DELAYED FEEDBACK CONTROL OF DELAYED CHAOTIC SYSTEMS

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Abstract
Although there are a number of theoretical methodologies to use time delayed feedback controller for controlling chaos, useful algorithms are still lacking. In this paper, some algorithms to adjust parameters of time delayed feedback controller derived from theoretical results in time domain and frequency domain are presented. Comparison of theses approaches based on computer simulations is included.

Key words
Delayed chaotic system, delayed feedback.

1 Introduction

Delay differential equations (DDEs) have attracted much attention in the field of nonlinear dynamics. They are used to model complex phenomena like physiological diseases by [Mackey & Glass(1977)], population dynamics by [May(1976)], neural networks by [Wei & Ruan(1999)], nonlinear optical devices by [Ikeda & Matsumoto(1987); Lang & Kobayashi(1980)] and [Lu & He(1996)]. Many of these systems are accurately described by a single scalar DDE of the form

$$\dot{x}(t) = -x(t) + f_b(x(t-\tau)), \qquad (1)$$

where b and τ are bifurcation parameters and $f_b(y)$ denotes a nonlinear function of y. Response of the system can be very rich ranging from periodic to high dimensional chaotic outputs in a certain range of bifurcation parameters.

Because of unpredictable nature of chaos, it is often viewed as an undesirable characteristic of practical devices.To utilize or to eliminate chaotic oscillations, researchers have recently been paying increasing attention to chaos control (cf. [Schöll et al.(2008)] for a recent overview of topics in chaos control). Among the chaos control methodologies, time delayed feedback controller (TDFC) has been receiving considerable attention since it was proposed by [Pyragas(1992)].

The TDFC consists of a linear feedback from the difference between the current state and the delayed state of the system

$$u(t) = k(x(t-T) - x(t)). \qquad (2)$$

Control input vanishes when the stabilization of the desired orbit or fixed point is attained.

From the control theory viewpoint, there are two important problems to use TDFC: The first to find the delay time T and control gain k analytically when the original system is known; the second is stability analysis of the closed loop system under TDFC.

Recently [Vasegh & Khaki-Sedigh(2008)] and (2009) have introduced two analytical approaches to study the behaviors of time delayed chaotic system under TDFC: Time domain approach and Frequency domain approach.

In this paper, first a brief introduction of the above approaches are presented. Also, some new results in the frequency domain are presented to determine critical bifurcation points in section 2. These consist of section 2. In section 3, these results are summarized in some useful practical algorithms to determine controller parameters T and k to stabilize unstable fixed points (FPs) and periodic orbits (POs). Some numerical examples are used to establish usefulness of the proposed algorithms. Section 4 includes comparison of the above two approaches. It is shown that although the analytical pedestal of the methods are different, the results confirm each other. Finally, the paper is closed with some conclusions.

2 Analytical Results

2.1 Time domain results

In this subsection τ is used as the bifurcation parameter. Let x_0 be the FP of (1) so that the linearized model around it becomes

$$\dot{z}(t) = -z(t) + \mu z(t - \tau), \tag{3}$$

where $\mu = f_b'(x_0)$. [Vasegh & Khaki-Sedigh(2008)] have shown that x_0 is locally asymptotically stable if $|\mu| < 1$ and unstable if $\mu > 1$. If $\mu < -1$, a sequence of Hopf bifurcation occurs at the critical delays:

$$\tau_j = \frac{1}{\omega_0}((j+1)\pi - \arctan(\omega_0)), \quad \omega_0 = \sqrt{\mu^2 - 1}. \tag{4}$$

When $\mu < -1$, the changes in the qualitative behavior of (1) as the parameter τ is varied are as follows: x_0 is locally asymptotically stable if $\tau < \tau_0$; For $\tau_0 < \tau < \tau_1$, there is a stable limit cycle. A period-doubling bifurcation sequence and chaos are observed when τ is more increased ([Giannakopoulos & Zapp(1999)]).

