $|b| = 1$ can be observed.

After taking the third approximation we obtain the differential equations for corresponding slow variables V

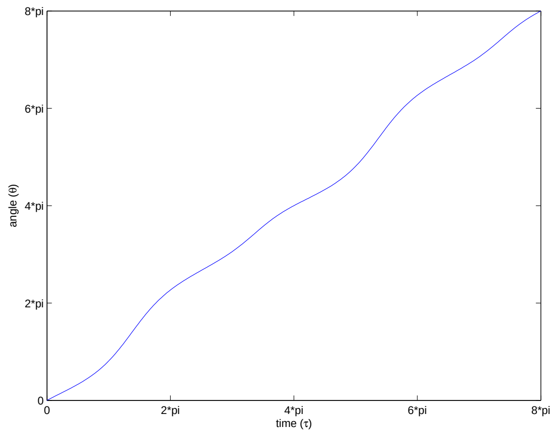

Figure 2. Regular rotation with the mean angular velocity being equal to the excitation frequency ($b = 1$) for the parameters: $\varepsilon = 0.28$, $\omega = 0.5$, and $\beta = 0.05$.

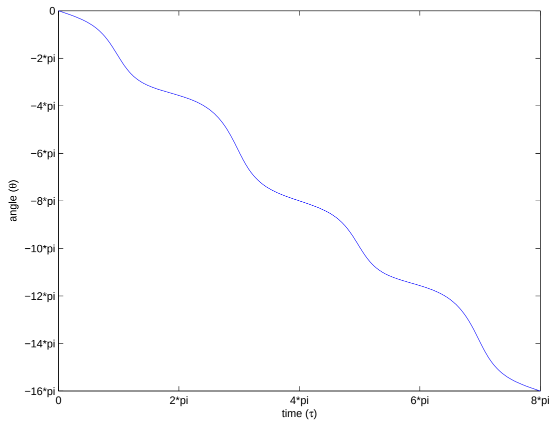

Figure 3. Regular rotation with the mean angular velocity twice as much as the excitation frequency ($b = -2$) for the parameters: $\varepsilon = 0.44$, $\omega = 0.5$, and $\beta = 0.05$.

and Ψ

$$\dot{\Psi} = \sqrt{\varepsilon} V, \tag{29}$$
$$\dot{V} = -\beta\omega V - \frac{\beta\omega b}{\sqrt{\varepsilon}} - \frac{3\sqrt{\varepsilon}\omega^2}{2}\sin(\Psi), \tag{30}$$

where $b = \pm 1$ i.e. the swing rotates counterclockwise or clockwise at the same frequency as the excitation (Fig. 2). From equations (29) and (30) we can write the averaged second order equation as

$$\ddot{\Psi} + \beta\omega\dot{\Psi} + \frac{3\varepsilon\omega^2}{2}\sin(\Psi) + \beta\omega b = 0. \tag{31}$$

This equation permits a stationary solution $\Psi = \Psi_0$ with the condition $\varepsilon\omega\sin(\Psi_0) = -2\beta b/3$. Thus, within the interval $(-\pi, \pi)$ the stationary solutions for (31) are

$$\Psi_{01} = -b \arcsin\frac{2\beta}{3\varepsilon\omega}, \tag{32}$$
$$\Psi_{02} = b\left(\arcsin\frac{2\beta}{3\varepsilon\omega} - \pi\right), \tag{33}$$

which exist only if the following condition is satisfied

$$\left|\frac{\beta}{\varepsilon\omega}\right| \leq \frac{3}{2}. \tag{34}$$

Conditions for stationary solutions with $|b| = 2$ (Fig. 3) and higher values could be obtained based on the higher order approximations of equation (24) and the averaging method. Condition (34) is compared with numerical results in Fig. 4 for mean angular velocity, where light blue points correspond to the regular swing rotation with $|b| = 1$ and yellow points denote that with

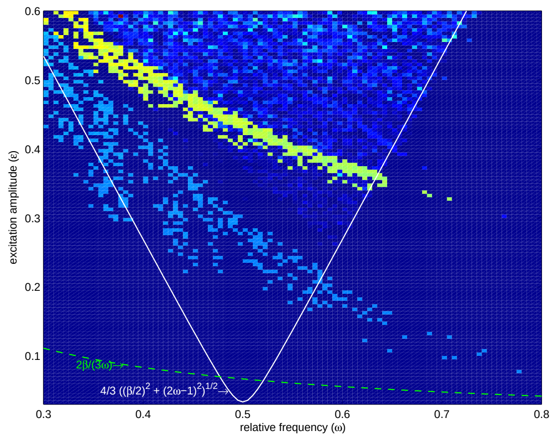

Figure 4. Mean angular velocity shown on the plane of parameters ε and ω at damping $\beta = 0.05$.

