On nonlinear resonance oscillations of a spring supported point particle.

M.M. Shunderyuk*
IPMecRAS
Moscow, Russia

Abstract— *Full research of flat small nonlinear oscillations of a spring pendulum with nonlinear dependence of a tension of a spring on its lengthening is conducted. The method of a hamiltonian normal form is used. For reduction to a hamiltonian normal form the method of invariant normalisation is used, what essentially reduces calculations. Solutions of the normal form equations have shown that periodic reorganisation between vertical and horizontal oscillations occurs only in case of resonances 1:1 and 2:1. At a resonance 2:1 this effect is shown in square-law members of the equation, and at a resonance 1:1 one should take into account cubic members. In all other cases, both in the presence of a resonance, and at its absence, oscillations have constant frequencies with a little different from frequencies of linear approach. For a resonance 2:1 it is found maximum detuning of frequencies at which the effect of swapping of energy from one kind of oscillation to another disappears. The resonance 1:1 is physically possible only for a spring possessing the negative cubic term in the law of deformation.*

Keywords: Spring pendulum, hamilton system, normal form, nonlinear oscillations.

I. A normal form of a Hamilton system [2].

To simplify our reasoning, we shall limit ourselves by two degrees of freedom, although all the conclusions are extended to the case of finite degrees of freedom. Let $(\mathbf{q},\mathbf{p}) \stackrel{\text{def}}{=} (q_1,q_2,p_1,p_2)$: be dependent variables, $H = H(\mathbf{q},\mathbf{p})$: be Hamilton function of Hamilton system

$$\dot{q}_i = \partial H/\partial p_i,\ \ \dot{p}_i = -\partial H/\partial q_i,\ \ i = 1,2, \qquad (0.1)$$

where the dot means d/dt. Let $\mathbf{q} = \mathbf{p} = 0$: be a fixed point of system (1) and function $H = H(\mathbf{q},\mathbf{p})$: in it be analytical. Then function H can be represented as an expansion in powers of q,p which starts with quadratic terms while power expansions of the right parts of system (0.1) start with linear members. Let R be a matrix of a linear part of system (0.1). Eigenvalues $\lambda_1,\ldots,\lambda_4$: of matrix R are split into pairs $\lambda_{j+2} = -\lambda_j$, $j = 1,2$:. By means of a canonic linear variable change: $(\mathbf{q},\mathbf{p})^* = B(\mathbf{x},\mathbf{y})^*$, where * means transposition, Matrix R can be reduced to Jordan complex-valued normal form, in which eigenvalues $\lambda_1,\ldots,\lambda_4$: are located diagonally. Then: $H(\mathbf{q},\mathbf{p}) = \tilde{H}(\mathbf{x},\mathbf{y})$. Let the canonic linear complex variable change:

$$(\mathbf{x},\mathbf{y}) = (\mathbf{u},\mathbf{v}) + \mathbf{N}(\mathbf{u},\mathbf{v}), \qquad (0.2)$$

where $\mathbf{N} \stackrel{\text{def}}{=} (N_1,\ldots,N_4)$, $N_j(\mathbf{u},\mathbf{v})$ are power series without constant and linear terms reduce Hamilton function: $\tilde{H}(\mathbf{x},\mathbf{y})$ to

$$h(\mathbf{u},\mathbf{v}) \stackrel{\text{def}}{=} \sum h_{\mathbf{s}}\,\mathbf{w}^{\mathbf{s}}, \qquad (0.3)$$

where $\mathbf{w} \stackrel{\text{def}}{=} (w_1,\ldots,w_4) \stackrel{\text{def}}{=} (\mathbf{u},\mathbf{v})$, $\mathbf{s} = (s_1,\ldots,s_4)$, $\mathbf{w}^{\mathbf{s}} \stackrel{\text{def}}{=} w_1^{s_1}\ldots w_4^{s_4} \stackrel{\text{def}}{=} u_1^{s_1}u_2^{s_2}v_1^{s_3}v_2^{s_4}$. Hamilton formal function is called *a complex normal form* provided that:

1) the linear part matrix of the relevant Hamilton system is of normal form with eigenvalues: $\lambda_1,\lambda_2,-\lambda_1,-\lambda_2$, located diagonally

2) in expansion (0.3) there are only resonance terms for which

$$(s_1 - s_3)\lambda_1 + (s_2 - s_4)\lambda_2 = 0. \qquad (0.4)$$

[1] proves that for every system (0.1) there is a formal change (0.2) which leads Hamilton function: $H(\mathbf{q},\mathbf{p})$ to a normal form (0.3), (0.4). In accordance with [4], if the initial system (0.1) is real, there is a real normal form which can be reduced to a complex normal form (0.3), (0.4) by a standard linear change of coordinates.