Now, assume that (1) behaves chaotically. Controller (2) can be used to stabilize unstable FPs or POs embedded in chaotic attractor. The closed loop model is as follows

$$\dot{x}(t) = -x(t) + f_b(x(t-\tau)) + k(x(t-T) - x(t)). \tag{5}$$

[Vasegh & Khaki-Sedigh(2008)] proved that (2) cannot stabilize x_0 if $\mu > 1$. But if $\mu < -1$, one has two choices to stabilize x_0: first set $T = \tau$ and choose $k \geq -(1+\mu)/2$; second set $T \ll 1$ and choose $k > -(1 + \mu\tau)/T$.

To stabilize unstable PO embedded in the chaotic attractor by TDFC, it has been proposed to set T such that the bifurcation critical point of open loop model preserves in the closed loop model. It means the open loop roots $\pm i\omega$ at $\tau = \tau_j$ must be the roots of the closed loop characteristic equation:

$$\lambda + \mu e^{-\tau_j \lambda} - k e^{-T\lambda} + k + 1 = 0 \tag{6}$$

The feedback gain k must be adjusted such that $\pm i\omega$ are the rightmost roots of (6).

2.2 Frequency domain results

To use frequency domain tools, model (1) is transformed into the equivalent feedback structure as Fig. 1 ([Vasegh & Khaki-Sedigh (2009)]), where a linear subsystem $L_o(s)$ is connected to a nonlinear one defined by

$$L_o(s) = \frac{e^{-\tau s}}{s+1}, \quad n(x) = -f_b(x). \tag{7}$$

Assume that b is the bifurcation parameter. If x_0 is the equilibrium point of (1) (or equivalently (7)), then

$$x_0 + L_o(0)n(x_0) = 0 \tag{8}$$

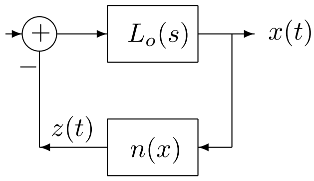

Figure 1. Lur'e form of the model (1).

The stability of x_0 can be obtained by the Nyquist plot of $L_o(j\omega)$ which is shown in Fig. 2 .

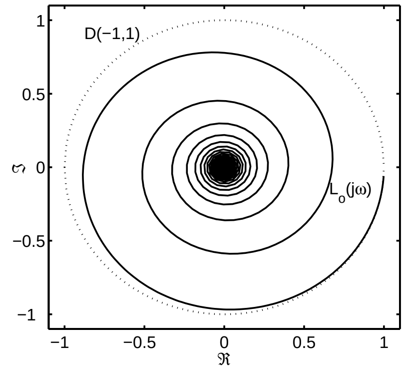

Figure 2. Nyquist plot of $L_o(j\omega)$ lies in disc $D(-1, 1)$.

Since $L_o(j\omega)$ is Hurwitz and its Nyquist plot lies in the disc $D(-1, 1)$, by using the circle criterion ([Cook(1994)]), x_0 is stable if $|\mu| = |f'(x_0)| < 1$ which is in good agreement with the results obtained in time domain. To study periodic solutions let

$$x(t) = A + B\cos(\omega t). \tag{9}$$

From the well known describing function method, A, bias of periodic solution; B, amplitude of periodic solution and ω the frequency of periodic solution are obtained from the following equations:

$$1 + N_0(A, B) = 0, \tag{10}$$

$$\tan(\tau\omega) = -\omega, \tag{11}$$

$$1 + \cos(\tau\omega)N_1(A, B) = 0. \tag{12}$$

where

$$N_0(A, B) = \frac{-1}{2\pi A}\int_{-\pi}^{\pi} f_b(A + B\cos(\omega t))d\omega t \tag{13}$$

$$N_1(A, B) = \frac{-1}{\pi B}\int_{-\pi}^{\pi} f_b(A + B\cos(\omega t))\cos(\omega t)d\omega t \tag{14}$$

One can see that (11) predicts many limit cycles.

