CONTROL OF LIMIT CYCLE BIFURCATIONS IN GENERAL LIÉNARD'S POLYNOMIAL SYSTEM

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Abstract
Applying a canonical system with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we control all possible limit cycle bifurcations and solve the limit cycle problem for general Liénard's polynomial system with an arbitrary (but finite) number of singular points.

Key words
General Liénard's polynomial system; limit cycle; field rotation parameter; bifurcation.

1 Introduction

In this paper, we consider Liénard equations

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0 \tag{1.1}$$

and the corresponding dynamical systems in the form

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y. \tag{1.2}$$

There are many examples in the natural sciences and technology in which this and related systems are applied [Agarwal and Ananthkrishnan, 2000; Bautin and Leontovich, 1990; Gasull and Torregrosa, 1999 — Smale, 1998]. Such systems are often used to model either mechanical or electrical, or biomedical systems, and in the literature, many systems are transformed into Liénard type to aid in the investigations. They can be used, e. g., in certain mechanical systems, where $f(x)$ represents a coefficient of the damping force and $g(x)$ represents the restoring force or stiffness, when modeling wind rock phenomena and surge in jet engines [Agarwal and Ananthkrishnan, 2000; Owens, Capone, Hall, Brandon and Chambers, 2004]. Such systems can be also used to model resistor-inductor-capacitor circuits with non-linear circuit elements. Recently, e. g., the Liénard system (1.2) has been shown to describe the operation of an optoelectronics circuit that uses a resonant tunnelling diode to drive a laser diode to make an optoelectronic voltage controlled oscillator [Slight, Romeira, Liquan, Figueiredo, Wasige and Ironside, 2008]. There are also some examples of using Liénard type systems in ecology and epidemiology [Moreira, 1992].

We suppose that system (1.2), where $f(x)$ and $g(x)$ are arbitrary polynomials of x, has an anti-saddle (a node or a focus, or a center) at the origin and write it in the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + \beta_1\,x + \ldots + \beta_{2l}\,x^{2l}) \\ &\quad + y\,(\alpha_0 + \alpha_1\,x + \ldots + \alpha_{2k}\,x^{2k}). \end{aligned} \tag{1.3}$$

In [Gaiko, 1997, 2001, 2003, 2005, 2008a, 2009a], we have already presented a solution of Hilbert's sixteenth problem in the quadratic case of polynomial systems proving that for quadratic systems four is really the maximum number of limit cycles and $(3:1)$ is their only possible distribution. We have also established some preliminary results on generalizing our ideas and methods to special cubic, quartic and other polynomial dynamical systems. In [Gaiko and van Horssen, 2004], e. g., we have constructed a canonical cubic dynamical system of Kukles type and have carried out the global qualitative analysis of its special case corresponding to a generalized Liénard equation. In particular, it has been shown that the foci of such a Liénard system can be at most of second order and that such system can have at most three limit cycles in the whole phase plane. Moreover, unlike all previous works on the Kukles-type systems, global bifurcations of limit and separatrix cycles using arbitrary (including as large as possible) field rotation parameters of the canonical system have been studied. As a result, a classification of all possible types of separatrix cycles for the generalized Liénard system has been obtained and all possible distributions of its limit cycles have been found. In [Gaiko and van Horssen, 2009a, 2009b], we have completed the global qualitative analysis of a planar Liénard-type dynamical system with a piecewise linear function containing an arbitrary number of dropping sections and approximating an arbitrary polynomial function. In [Botelho and Gaiko, 2006; Broer and Gaiko, 2010], we have carried out the global qualitative analysis of a centrally symmetric cubic system which is used as a learning model of a planar neural network and a quartic dynamical system which models the dynamics of the populations of predators and their prey in a given ecological system, respectively. In [Gaiko, 2011b], we have also completed the study of multiple limit cycle bifurcations in the well-known FitzHgh–Nagumo neuronal model. In [Gaiko, 2008b, 2009b, 2011a, 2012], we have presented a solution of Smale's Thirteenth Problem [Smale, 1998] proving that classical Liénard's system with a polynomial $f(x)$ of degree $2k+1$ and $g(x) \equiv x$ can have at most k limit cycles. Generalizing this result, we have also presented a solution of Hilbert's Sixteenth Problem [Gaiko, 2003] on the maximum number of limit cycles surrounding a singular point for an arbitrary polynomial system [Gaiko, 2011a, 2012].