Special cases of such a normal form are those of Birkhoff [3] and of Cherry and Gustavson. Birkhoff [3] analyzed a case with all eigenvalues incommensurable, so equation (0.4) in which $s_1,\ldots,s_4$ are integers, has a trivial solution $s_1 - s_3 = s_2 - s_4 = 0$. In this case expansion (0.4) is a series of products: $(u_1v_1)^{s_1}(u_2v_2)^{s_2}$ and every such a product is a formal integral of the relevant Hamilton system. Cherry considered a case where eigenvalues λ_1 and λ_2 are different. Gustavson came to the same result. Belitskii suggested an expanded normal form in which Jordan boxes of matrix linear part are used to further reduce the number of nonlinear members. A more detailed review of other expanded normal forms is given in [4],[2].

*E-mail: shunderyuk@gmail.com

II. Methods of normal form computation

Computation algorithms of canonical normalizing transformations (0.2) and normal forms (0.3), (0.4) are classified into three groups according to the form of canonical transformation. There are now three forms of canonical transformations: A. by means of a generating function; B. by means of Lie series; C. parametric. Thus, we will refer algorithms to one of the three groups above depending on the canonical transformation used.

Description of algorithms.

A. The generating function to compute a normal form was first introduced by Jacobi [3]-[8]. According to this method, vector series: $\mathbf{N}(\mathbf{u},\mathbf{v})$ in nonlinear formal transformation is computed using generating function $g(\mathbf{x},\mathbf{v}) = x_1v_1 + \ldots + x_2v_2 + \ldots$ of mixed variables $\mathbf{x} = (x_1, x_2)$ $\mathbf{v} = (v_1, v_2)$, while

$$\begin{aligned} u_j = \partial g/\partial v_j = x_j + \ldots, \\ y_j = \partial g/\partial x_j = v_j + \ldots, j = 1, 2. \end{aligned} \tag{0.5}$$

If the generating series $g(\mathbf{x},\mathbf{v})$ is computed, it is necessary to express x_j with the help of $\mathbf{u},\mathbf{v}$ to obtain transformation (0.2), thus to invert power series for u_j. This results in a highly complicated computation, however always applicable (with no matrix R limitations).

B. For normalizing with the help of Lie series, scaling $\mathbf{q} = \varepsilon\mathbf{q}'$, $\mathbf{p} = \varepsilon\mathbf{p}'$ and $\mathbf{x} = \varepsilon\mathbf{x}'$, $\mathbf{y} = \varepsilon\mathbf{y}'$, $t' = \varepsilon^2 t$, $\mathbf{w} = \varepsilon\mathbf{w}'$ is usually applied in which case $\tilde{H}(\mathbf{x},\mathbf{y})$: Hamiltonian, $h(\mathbf{w})$: normal form, and Lie $G(\mathbf{w})$ generator can be considered a series on a small parameter ε
$\tilde{H}(\mathbf{x}',\mathbf{y}') = \sum\limits_{k=0}^{\infty} \varepsilon^k \tilde{H}_k(\mathbf{x}',\mathbf{y}')$, $h(\mathbf{w}') = \sum\limits_{k=0}^{\infty} \varepsilon^k G_k(\mathbf{w}')$
The normalizing coordinate transformation and the normal form $h(\mathbf{w}')$: can be found in the form of Lie series
$\mathbf{z}' = \mathbf{w}' + \varepsilon\{\mathbf{w}', G\} + \frac{\varepsilon^2}{2!}\{\{\mathbf{w}', G\}, G\} + \ldots$,
$h(\mathbf{w}') = \tilde{H}(\mathbf{w}') + \varepsilon\{\tilde{H}, G\} + \frac{\varepsilon^2}{2!}\{\{\tilde{H}, G\}, G\} + \ldots$, where curly brackets mean Puasson brackets.

Functions h_k and G_{k-1} in their turn are computed successively following k growth with the help of a homologous equations

$$h_k(\mathbf{w}) = \{\tilde{H}_0(\mathbf{w}), G_{k-1}(\mathbf{w})\} + M_k(\mathbf{w}), \tag{0.6}$$

There are two algorithms to solve homologous equations (0.6), and consequently two normalization algorithms.

B.1. Equation (0.6) is computed as a system of linear equations for form h_k G_{k-1} coefficients. This method was developed by Hori and Deprit. Similar to the previous method, there are no R matrix limitations in this method either.

B.2. Zhuravlev [8], [6] proposed to solve the homologous equation by means of integration. If matrix R is diagonalizable $\{H_0, G_{k-1}\} = dG_{k-1}/dt$, Puasson bracket equals to the derivative of G with respect to t along the solution of system $\dot{\mathbf{q}} = \partial H_0/\partial\mathbf{p}$, $\dot{\mathbf{p}} = -\partial H_0/\partial\mathbf{q}$. Therefore, h_k is an average of M_k function along the solutions of the system, and function minus G_{k-1} is a constant in $\int\limits_0^t M_k\, dt$ integral. Thus, for the first two approximations, functions $M_k(\mathbf{w})$ are as follows
$M_1 = H_1$,
$M_2 = H_2 + \{H_1, G_1\} + \frac{1}{2}\{\{H_0, G_1\}, G_1$.