Now, we find bifurcation critical points of μ. Assume the bifurcation parameter μ (or equivalently b) is near by its critical point μ^*. So the amplitude of periodic solution B is almost small and $A \approx x_0$. By using Taylor expansion for $f_b(x)$ around x_0 we have

$$f_b(x_0 + B\cos(\omega t)) = f(x_0) + f'(x_0)B\cos(\omega t) + \cdots$$

$$= x_0 + \mu^* B\cos(\omega t) + \cdots .$$

Substitute it in (14), we have

$$N_1 \approx \frac{-1}{\pi B}\int_{-\pi}^{\pi}[x_0 + \mu^* B\cos(\omega t)]\cos(\omega t)d\omega t = -\mu^*$$

Compare this with (12) we find that

$$\mu_i^* = -1/\cos(\tau\omega_i) \tag{15}$$

where ω_i is obtained by (11). Similarly we can find that

$$N_0 \approx \frac{-1}{2\pi x_0}\int_{-\pi}^{\pi}[x_0 + \mu^* B\cos(\omega t)]d\omega t = -1$$

which satisfies (10).

Alternatively, if (A_i, B_i, ω_i) is a solution of (10)-(12), the stability of predicted PO can be determined by the following condition ([Cook(1994)]):

$$\frac{\partial}{\partial B}N_1(A,B)|_{(A_i,B_i)} \operatorname{Im}\left\{\frac{d}{d\omega}(L_o(j\omega))\Big|_{\omega=\omega_i}\right\} < 0 \tag{16}$$

If the system is chaotic, one can stabilize the FP x_0 or predicted PO embedded in chaotic attractor by TDFC. To do this, consider the closed loop model (5) which can be modeled as the following feedback structure ([Vasegh & Khaki-Sedigh (2009)])

$$L_c(s) = \frac{e^{-\tau s}}{s + 1 + k - ke^{-Ts}}, \qquad n(x) = -f_b(x). \tag{17}$$

Since $L_c(j\omega)$ is Hurwitz we can use the circle criterion ([Cook(1994)]). Let

$$h(T,k) = \inf_{\omega\geq 0} \operatorname{Re}\big(L_c(j\omega)\big). \tag{18}$$

Then the Nyquist plot of $L_c(j\omega)$ lies in the right half plane of $\operatorname{Re}(s) = h(T,k)$. If for some T and k we have $\mu h(k,T) < 1$, then x_0 will become locally asymptotically stable.

To extract one of the periodic solution embedded in chaotic attractor, first we find the frequency of PO of the open loop model ω_i. So that T must be set $2\pi/\omega_i$. The feedback gain k is determined such that $L_c(j\omega)$ filters other periodic solutions.

3 Algorithms

In this section, some algorithms are affirmed based on the analytical results of last section. First, we use the time domain outcomes.

Algorithm 1 Stabilizing unstable periodic orbit
Data: $\tau > 0$ and $\mu < -1$;
Results: Time delay T and a range for feedback gain k;
Steps:

1. Let $\omega_0 = \sqrt{\mu^2 - 1}$;
2. Set $T = \frac{2\pi}{\omega_0}$;
3. Find τ_j such that $\tau_j \leq \tau < \tau_{j+1}$;
4. Find a range for k such that for τ_j and T of step (1), the rightmost roots of (6) being $\pm i\omega_0$.

Remark 1. $T = \frac{2\pi}{\omega_0}$ preserves the bifurcation critical points τ_j.

Remark 2. To find a suitable range of k one can use some numerical approach such as continuous pole placement ([Michielsl et al.(2002)]).

Remark 3. The step (4) ensured that all dynamics are independent of periodic solution and the periodic solution corresponding to characteristic roots $\pm i\omega_0$ are stable.

The following example illustrates the controller design procedure to stabilize unstable FP and PO. Consider Logistic model, which is one of the famous systems in the class of (1),

$$\dot{x}(t) = -x(t) + bx(t-\tau)(1 - x(t-\tau)). \tag{19}$$

This model has two FPs $x_1 = 0, x_2 = b^{-1}(b-1)$. Also, we have $f'(0) = b$ and $f'(x_2) = 2 - b$. [Jiang et al.(2006)] have shown that for $b = 4$, (19) may behave chaotically for some values of τ. Chaotic attractor of (19) is shown in Fig. 3 for $\tau = 5$.