In Section 2 of this paper, applying a canonical system with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we show how to control all possible limit cycle bifurcations of the general Liénard polynomial system (1.3) and present a solution of Hilbert's Sixteenth Problem for system (1.3) with an arbitrary (but finite) number of singular points.

2 Limit cycles of Liénard's polynomial system

By means of our bifurcationally geometric approach [Gaiko, 2008b, 2009b, 2011a, 2012], we will study the Liénard polynomial system (1.3). Its finite singularities are determined by the algebraic system

$$x\,(1+\beta_1\,x+\ldots+\beta_{2l}\,x^{2l})=0, \quad y=0. \tag{2.1}$$

It always has an anti-saddle at the origin and, in general, can have at most $2l+1$ finite singularities which lie on the x-axis and are distributed so that a saddle (or saddle-node) is followed by a node or a focus, or a center and vice versa [Bautin and Leontovich, 1990]. At infinity, system (1.3) has two singular points: a node at the "ends" of the x-axis and a saddle at the "ends" of the y-axis. For studying the infinite singularities, the methods applied in [Bautin and Leontovich, 1990] for Rayleigh's and van der Pol's equations and also Erugin's two-isocline method developed in [Gaiko, 2003] can be used; see also [Gaiko, 2008b, 2009b, 2011a, 2012].

Following [Gaiko, 2003], we will study limit cycle bifurcations of (1.3) by means of a canonical system containing field rotation parameters of (1.3) [Bautin and Leontovich, 1990; Gaiko, 2003].

Theorem 2.1. *The Liénard polynomial system* (1.3) *with limit cycles can be reduced to the canonical form*

$$\begin{aligned} &\dot{x}=y\equiv P(x,y),\\ &\dot{y}=-x\,(1+\beta_1 x\pm x^2+\ldots\\ &+\beta_{2l-1}x^{2l-1}\pm x^{2l})+y\,(\alpha_0+x+\alpha_2x^2\\ &+\ldots+x^{2k-1}+\alpha_{2k}x^{2k})\equiv Q(x,y), \end{aligned} \tag{2.2}$$

where $\beta_1,\beta_3,\ldots,\beta_{2l-1}$ *are fixed and* $\alpha_0,\alpha_2,\ldots,\alpha_{2k}$ *are field rotation parameters of* (2.2).

Proof. Let all the parameters α_i, $i=0,1,\ldots,2k$, vanish in system (2.2),

$$\begin{aligned} &\dot{x}=y,\\ &\dot{y}=-x\,(1+\beta_1\,x+\beta_2\,x^2+\ldots+\beta_{2l}\,x^{2l}), \end{aligned} \tag{2.3}$$

and consider the corresponding equation

$$\begin{aligned} \frac{dy}{dx}&=\frac{-x\,(1+\beta_1\,x+\beta_2\,x^2+\ldots+\beta_{2l}\,x^{2l})}{y}\\ &\equiv F(x,y). \end{aligned} \tag{2.4}$$

Since $F(x,-y)=-F(x,y)$, the direction field of (2.4) (and the vector field of (2.3) as well) is symmetric with respect to the x-axis. It follows that for arbitrary values of the parameters β_j, $j=1,2,\ldots,2l$, system (2.3) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, without loss of generality, all the even parameters β_j of system (1.3) can be supposed to be equal, e. g., to ±1: $\beta_2=\beta_4=\beta_6=\ldots=\pm1$.

Let now all the parameters α_i with even indexes and β_j with odd indexes vanish in system (2.2),

$$\begin{aligned} &\dot{x}=y,\\ &\dot{y}=-x\,(1\pm\,x^2\pm\ldots\pm\,x^{2l})\\ &+y\,(\alpha_1\,x+\alpha_3\,x^3+\ldots+\alpha_{2k-1}\,x^{2k-1}), \end{aligned} \tag{2.5}$$

and consider the corresponding equation

$$\begin{aligned} \frac{dy}{dx}&=\frac{-x\,(1\pm\,x^2\pm\ldots\pm\,x^{2l})}{y}\\ &+\alpha_1\,x+\alpha_3\,x^3+\ldots+\alpha_{2k-1}\,x^{2k-1}\\ &\equiv G(x,y). \end{aligned} \tag{2.6}$$

Since $G(-x,y)=-G(x,y)$, the direction field of (2.6) (and the vector field of (2.5) as well) is symmetric with respect to the y-axis. It follows that for arbitrary values of the parameters $\alpha_1,\alpha_3,\ldots,\alpha_{2k-1}$ system (2.3) has centers as anti-saddles and cannot have limit cycles surrounding these points. Therefore, without loss of generality, all the odd parameters α_i of system (1.3) can be supposed to be equal, e. g., to 1: $\alpha_1 = \alpha_3 = \ldots = \alpha_{2k-1} = 1$.