C. Petrov [9], [10] proposed a parametrical form of canonical transformation
$(\mathbf{q},\mathbf{p}) \to (\mathbf{Q},\mathbf{P})$. The general result concerning the parametrization of canonical change of variables can be stated as follows [10]

Theorem. *Suppose that transformation* $(\mathbf{q},\mathbf{p}) \to (\mathbf{Q},\mathbf{P})$ *of variables is represented in the parametric form .30*

$$\begin{aligned} \mathbf{q} = \mathbf{x} - \frac{1}{2}\Psi_{\mathbf{y}}, \;\; \mathbf{Q} = \mathbf{x} + \frac{1}{2}\Psi_{\mathbf{y}}, \\ \mathbf{p} = \mathbf{y} + \frac{1}{2}\Psi_{\mathbf{x}}, \;\; \mathbf{P} = \mathbf{y} - \frac{1}{2}\Psi_{\mathbf{x}}. \end{aligned} \tag{0.7}$$

where $\Psi(t,\mathbf{x},\mathbf{y})$ *is a twice continuously differentiable function in a neighborhood of the point* $(t_0,\mathbf{x}_0,\mathbf{y}_0)$. *Then the following assertions are valid.*

1) The Jacobians of two transformations $(\mathbf{x},\mathbf{y}) \to (\mathbf{q},\mathbf{p})$ *and* $(\mathbf{x},\mathbf{y}) \to (\mathbf{Q},\mathbf{P})$ *identically coincide:*

$$\frac{\partial(\mathbf{q},\mathbf{p})}{\partial(\mathbf{x},\mathbf{y})} = \frac{\partial(\mathbf{Q},\mathbf{P})}{\partial(\mathbf{x},\mathbf{y})} = J(t,\mathbf{x},\mathbf{y}). \tag{0.8}$$

2) For $J(t,\mathbf{x},\mathbf{y}) \neq 0$, *there exists a neighborhood of the point* $(t_0,\mathbf{x}_0,\mathbf{y}_0)$ *in which the transformation (0.2)* $(\mathbf{q},\mathbf{p}) \to (\mathbf{Q},\mathbf{P})$ *brings a Hamiltonian* $H(t,\mathbf{q},\mathbf{p})$ *to Hamiltonian*
$\bar{H}(t,\mathbf{Q},\mathbf{P})$ *such that*

$$\Psi_t(t,\mathbf{x},\mathbf{y}) + H(t,\mathbf{q},\mathbf{p}) = \bar{H}(t,\mathbf{Q},\mathbf{P}),$$

where the arguments $\mathbf{q},\mathbf{p}$ *and* $\mathbf{Q},\mathbf{P}$ *of the Hamiltonians* H *and* $\bar{H}$ *can be expressed via parameters* $\mathbf{x}$ *and* $\mathbf{y}$ *by formulas (0.7).*

If a Hamiltonian system is autonomous, then the function
$\Phi(\mathbf{q},\mathbf{p}) = \Psi\left(\frac{1}{2}(\mathbf{q} + \mathbf{Q}(\mathbf{q},\mathbf{p})), \frac{1}{2}(\mathbf{p} + \mathbf{P}(\mathbf{q},\mathbf{p}))\right)$ is a generating function, which Poincare introduced [13]. Thus, function $\Psi(t,\mathbf{x},\mathbf{y})$ can be called parametrical Poincare function.

Function $\Psi(\mathbf{x},\mathbf{y})$ and parametric canonical normalizing transformation of variables in the form of (0.7) is used instead of G generator in the algorithm of constructing a normal form [11], [12]. The first two approximations for G and Ψ are the same, while the ones that follow are different. To simplify the computation, it is possible to use integration similar to the method described in B.2.

Methods B.2 and C simplify the normal form computation significantly. In addition to this, there is a notion of Hamiltonian symmetrization introduced which expands the notion of the normal form. This is done using property of commutation of perturbed and non-perturbed parts only.

III. Hamilton symmetric form [6]

Definition. *Perturbed Hamiltonian* $H_0 + F$*: is a symmetric form if perturbation* $F(t, \mathbf{q}, \mathbf{p}, \varepsilon)$ *is the first integral of non-perturbed part* $\frac{\partial F}{\partial t} + \{H_0, F\} = 0$.