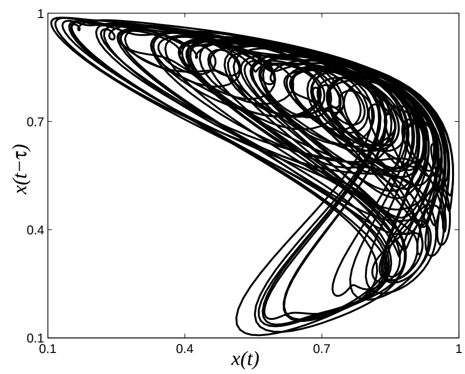

Figure 3. Chaotic attractor of Logistic model for $\tau = 5$

The origin cannot be stabilized by TDFC but x_2 becomes stable if we set ($T = 0.5$, $k = 10$) or ($T = 4$, $k = 0.5$). The response of closed loop are shown in Fig. 4 for both selections of controller parameters.

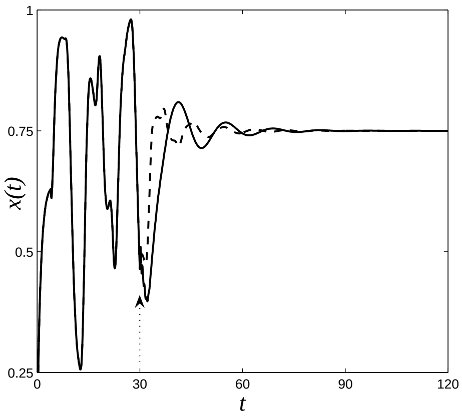

Figure 4. $x_2 = 0.75$ is stabilized by TDFC where $T = .5,\ k = 10$ (dash line) and $T = 5,\ k = 0.5$ (solid line).

Now to stabilize the unstable PO, we apply algorithm 1. Here $\omega_0 = \sqrt{(2-b)^2 - 1} = \sqrt{3}$ so we may set $T = 2\pi/\sqrt{3} = 3.6276$. Since

$$\tau_5 \simeq 4.8 < \tau < \tau_6 \simeq 5.7,$$

we must find the range of k such that $\pm i\omega_0$ are the rightmost roots of (6) for $\tau = \tau_5$ and $T = 3.6276$. By numerical method we find that the above conditions are satisfied by $0.75 \leq k \leq 4$. We choose $k = 1$ and apply TDFC to the model. The stabilized PO in phase space of the closed loop system is shown in Fig. 5.

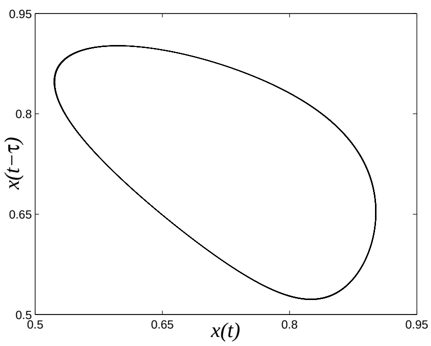

Figure 5. Stable periodic solution of the closed loop model with $T = 3.6276$ and $k = 1$

Now, the frequency domain approaches to stabilize FP and PO are stated in the following two algorithms:

Algorithm 2 Stabilizing x_0
Data: $\tau > 0$ and $\mu < -1$
Results: The suitable ranges of parameters T and k;
Steps:

1. Let $T_{\max} = 2\tau$, $k_{max} = -3\mu$;
2. Plot surface $h((T,k)$ for $0 < T \leq T_{max}$ and $0 < k \leq k_{max}$;
3. Plot plane $z(T,k) = -1/\mu$;
4. All the points of $h(T,k)$ on the top of the plane $z(T,k)$ determine suitable parameters of TDFC.

Remark 4. This choice of $T_{\max}$ and k_{max} guarantees the existence of T and k to stabilize model (1).

By applying this algorithm to model (19) almost all $(T,\ k)$ that stabilize FP, are obtained. These solutions are plotted in Fig. 6.

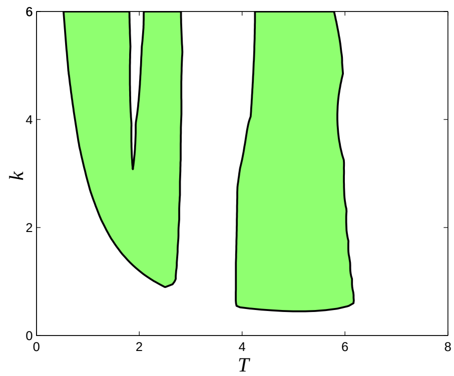

Figure 6. Projection of $h(k,T) > -1/\mu$ on the plane(k,T)

Time response of the closed loop model for $k = 3$, $T = 1.5$ is shown in Fig. 7. These parameters cannot be obtained by the proposed time domain method.