Inputting the odd parameters $\beta_1,\, \beta_3, \ldots,\, \beta_{2l-1}$ into system (2.5),

$$\begin{aligned} \dot{x} &= y \equiv R(x,y), \\ \dot{y} &= -x\,(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \ldots \\ &\quad + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y\,(x + x^3 + \ldots \\ &\quad + x^{2k-1}) \equiv S(x,y), \end{aligned} \tag{2.7}$$

and calculating the determinants

$$\begin{aligned} \Delta_{\beta_1} &= R\,S'_{\beta_1} - SR'_{\beta_1} = -x^2 y, \\ \Delta_{\beta_3} &= R\,S'_{\beta_3} - SR'_{\beta_3} = -x^4 y, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Delta_{\beta_{2l-1}} &= R\,S'_{\beta_{2l-1}} - SR'_{\beta_{2l-1}} = -x^{2l} y, \end{aligned}$$

we can see that the vector field of (2.7) is rotated symmetrically (in opposite directions) with respect to the x-axis and that the finite singularities (centers and saddles) of (2.7) moving along the x-axis (except the center at the origin) do not change their type or join in saddle-nodes. Therefore, we can fix the odd parameters $\beta_1, \beta_3, \ldots, \beta_{2l-1}$ in system (2.2), fixing the position of its finite singularities on the x-axis.

To prove that the even parameters $\alpha_0,\, \alpha_2, \ldots,\, \alpha_{2k}$ rotate the vector field of (2.2), let us calculate the following determinants:

$$\begin{aligned} \Delta_{\alpha_0} &= P\,Q'_{\alpha_0} - QP'_{\alpha_0} = y^2 \geq 0, \\ \Delta_{\alpha_2} &= P\,Q'_{\alpha_2} - QP'_{\alpha_2} = x^2 y^2 \geq 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Delta_{\alpha_{2k}} &= P\,Q'_{\alpha_{2k}} - QP'_{\alpha_{2k}} = x^{2k} y^2 \geq 0. \end{aligned}$$

By definition of a field rotation parameter [Bautin and Leontovich, 1990; Gaiko, 2003], for increasing each of the parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$, under the fixed others, the vector field of system (2.2) is rotated in the positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (2.2) is rotated in the negative direction (clockwise).

Thus, for studying limit cycle bifurcations of (1.3), it is sufficient to consider the canonical system (2.2) containing only its even parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ which rotate the vector field of (2.2) under the fixed others. The theorem is proved. $\square$

By means of the canonical system (2.2), let us study global limit cycle bifurcations of (1.3) and prove the following theorem.

Theorem 2.2. *The general Liénard polynomial system* (1.3) *can have at most $k + l$ limit cycles, k surrounding the origin and l surrounding one by one the other singularities of* (1.3).

Proof. According to Theorem 2.1, for the study of limit cycle bifurcations of system (1.3), it is sufficient to consider the canonical system (2.2) containing the field rotation parameters $\alpha_0,\, \alpha_2, \ldots,\, \alpha_{2k}$ of (1.3) under the fixed its parameters $\beta_1,\, \beta_3, \ldots,\, \beta_{2l-1}$.

Let all these parameters vanish:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 \pm x^2 \pm \ldots \pm x^{2l}) \\ &\quad + y\,(x + x^3 + \ldots + x^{2k-1}). \end{aligned} \tag{2.8}$$

System (2.8) is symmetric with respect to the y-axis and has centers as anti-saddles. Its center domains are bounded by either separatrix loops or digons of the saddles of (2.8) lying on the x-axis. If to input the parameters $\beta_1,\, \beta_3, \ldots,\, \beta_{2l-1}$ into (2.8) successively, we will get again system (2.7) the vector field of which is rotated symmetrically (in opposite directions) with respect to the x-axis. The finite singularities (centers and saddles) of (2.7) moving along the x-axis (except the center at the origin) do not change their type or join in saddle-nodes and the center domains will be bounded by separatrix loops of the saddles (or saddle-nodes) of (2.7) [Bautin and Leontovich, 1990; Gaiko, 2003].