There are three advantages of this definition over the previous ones [3], [7]. They are as follows:

1. To solve the whole system of Hamilton equations in its symmetrical form, a superposition of solutions of a non-perturbed system and a solution of an autonomous Hamiltonian, which equals $F(0, \mathbf{q}, \mathbf{p}, \varepsilon)$, is used.

2. The invariant character of the definition allows symmetrization both without a preliminary simplification of a non-perturbed part and specification of autonomous/non-autonomous, resonance/non-resonance cases.

3. Asymptotic of a normal form and transformation of variables which lead Hamiltonian to its normal form can be found by consequent quadratures of the functions known at every step (algorithms II.2 and III).

IV. Algorithm of symmetrization with the help of generating Hamiltonian [6]

Let the Hamiltonian under consideration has the following form

$$H(\mathbf{q}, \mathbf{p}, \varepsilon) = H_0(\mathbf{q}, \mathbf{p}) + F(\mathbf{q}, \mathbf{p}, \varepsilon),$$

$$F = \varepsilon F_1 + \varepsilon^2 F_2 + \ldots,$$

where H_0 and F are non-perturbed and perturbed parts of the Hamiltonian, ε is small parameter.

To construct generating Hamiltonian $G = \varepsilon G_1 + \varepsilon^2 G_2 +, \ldots$ (Lee generator) and symmetrical part $\bar{F} = \varepsilon \bar{F}_1 + \varepsilon^2 \bar{F}_2 +, \ldots$ we should

1. find solution $\mathbf{q}(t, \mathbf{Q}, \mathbf{P})$, $\mathbf{p}(t, \mathbf{Q}, \mathbf{P})$ of a non-perturbed system;

$$\dot{\mathbf{q}} = \frac{\partial H_0}{\partial \mathbf{p}}, \quad \dot{\mathbf{p}} = -\frac{\partial H_0}{\partial \mathbf{q}}, \quad \mathbf{q}(0) = \mathbf{Q}, \quad \mathbf{p}(0) = \mathbf{P},$$

2. find functions: $m_k(t, \mathbf{Q}, \mathbf{P}) = M_k(\mathbf{q}(t, \mathbf{Q}, \mathbf{P}), \mathbf{p}(t, \mathbf{Q}, \mathbf{P}))$, $k = 1, 2, \ldots$, where $M_1 = F_1$, $M_2 = F_2 + \{F_1, G_1\} + \frac{1}{2}\{\{H_0, G_1\}, G_1\}$, ($\{f, g\}$ are Poisson brackets).

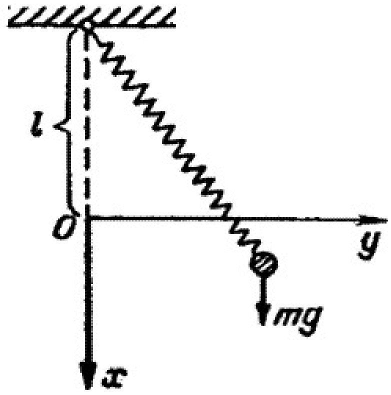

Fig. 1. Spring pendulum

3. Using identity of $\int\limits_0^t m_k(t, \mathbf{Q}, \mathbf{P}) dt = t\bar{F}_k(\mathbf{Q}, \mathbf{P}) + G_k(\mathbf{Q}, \mathbf{P}) + f(t)$ we find asymptotics coefficients of the symmetrized part $\bar{F}_k(\mathbf{Q}, \mathbf{P})$ and generator $G_k(\mathbf{Q}, \mathbf{P})$ from the quadrature $\int\limits_0^t m_k(t, \mathbf{Q}, \mathbf{P}) dt$. In particular, H_0 is a quadratic normal form, the symmetrization algorithm equals II.3 normal form algorithm. However, even in this classical case, the algorithm is significantly less complicated than classical algorithms I. and II.1.

V. Spring pendulum. The pendulum with two degrees of freedom is considered: the heavy point shaking in a vertical plane on a spring (fig. 1), the spring is weightless. We enter following designations: k - rigidity of a spring, l - its length position of rest of the point, m - weight of the point, lx, ly - point co-ordinates, lR - length of a spring, where

$$R = \sqrt{(1 + x)^2 + y^2}.$$

The Cartesian system of coordinates has the beginning in a point O - position of rest of the point - and axes x and y, directed vertically and horizontally correspondingly (see fig. 1). The spring tension obeys the following nonlinear law:

$$T = \frac{k\varepsilon \left(lR - l_0\right)^3}{l_0^3} + \frac{k \left(lR - l_0\right)}{l_0} \tag{1}$$

Where l_0 - length of not loaded spring.

Potential E_p and kinetic E_k energy of system:

$$E_p = \frac{k\varepsilon \left(lR - l_0\right)^4}{4 l_0^3} + \frac{k \left(lR - l_0\right)^2}{2 l_0} - mglx$$

$$E_c = \frac{m}{2}\left[\left(\frac{dx}{dt'}\right)^2 + \left(\frac{dy}{dt'}\right)^2\right] = \frac{mgl}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]$$

Here t' - dimensional time, $t = \omega t'$ - dimensionless time.