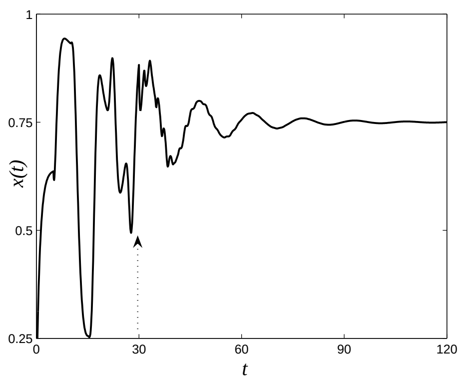

Figure 7. Time response of closed loop model for $T = 1.5$ and $k = 3$.

The frequency method to take out PO of chaotic attractor is summarized as follows:

Algorithm 3 Stabilizing Periodic Orbits
Data: $\tau > 0$ and $f_b(x)$
Results: Time delay T and a range for feedback gain k;

1. Compute N_0 and N_1 using (13) and (14);

2. Compute all solutions of (11) satisfy $0 < \omega_i < 2$;
3. Find A_i and B_i, bias and amplitude of the corresponding ω_i by (10) and (12);
4. Check stability of the PO:(A_i, B_i, ω_i);
5. Select one of the stable PO and set $T = 2\pi/\omega_i$;
6. Determine k such that $2|L_c(j\omega_i)|^2 < |L_o(j\omega_i)|^2$;

Remark 5. The limitation of ω_i in step 2 is applied because $L_o(j\omega)$ is a low pass filter with bandwidth $\omega_c = 1$.
Remark 6. TDFC dose not change the stability of PO. ([Vasegh & Khaki-Sedigh (2009)]).
Remark 7. By Step 6 one ensures that the closed loop model rejects undesired PO.
By applying this algorithm to model (19) we have

$$N_0 = b(A^2 - A + B^2/2)/A, \quad N_1 = b(2A-1) \quad (20)$$

and $\omega_1 = 0.531$, $\omega_3 = 1.6785$. So we select the corresponding POs from chaotic attractor with period $T_1 = 11.83$ and $T_3 = 3.74$. $k = 1$ satisfies step 6. Fig. 8 shows the phase space of the closed loop model after transition time.

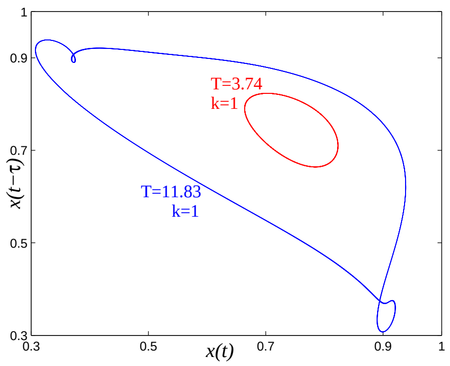

Figure 8. Phase space of the closed loop model after transition time.

4 Comparison study

The time and the frequency domain methods presented in section 2 are different methods to solve the same problem. Different approaches to design TDFC, provide different properties of the controller. In this section we provide a comparison study of the two proposed methods. Moreover it is shown that these two methods confirm each other.

4.1 Robustness of the methods

In the time domain method, it is assumed that b is fixed and τ is the bifurcation parameter. It is found that the proposed method to find controller parameters T and k is robust against the variation of bifurcation parameter τ. Fig 9 shows the bifurcation diagram of the closed loop model with respect to τ. It shows that although the controller is designed for fixed $\tau = 5$, but the period one is remained stable for a large range of τ. In

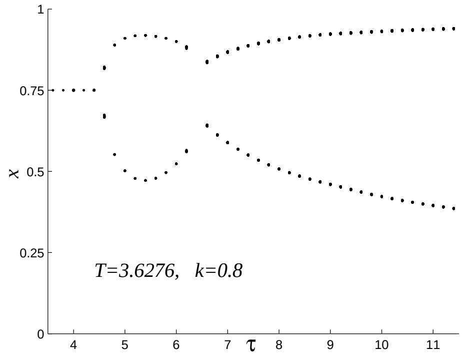

Figure 9. Bifurcation diagram of the closed loop system with respect to τ.