$|b| = 2$. Dark blue points correspond to zero mean angular velocity of stable equilibria points and limit cycles. The boundary approximation for the regime with $|b| = 1$ is drawn with green dashed line. We see that this line is closer to the lower edge of the light blue points manifold near $\omega = 0.8$ than it is near $\omega = 0.3$. It is because the accuracy decreases with the increase of the small parameter ε in the asymptotic method. It is possible to obtain more accurate boundary for rotations with $|b| = 1$ based on the higher order approximations.

The white solid line bounds the instability domain (12) of the lower vertical position of the swing. There are some regular rotation points located outside this domain. It means that at these points two stable regimes coexist: the regular rotation and the stationary position.

Fig. 4 contains points with significantly noninteger mean angular velocities which is a feature of more complicated rotational motion. For example, when pendulum repeatedly rotates twice clockwise and once counterclockwise, its mean angular velocity can be

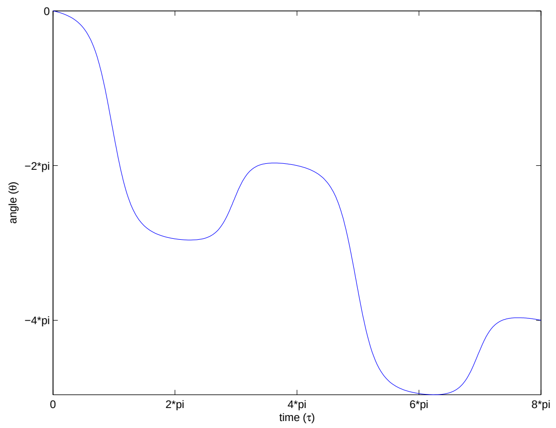

Figure 5. Regular rotation with the mean angular velocity being equal to one half of the excitation frequency ($b = -1/2$) for the parameters: $\varepsilon = 0.555$, $\omega = 0.5$, and $\beta = 0.05$.

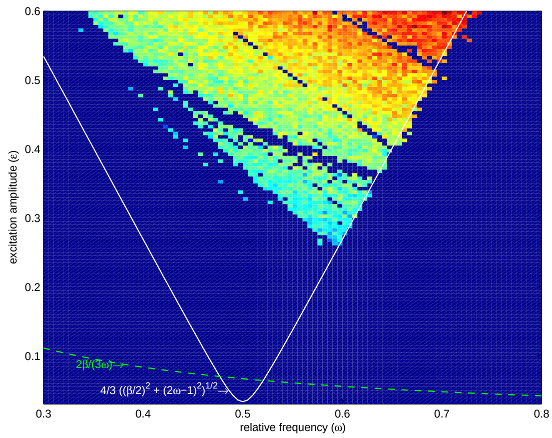

Figure 6. Maximal Lyapunov's exponents shown on the plane of parameters ε and ω at damping $\beta = 0.05$.

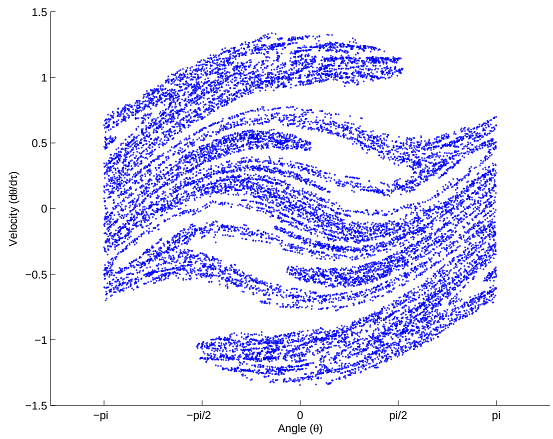

Figure 7. The Poincare map for the parameters $\varepsilon = 0.3$, $\omega = 0.55$ and $\beta = 0.05$.

noninteger ($b = -1/2$) as in Fig. 5.