We enter dimensionless impulses $u = \dot{x}$, $v = \dot{y}$ and write down through them function of Hamilton $H = (E_k + E_p)/(mgl)$

$$H = \frac{1}{2}\left(u^2 + v^2\right) - x + \frac{k\varepsilon\left(lR - l_0\right)^4}{4mgll_0^3} + \frac{k\left(lR - l_0\right)^2}{2mgll_0}$$

The movement equations of hamilton system:

$$\frac{dx}{dt} = \frac{\partial H}{\partial u}, \quad \frac{du}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{dy}{dt} = \frac{\partial H}{\partial v}, \quad \frac{dv}{dt} = -\frac{\partial H}{\partial y}$$

We study movement near to rest position on the big times t and expand hamiltonian in a vicinity of position of balance $H = H_1 + H_2 + F_1 + F_2$, where H_1, H_2, F_1, F_2 - polynoms of the first, second, third and fourth degrees accordingly.

$$H_1 = x\left(K\left(\varepsilon\lambda^3 + \lambda\right) - 1\right),$$

$$H_2 = \frac{u^2 + v^2 + x^2 K(\lambda+1)\left(3\varepsilon\lambda^2+1\right) + y^2 K\left(\varepsilon\lambda^3+\lambda\right)}{2}, \tag{2}$$

$$F_1 = ..., \quad F_2 = ...$$

The expressions for F_1 and F_2 are too lengthy to be presented here.

$$K = \frac{k}{mg}, \quad \lambda = \frac{l}{l_0} - 1$$

The linear part of the hamiltonian is equal in balance position to zero, whence we receive

$$K\left(\varepsilon\lambda^3 + \lambda\right) = 1 \tag{3}$$

It follows from this that the factor at y^2 in the hamiltonian is equal in position of balance $\frac{1}{2}$. It means that frequency of horizontal oscillations is equal 1. Let us fixate some any frequency of fluctuations on a vertical: $\omega_{vert} = \omega$, that is

$$(\lambda + 1)\left(3\varepsilon\lambda^2 + 1\right) = \left(\varepsilon\lambda^3 + \lambda\right)\omega^2 \tag{4}$$

The soliton of the system (3) (4) with the variables K and ε:

$$\varepsilon = \frac{-\lambda\omega^2 + \lambda + 1}{\lambda^2\left(\lambda\left(\omega^2 - 3\right) - 3\right)}, \quad K = \frac{3}{2\lambda} - \frac{\omega^2}{2(\lambda+1)} \tag{5}$$

Besides one can present λ as a series on the parameter of spring nonlinearity ε. To within the first degree ε we will receive:

$$\lambda = \frac{1}{\omega^2 - 1} + \varepsilon\frac{2\omega^2}{\left(\omega^2 - 1\right)^4} + O(\varepsilon^2) \tag{6}$$

In case of resonance 1:1, that is when $\omega = 1$, we have

$$\lambda = -\frac{1}{\sqrt[3]{2\varepsilon}} - \frac{1}{2} + O(\sqrt[3]{\varepsilon}) \tag{7}$$

By means of (3) and (4) one can exclude parameters K ε. Then in the vicinity of balance point we can expand (2) with the dependence on two parameters: ω λ

$$H_1 = 0$$

$$H_2 = \frac{u^2}{2} + \frac{v^2}{2} + \frac{y^2}{2} + \frac{x^2\omega^2}{2} \tag{8}$$

$$F_1 = ..., \quad F_2 = ...$$

Further to the hamiltonian expansion the algorithm of invariant normalization will be applied knowing value of frequency ω.

VI. Non-resonance

At first we formally apply algorithm of invariant normalization not imposing any restrictions on the vertical frequency. The received hamiltonian normal form is convenient for writing down in Birkhoff variables. Normal form factors are expressed precisely through the dimensionless lengthening λ, with the help (6) in the form of series on ε to within members of an order ε^2

$$\begin{gathered} Z_1 = \frac{U}{\sqrt{\omega}} + i\sqrt{\omega}X, \quad Z_2 = V + iY \\ \hat{H}_2 = i\left(\omega Z_1\bar{Z}_1 + Z_2\bar{Z}_2\right), \quad \hat{F}_1 = 0 \\ \hat{F}_2 = i\left(\alpha_{11}Z_1^2\bar{Z}_1^2 + \alpha_{12}Z_1\bar{Z}_1Z_2\bar{Z}_2 + \alpha_{22}Z_2^2\bar{Z}_2^2\right) \end{gathered} \tag{9}$$

where α_{ij} is known coefficients.