frequency domain method, τ is fixed and b is the bifurcation parameter and the proposed method is robust against the variation of bifurcation parameter b. Fig 10 shows the bifurcation diagram of the closed loop model with respect to b. Again, although the controller is designed for $b = 4$, the period one is remained stable for a large range of b. Note that the Lgistic model is unstable and the trajectories of the open loop model goes to infinity for $b > 4.2$, but period one is stable in closed loop model for $b \leq 5$

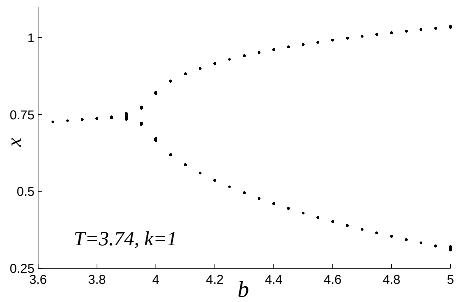

Figure 10. Bifurcation diagram of the closed loop system with respect to b.

4.2 Similarity of the methods

In this subsection it is shown that the two proposed methods lead us to the unique relations to find the controller parameters based on the free system analysis. Again see (4) to find critical delay time τ and (15)-(11)

to find critical μ:

$$\tau_j = \frac{1}{\omega_0}((j+1)\pi - \arctan(\omega_0)), \quad \omega_0 = \sqrt{\mu^2 - 1}, \tag{21}$$

and

$$\mu_i = -1/\cos(\tau\omega_i), \quad \tan(\tau\omega_i) = -\omega_i. \tag{22}$$

By (21) we have $\tan(\tau_j\omega_0) = -\omega_0$ which confirms (22).

Also (22) implies that $\mu_i^2 = 1 + \omega_i^2$ which confirms (21).

Note that in (21), μ is fixed and τ_j is calculated in term of μ but in (22), τ is fixed and μ_i is calculated in term of τ. It means two presented approach validate each other.

Although, open loop analysis leads to similar results, there is significant difference in the delay time of the controllers. By using algorithm 1, the period of the closed loop is the same as T just for $\tau = \tau_j$, the shape of the stabilized periodic orbit and the periodic orbit of the open loop model at bifurcation critical point τ_0 are alike which is shown in Fig. 11. For other τ the control

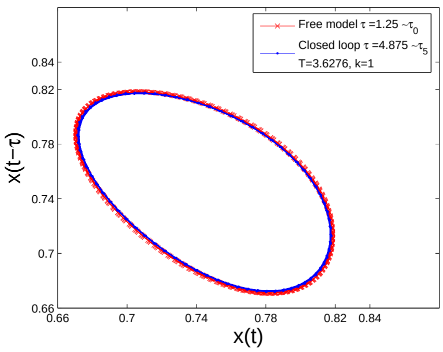

Figure 11. Periodic solutions of the free and closed loop model near bifurcation critical points.

effort $u(t)$ in (2) dose not converge to zero.

If we use algorithm 3 to find the controller parameters and then apply it to the chaotic model, the period of the stabilized PO dose not change by varying bifurcation parameter b. But strongly depends on the delay time τ of the model since the period of PO of the free model is determined by solving the nonlinear equation (11).

5 Conclusion

In this paper chaos control is studied from two viewpoints: Time and Frequency domains. Practical algorithms are given in each approach.

Almost all the suitable parameters of TDFC are determined by the fixed point stabilizing algorithm (algorithm 2). In applying algorithms 1 and to stabilize unstable POs. Note that: if delay time τ is fixed and b varies, algorithm 3 is suitable and if b is fixed and τ varies, algorithm 1 is appropriate.

In both methods adjusting delay time T is more important than feedback gain k. In many works ([Schöll et al.(2008)]) it is shown that a suitably chosen T can broaden the allowed range of k. So one can use (9) or (11) to find the frequency of unstable POs of systems (and also period of solutions) and then adjust k practically. Also by using (11), more than one periodic solution can be found.

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