Let us input successively the field rotation parameters $\alpha_0,\, \alpha_2, \ldots,\, \alpha_{2k}$ into system (2.7) beginning with the parameters at the highest degrees of x and alternating with their signs; see [Gaiko, 2008b, 2009b, 2011a, 2012]. So, begin with the parameter α_{2k} and let, for definiteness, $\alpha_{2k} > 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + \beta_1 x \pm x^2 \\ &\quad + \beta_3 x^3 \pm x^4 + \ldots + \beta_{2l-1} x^{2l-1} \pm x^{2l}) \\ &\quad + y\,(x + x^3 + \ldots + x^{2k-1} + \alpha_{2k} x^{2k}). \end{aligned} \tag{2.9}$$

In this case, the vector field of (2.9) is rotated in the positive direction (counterclockwise) turning the center at the origin into a nonrough (weak) unstable focus. All the other centers become rough unstable foci [Bautin and Leontovich, 1990; Gaiko, 2003].

Fix α_{2k} and input the parameter $\alpha_{2k-2} < 0$ into (2.9):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \ldots \\ &\quad + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y\,(x + x^3 + \ldots \\ &\quad + \alpha_{2k-2} x^{2k-2} + x^{2k-1} + \alpha_{2k} x^{2k}). \end{aligned} \tag{2.10}$$

Then the vector field of (2.10) is rotated in the opposite direction (clockwise) and the focus at the origin immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of x changes) generating a stable limit cycle. All the other foci will also generate stable limit cycles for some values of α_{2k-2} after changing the character of their stability. Under further decreasing α_{2k-2}, all the limit cycles will expand disappearing on separatrix cycles of (2.10) [Bautin and Leontovich, 1990; Gaiko, 2003].

Denote the limit cycle surrounding the origin by Γ_1, the domain outside the cycle by D_1, the domain inside the cycle by D_2 and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a "trajectory concentration" surrounding this singular point. It is clear that, under decreasing the parameter α_{2k-2}, a semi-stable limit cycle cannot appear in the domain D_2, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation [Gaiko, 2008b, 2009b, 2011a, 2012].

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain D_1. Suppose it appears in this domain for some values of the parameters $\alpha^*_{2k} > 0$ and $\alpha^*_{2k-2} < 0$. Return to system (2.7) and change the inputting order for the field rotation parameters. Input first the parameter $\alpha_{2k-2} < 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \ldots \\ &\quad + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y\,(x + x^3 + \ldots \\ &\quad + \alpha_{2k-2} x^{2k-2} + x^{2k-1}). \end{aligned} \tag{2.11}$$

Fix it under $\alpha_{2k-2} = \alpha^*_{2k-2}$. The vector field of (2.11) is rotated clockwise and the origin turns into a nonrough stable focus. Inputting the parameter $\alpha_{2k} > 0$ into (2.11), we get again system (2.10) the vector field of which is rotated counterclockwise. Under this rotation, a stable limit cycle Γ_1 will appear from a separatrix cycle for some value of α_{2k}. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing α_{2k} to the value α^*_{2k}. It follows that there are no values of $\alpha^*_{2k-2} < 0$ and $\alpha^*_{2k} > 0$ for which a semi-stable limit cycle could appear in the domain D_1.

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (2.10) for any values of the parameters α_{2k-2} and α_{2k} of different signs. Obviously, if these parameters have the same sign, system (2.10) has no limit cycles surrounding the origin at all. On the same reason, this system cannot have more than l limit cycles surrounding the other singularities (foci or nodes) of (2.10) one by one.