4.1 Stability of the regular rotation

In order to get a stability condition for the stationary solutions (32) and (33) with $b = \pm 1$ we add to them a small perturbation $\Psi = \Psi_0 + \Delta$. Then, from (31), (32) and (33) we obtain a linearized equation for Δ

$$\ddot{\Delta} + \beta\omega\dot{\Delta} + \frac{3\varepsilon\omega^2}{2}\cos(\Psi_0)\Delta = 0. \tag{35}$$

The roots of a corresponding characteristic equation

$$p^2 + \beta\omega p + \frac{3\varepsilon\omega^2}{2}\cos(\Psi_0) = 0 \tag{36}$$

all have negative real parts only if $\varepsilon\omega^2\cos(\Psi_0) > 0$, and one root of the characteristic equation has a positive real part if $\varepsilon\omega^2\cos(\Psi_0) < 0$. Hence, for all parameters such that

$$0 < \left|\frac{\beta}{\varepsilon\omega}\right| < \frac{3}{2} \tag{37}$$

solution (32) is stable while solution (33) is unstable. Thus, we conclude that if the parameters satisfy (37) there are two stable regular rotations: $b = 1$ with $\Psi_{01} \in (-\pi/2, 0)$ and $b = -1$ with $\Psi_{01} \in (0, \pi/2)$; and two unstable rotations: $b = 1$ with $\Psi_{02} \in (-\pi, -\pi/2)$ and $b = -1$ with $\Psi_{02} \in (\pi/2, \pi)$.

The similar stability condition to (37) has been found in the book [Bogolyubov and Mitropol'skii, 1974] for the rotational motion of the pendulum with periodically moving support.

5 Chaotic motion

Location of chaotic regimes in the space of parameters ω and ε is shown in Fig. 6 for Liapunov's exponents, where dark blue regions have zero maximal Liapunov's exponent which implies regular motion or stationary position while all other colors correspond to positive Liapunov's exponents which means chaos. The closer color is to red the greater is the maximal Lyapunov's exponent. To be sure that this is chaos indeed rather than a long transition process, we plot the Poincare map. In Fig. 7 the Poincare map is shown for the parameters $\varepsilon = 0.3$, $\omega = 0.55$, which reveals a typical attractor structure.

6 Conclusion

"Child's swing" (a pendulum with periodically varying length) exhibits the diversity of behavior types. We recognized that the analytical stability boundaries of the vertical position of the swing and the frequency-response curve for limit cycles are in a good agreement with the numerical results. We found regular rotations of the swing and derived their stability conditions. These results are also approved numerically. It is shown that the limit cycles and regular rotations can coexist with the stable stationary attractor in contrast with the chaotic regimes which occur only inside the instability domain of the vertical position.

Acknowledgement
This research is supported by INTAS grant No. 06-1000013-9019

References

Kauderer, H. (1958). *Nichtlineare Mechanik*. Springer, Berlin.

Bogolyubov, N. N. and Mitropol'skii, Yu. A. (1974). *Asymptotic Methods in the Theory of Nonlinear Oscillations*. Nauka, Moscow.

Magnus, K. (1976). *Schwingungen. Eine Einfuhrung in die theoretische Behandlung von Schwingungensproblemen*. J.Teubner, Stuttgart.

Panovko, Ya.G. and Gubanova, I.I. (1987). *Stability and Oscillations of Elastic Systems. Modern Concepts, Paradoxes and Mistakes*. Nauka, Moscow.

Arnold, V.I. (1989). *Mathematical Methods of Classical Mechanics*. Nauka, Moscow.

Bolotin, V.V. (1999). *Vibrations in Engineering. A Handbook,* Vol. 1. *Oscillations of Linear Systems*. Mashinostroenie, Moscow.

Seyranian, A. P. (2004). The swing: parametric resonance. *Journal of Applied Mathematics and Mechanics*. **68**(5), pp. 757–764.

Seyranian, A. P. (2001). Resonance domains for the Hill equation with damping," *Doklady Physics*. **46**(1), pp. 41–44.