The decision of such normal form are the usual harmonious oscillation occurring separately on a vertical and a horizontal which frequency receives the small amendment of an order of a square of amplitudes.

VII. Resonance 1:1

Let's consider a resonance case 1:1. Believing in (8) $\omega = 1$ we find a normal form in Birkhoff variables $Z_1 = U + iX$, $Z_2 = V + iY$:

$$\hat{H}_2 = i\left(Z_1\bar{Z}_1 + Z_2\bar{Z}_2\right), \quad \hat{F}_1 = 0$$

$$\hat{F}_2 = i(\alpha_1 Z_1^2\bar{Z}_1^2 +$$

$$+\alpha_2 Z_2^2\bar{Z}_2^2 + \alpha_3 Z_1\bar{Z}_1Z_2\bar{Z}_2 + \alpha_4\left(Z_2^2\bar{Z}_1^2 + Z_1^2\bar{Z}_2^2\right))$$

where

$$\alpha_1 = -\frac{3(\lambda+1)^2(\lambda+5)}{32\lambda^4} \approx \frac{3\sqrt[3]{\varepsilon}}{16 * 2^{2/3}}, \quad \alpha_2 = 0$$

$$\alpha_3 = -\frac{3(\lambda+1)}{8\lambda^2} \approx \frac{3\sqrt[3]{\varepsilon}}{4 * 2^{2/3}}, \quad \alpha_4 = -\frac{3(\lambda+1)}{32\lambda^2} \approx \frac{3\sqrt[3]{\varepsilon}}{16 * 2^{2/3}}$$

This normal form in contrast to the normal form at not resonant case posses component $Z_2^2\bar{Z}_1^2 + Z_1^2\bar{Z}_2^2$ which bring essential changes in behavior of system.

With a view of simplification of the analysis perturbation of the second order has been broken into two components:

$$\hat{F}_2 = \hat{F}_{21} + \hat{F}_{22}, \quad \hat{F}_{21} = ik\left(Z_1\bar{Z}_1 + Z_2\bar{Z}_2\right)^2$$

$$\hat{F}_{22} = -ikZ_2^2\bar{Z}_2^2 + inZ_1Z_2\bar{Z}_1\bar{Z}_2 + im\left(Z_1\bar{Z}_2 - Z_2\bar{Z}_1\right)^2$$

$$k = -\frac{3\left(\lambda^3 + 7\lambda^2 + 11\lambda + 5\right)}{32\lambda^4}, \quad m = -\frac{3(\lambda+1)}{32\lambda^2},$$

$$n = -\frac{3\left(2\lambda^3 - 4\lambda^2 - 11\lambda - 5\right)}{16\lambda^4}$$

All three parts of the hamiltonian ($\hat{H}_2$, $\hat{F}_{21}$, $\hat{F}_{22}$) commutates with each other. That is why we can search solutions for every hamiltonian separately, and then build combined solution by means of the algorithm described in [6].

The solution for hamiltonian $\hat{H}_2 + \hat{F}_{21}$ is the same as in the case of absence of resonance. And for the $\hat{F}_{22}$ the solution is found for the particular case, when $\lambda = 3 + \sqrt{14} \approx 6.74$. For that λ we can see $k = n = 2m$, that greatly simplifies system.

Then the asymptotical decision in case $|y_0/x_0| = \nu \to 0$ assumes the following form:

$$x(t) = X(t)\cos(t), \quad y(t) = Y(t)\cos(t)$$

$$2[ex]Y(t) = \frac{x_0}{\sqrt{2}} Sech\left(2m\left(t - \frac{3T}{2}\right)x_0^2\right) +$$

$$+\frac{x_0}{\sqrt{2}} Sech\left(2m\left(t - \frac{T}{2}\right)x_0^2\right) + (...)$$

$$X(t) = \sqrt{c^2 - Y^2(t)}$$

$$T = \frac{\tau_0}{2\sqrt{m\left(c^4 m - a\right)}} \approx \frac{1}{2mx_0^2}\ln\left(\frac{4x_0^2}{y_0^2}\right)$$

As can be seen from the formulas here takes place process of periodical energy swapping between horizontal and vertical oscillations. The frequency of this reorganization is depends on initial conditions and can be smaller then than frequencies of oscillations in tens thousand times. The graph of the solution is presented on fig.2. Visual frequency of oscillations on the graph is considerably smaller than real for the sake of clearness.

It is possible to solve system and for other values of λ, but the decision will turn out very inconvenient.