Let system (2.10) have the unique limit cycle Γ_1 surrounding the origin and l limit cycles surrounding the other antisaddles of (2.10). Fix the parameters $\alpha_{2k} > 0$, $\alpha_{2k-2} < 0$ and input the third parameter, $\alpha_{2k-4} > 0$, into this system:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + \beta_1 x \pm x^2 + \ldots + \beta_{2l-1} x^{2l-1} \\ &\quad \pm x^{2l}) + y\,(x + x^3 + \ldots + \alpha_{2k-4} x^{2k-4} \\ &\quad + \alpha_{2k-2} x^{2k-2} + x^{2k-1} + \alpha_{2k}\, x^{2k}). \end{aligned} \tag{2.12}$$

The vector field of (2.12) is rotated counterclockwise, the focus at the origin changes the character of its stability and the second (unstable) limit cycle, Γ_2, immediately appears from this point. The limit cycles surrounding the other singularities of (2.12) can only disappear in the corresponding foci (because of their roughness) under increasing the parameter α_{2k-4}. Under further increasing α_{2k-4}, the limit cycle Γ_2 will join with Γ_1 forming a semi-stable limit cycle, Γ_{12}, which will disappear in a "trajectory concentration" surrounding the origin. Can another semi-stable limit cycle appear around the origin in addition to Γ_{12}? It is clear that such a limit cycle cannot appear either in the domain D_1 bounded on the inside by the cycle Γ_1 or in the domain D_3 bounded by the origin and Γ_2 because of the increasing distance between the spiral coils filling these domains under increasing the parameter α_{2k-4} [Gaiko, 2008b, 2009b, 2011a, 2012].

To prove the impossibility of the appearance of a semi-stable limit cycle in the domain D_2 bounded by the cycles Γ_1 and Γ_2 (before their joining), suppose the contrary, i. e., that for some set of values of the parameters, $\alpha^*_{2k} > 0$, $\alpha^*_{2k-2} < 0$, and $\alpha^*_{2k-4} > 0$, such a semi-stable cycle exists. Return to system (2.7) again and input first the parameters $\alpha_{2k-4} > 0$ and $\alpha_{2k} > 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \ldots \\ &\quad + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y\,(x + x^3 + \ldots \\ &\quad + \alpha_{2k-4} x^{2k-4} + x^{2k-3} + \alpha_{2k} x^{2k}). \end{aligned} \tag{2.13}$$

Both parameters act in a similar way: they rotate the vector field of (2.13) counterclockwise turning the origin into a nonrough unstable focus.

Fix these parameters under $\alpha_{2k-4} = \alpha^*_{2k-4}$, $\alpha_{2k} = \alpha^*_{2k}$ and input the parameter $\alpha_{2k-2} < 0$ into (2.13) getting again system (2.12). Since, by our assumption, this system has two limit cycles surrounding the origin for $\alpha_{2k-2} > \alpha^*_{2k-2}$, there exists some value of the parameter, α^{12}_{2k-2} ($\alpha^*_{2k-2} < \alpha^{12}_{2k-2} < 0$), for which a semi-stable limit cycle, Γ_{12}, appears in system (2.12) and then splits into a stable cycle, Γ_1, and an unstable cycle, Γ_2, under further decreasing α_{2k-2}. The formed domain D_2 bounded by the limit cycles Γ_1, Γ_2 and filled by the spirals will enlarge since, on the properties of a field rotation parameter, the interior unstable limit cycle Γ_2 will contract and the exterior stable limit cycle Γ_1 will expand under decreasing α_{2k-2}. The distance between the spirals of the domain D_2 will naturally increase, which will prevent the appearance of a semi-stable limit cycle in this domain for $\alpha_{2k-2} < \alpha^{12}_{2k-2}$ [Gaiko, 2008b, 2009b, 2011a, 2012].

Thus, there are no such values of the parameters, $\alpha^*_{2k} > 0$, $\alpha^*_{2k-2} < 0$, and $\alpha^*_{2k-4} > 0$, for which system (2.12) would have an additional semi-stable limit cycle surrounding the origin. Obviously, there are no other values of the parameters α_{2k}, α_{2k-2}, and α_{2k-4} for which system (2.12) would have more than two limit cycles surrounding this singular point. On the same reason, additional semi-stable limit cycles cannot appear around the other singularities (foci or nodes) of (2.12). Therefore, $2 + l$ is the maximum number of limit cycles in system (2.12).