IIX. Vicinity of the resonance 2:1 Solutions for the spring pendulum in the case of resonance 2:1 were obtained in [17]. Unperturbed hamiltonian for that case:

$$H_2 = \frac{u^2}{2} + \frac{v^2}{2} + \frac{4x^2}{2} + \frac{y^2}{2}, \quad F_1 = \frac{\varepsilon(\lambda+1)^2 x^3}{\varepsilon\lambda^2 + 1} + \frac{3y^2 x}{2}$$

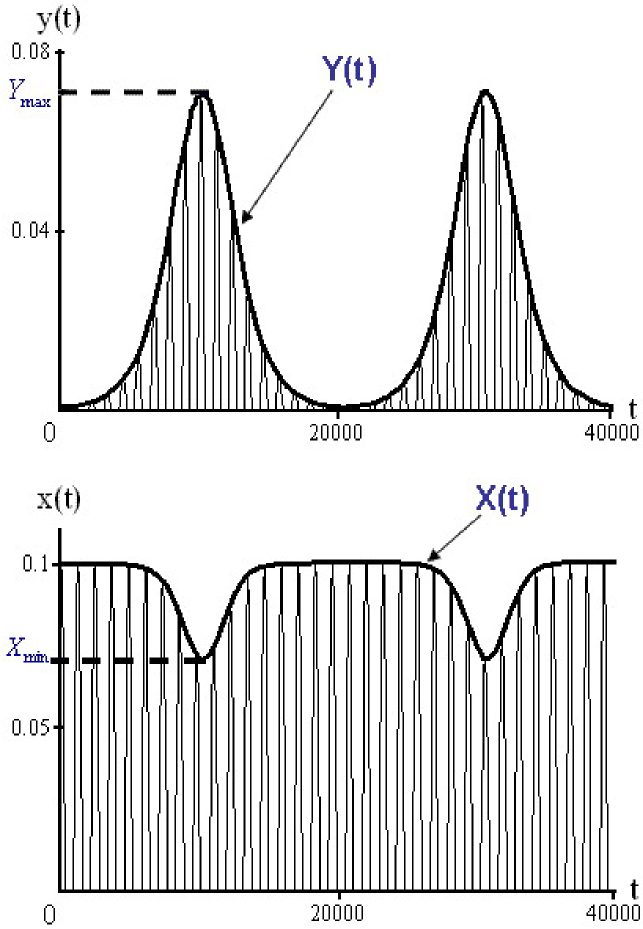

Fig. 2. Resonance 1:1

Here we consider, how much quickly disappears effect of swapping of energy when we introduce small frequency detuning. We consider for this purpose, that frequency of horizontal oscillations is equaled not 2, but $\omega = 2 + \mu$, where μ - small parameter of 1st order to hamiltonian variables. Thus composed in hamiltonian μx^2 we will carry not to a square-law part, but to perturbation of the first order. Component $\mu^2 x^2$ thus it is necessary to carry to the second order of perturbation and due to fact that at normalization we will count up a normal form only up to members of 3rd order, this component can be excluded from consideration. Thus, the following system should be subjected normalization:

$$H_2 = \frac{u^2}{2} + \frac{v^2}{2} + \frac{4x^2}{2} + \frac{y^2}{2},$$

$$F_1 = \frac{\varepsilon(\lambda+1)^2 x^3}{\varepsilon\lambda^2+1} + \frac{3y^2 x}{2} + \mu x^2$$

After normalization

$$\hat{H}_2 = i\left(2Z_1\bar{Z}_1 + Z_2\bar{Z}_2\right),$$
$$\hat{F}_1 = \frac{3\bar{Z}_1 Z_2^2}{8\sqrt{2}} - \frac{3Z_1\bar{Z}_2^2}{8\sqrt{2}} + \frac{1}{2}i\mu Z_1\bar{Z}_1 \qquad (10)$$

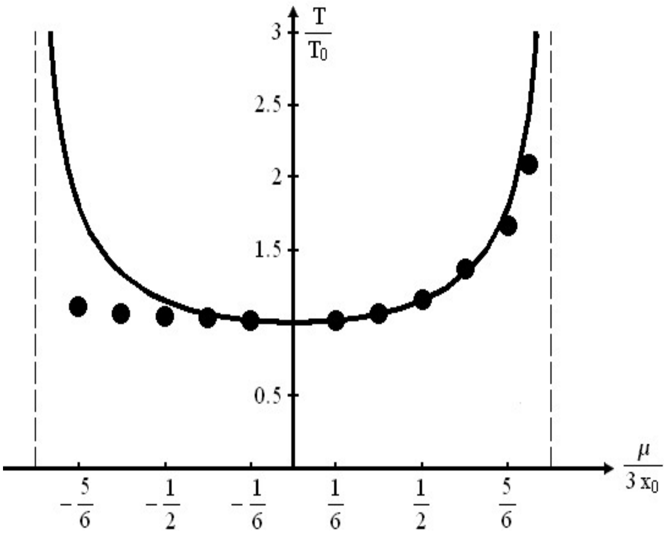

Fig. 3. Detuning at the resonance 2:1

The solution of the system gives us energy swapping between horizontal and vertical oscillation with the period

$$T = T_0 / \sqrt{1 - \left(\frac{\mu}{3x_0}\right)^2},$$
$$T_0 = \frac{4\ln\left(\frac{32}{\nu^2}\right)+3\nu^2+(3/32)(-17+5\ln(32/\nu^2))\nu^4+O(\nu^6\ln\nu)}{3x_0}$$
$$|y_0/x_0| = \nu \to 0 \tag{11}$$

where T_0 - period at the resonance.

The period finite only when $\mu \in (-3|x_0|;3|x_0|)$, and that determines the area, where reorganization of oscillations can be observed. The comparison with the numerical evaluations is presented at fig.3. The results of the numerical solutions are denoted as filled circles.

IX. Conclusions

It is proved that reorganization between vertical and horizontal oscillations is possible only at resonances 1:1 and 2:1. The resonance 1:1 is possible only for a nonlinear spring possessing the negative cubic term in the law of deformation. Besides the effect for a resonance 1:1 is much weaker. At these resonances the amplitude envelope for both oscillations is defined. It is described by means of simple function Sech t and well co-ordinated with numerical calculations.

Dependence of the period T of reorganization from a deviation of resonant frequency $\mu = \omega_x/\omega_y - 2$.

Received numerical evaluations show good accuracy of these analytical dependences (see fig. 2 and fig. 3)

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References

[1] A.D. Bruno, *Analytic form of differential equations.* Tr. Mosk. Mat. Obshc. **26** (1972), 199-239.

[2] A.D. Bruno and A. G. Petrov, *Calculation of the Hamiltonian Normal Form.* Dokl. Akad. Nauk **410**(4), (2006), 474-478 [Dokl. Phys. **51**(10) (2006), 555-559].

[3] G.D. Birkhoff, Dynamical Systems (American Mathematical Society, New York, 1927).

[4] A.D. Bruno, *The Restricted Three Body Three Problem: Periodic Orbits* (Nauka, Moscow, 1990), [in Russian].

[5] V.M. Zhuravlev, D.M. Klimov, Applied Methods of Vibration Theory (Nauka, Moscow, 1988) [in Russian].

[6] V.F. Zhuravlev, *Invariant normalization of non-autonomus Hamiltonian systems.* Prikl. Mat. Mekh. **66**(3) (2002), 356-365.

[7] V.I. Arnold, Mathematical Methods of Classical Mechanics. (Nauka, Moscow, 1978), [in Russian].

[8] V.F. Zhuravlev, Introduction to Theoretical Mechanics. (Nauka Fizmatlit, Moscow, 1997) [in Russian].

[9] A.G. Petrov, *On average of Hamiltonian systems with periodic Hamiltonian.* Dokl. Akad. Nauk **368**(4), (1999), 483-488 [Dokl. Phys. **44**(10) (1999)].

[10] A.G. Petrov, *Asymptotic Methods for Solving the Hamilton Equations with the Use of a Parametrization of Canonical Transformations.* Differentsial'niya Uravnenia **40**(5), (2004), 626-638. [Differencial Equations **40**(5), (2004), 672-685.

[11] A.G. Petrov, *Modification of invariant normalization method of Hamiltonians with help parametrization of the canonical transformations.* Dokl. Akad. Nauk. **386**(4) (2002) 343-345 [Dokl. Phys. **47**(10) (2002), 742-744].

[12] A.G. Petrov, *Invariant normalization of non-autonomus Hamiltonian systems.* Prikl. Mat. Mekh. **68**(3) (2004), 402-413. [J. Appl. Math. Mech. **68** (3) (2004), 357-367].

[13] H. Poincare, Les methods nouvelles de la mecanique celeste. Paris: Gauthier-Villars, 1892, 385 p.; 1893, 472 p.; 1899, 414 p.

[14] A.P. Markeev, Libration Points in Celestial Mechanics and Astrodynamics (Fizmatlit, Moscow, 1978) [in Russian].

[15] A.G. Petrov, A.V. Fomichev, *On Nonlinear threedimentional oscillations of a havy swinging Spring.* (unpublished)

[16] M.N. Zaripov, A.G. Petrov, *Nonlinear Oscillations of a Swinging Spring.* Dokl. Phys. **49**(11) (2004), 691-696 [Dokl. Akad. Nauk **399**(3) (2006), 347-352].

[17] A.G. Petrov, *Nonlinear oscillations of a swinging Spring for resonance.* Mechanics of Solids **40**(5) (2006) [Mekh. Tverd. Tela (5) (2006), 18-28]

[18] A.G. Petrov, *On the Equations of Motion of a Spherical Pendulum with a Fluctuating Support.* Dokl. Phys. **50**(11) (2005), 588-591 [Dokl. Akad. Nauk V. 405, No 1, 51 (2005)].