Suppose that system (2.12) has two limit cycles, Γ_1 and Γ_2, surrounding the origin and l limit cycles surrounding the other antisaddles of (2.12) (this is always possible if $\alpha_{2k} \gg -\alpha_{2k-2} \gg \alpha_{2k-4} > 0$). Fix the parameters α_{2k}, α_{2k-2}, α_{2k-4} and consider a more general system inputting the fourth parameter, $\alpha_{2k-6} < 0$, into (2.12):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + \beta_1 x \pm x^2 + \beta_3 x^3 \pm x^4 + \ldots \\ &\quad + \beta_{2l-1} x^{2l-1} \pm x^{2l}) + y\,(x + x^3 + \ldots \\ &\quad + \alpha_{2k-6} x^{2k-6} + x^{2k-5} + \ldots + \alpha_{2k} x^{2k}). \end{aligned} \tag{2.14}$$

For decreasing α_{2k-6}, the vector field of (2.14) will be rotated clockwise and the focus at the origin will immediately change the character of its stability generating a third (stable) limit cycle, Γ_3. With further decreasing α_{2k-6}, Γ_3 will join with Γ_2 forming a semi-stable limit cycle, Γ_{23}, which will disappear in a "trajectory concentration" surrounding the origin; the cycle Γ_1 will expand disappearing on a separatrix cycle of (2.14).

Let system (2.14) have three limit cycles surrounding the origin: Γ_1, Γ_2, Γ_3. Could an additional semi-stable limit cycle appear with decreasing α_{2k-6} after splitting of which system (2.14) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear either in the domain D_2 bounded by the cycles Γ_1 and Γ_2 or in the domain D_4 bounded by the origin and Γ_3 because of the increasing distance between the spiral coils filling these domains after decreasing α_{2k-6}. Consider two other domains: D_1 bounded on the inside by the cycle Γ_1 and D_3 bounded by the cycles Γ_2 and Γ_3. As before, we will prove the impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters $\alpha^*_{2k} > 0$, $\alpha^*_{2k-2} < 0$, $\alpha^*_{2k-4} > 0$, and $\alpha^*_{2k-6} < 0$ such a semi-stable cycle exists. Return to system (2.7) again, input first the parameters $\alpha_{2k-6} < 0$, $\alpha_{2k-2} < 0$ and then the parameter $\alpha_{2k} > 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x\,(1 + \beta_1 x \pm x^2 + \ldots + \beta_{2l-1} x^{2l-1} \\ &\quad \pm x^{2l}) + y\,(x + x^3 + \ldots + \alpha_{2k-6} x^{2k-6} \\ &\quad + \ldots + \alpha_{2k-2} x^{2k-2} + x^{2k-3} + \alpha_{2k} x^{2k}). \end{aligned} \tag{2.15}$$

Fix the parameters α_{2k-6}, α_{2k-2} under the values α^*_{2k-6}, α^*_{2k-2}, respectively. With increasing α_{2k}, a separatrix cycle formed around the origin will generate a stable limit cycle, Γ_1. Fix α_{2k} under the value α^*_{2k} and input the parameter $\alpha_{2k-4} > 0$ into (2.15) getting system (2.14).

Since, by our assumption, (2.14) has three limit cycles for $\alpha_{2k-4} < \alpha^*_{2k-4}$, there exists some value of the parameter α^{23}_{2k-4} $(0 < \alpha^{23}_{2k-4} < \alpha^*_{2k-4})$ for which a semi-stable limit cycle, Γ_{23}, appears in this system and then splits into an unstable cycle, Γ_2, and a stable cycle, Γ_3, with further increasing α_{2k-4}. The formed domain D_3 bounded by the limit cycles Γ_2, Γ_3 and also the domain D_1 bounded on the inside by the limit cycle Γ_1 will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there [Gaiko, 2008b, 2009b, 2011a, 2012].

All other combinations of the parameters α_{2k}, α_{2k-2}, α_{2k-4}, and α_{2k-6} are considered in a similar way. It follows that system (2.14) can have at most $3 + l$ limit cycles.

If we continue the procedure of successive inputting the even parameters, $\alpha_{2k}, \ldots, \alpha_2, \alpha_0$, into system (2.7), it is possible first to obtain k limit cycles surrounding the origin $(\alpha_{2k} \gg -\alpha_{2k-2} \gg \alpha_{2k-4} \gg -\alpha_{2k-6} \gg \alpha_{2k-8} \gg \ldots)$ and then to conclude that the canonical system (2.2) (i. e., the Liénard polynomial system (1.3) as well) can have at most $k + l$ limit cycles, k surrounding the origin and l surrounding one by one the antisaddles (foci or nodes) of (2.2) (and (1.3) as well). The theorem is proved. □

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