State Estimation Problems For Differential Inclusions

Tatiana F. Filippova

***Abstract*— The paper is devoted to the problems of evolution modelling for nonlinear uncertain dynamic systems with system states being compact sets. Applying results related to discrete-time versions of the funnel equations and techniques of ellipsoidal estimation theory developed for linear control systems we present new approaches that allow to find the upper estimates for such set-valued states of the uncertain nonlinear control system. Numerical simulations are also given.**

I. INTRODUCTION

The topics of this paper come from the control theory for systems with unknown but bounded uncertainties related to the case of set-membership description of uncertainty [1], [2], [3], [4], [5], [6], [7]. The motivations for these studies come from applied areas ranged from engineering problems in physics to economics as well as to ecological and biomedical modelling. The paper presents recent results in the theory of tubes of solutions (trajectory tubes) to differential control systems modelled by nonlinear differential inclusions with uncertain parameters or functions.

Of particular interest is the description of the behavior of these tubes when the system is subjected to state constraints. Such constraints may be induced by given state constraints defined for a plant model or by current state measurements with unknown but bounded noises. The objects under investigation are then known as the viability tubes and their time cross-sections turn to be the attainability domains (or reachable sets) for the original differential system with state constraints. Starting at a specified initial set represented the uncertainty in initial state, the overall system generates a set-valued map (the trajectory tube) that defines a generalized dynamic system.

This paper is devoted to the problems of evolution modelling for nonlinear uncertain dynamic systems with set-valued system states. Applying results related to ellipsoidal calculus [6], [5] and discrete-time versions of the funnel equations we find estimates for such set-valued states of nonlinear dynamical control systems. The applications of the problems studied in this paper are in guaranteed state estimation for nonlinear systems with unknown but bounded errors and in nonlinear control theory. The numerical simulation schemes developed for such problems are also presented.

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T. F. Filippova is with Department of Optimal Control, Institute of Mathematics and Mechanics, Ural Branch of Russian Academy of Sciences, 16 S.Kovalevskaya Str., GSP-384, 620219 Ekaterinburg, Russia; ftf@imm.uran.ru

II. PROBLEM STATEMENT

The paper deals with the problems of control and state estimation for a dynamical control system

$$\dot{x}(t) = A(t)x(t) + f(x(t)) + G(t)u(t), \qquad (1)$$
$$x \in R^n, \quad t_0 \leq t \leq T,$$

with unknown but bounded initial condition

$$x(t_0) = x_0, \quad x_0 \in X_0, \quad X_0 \subset R^n, \qquad (2)$$

$$u(t) \in U, \quad U \subset R^m, \quad \text{for a.e. } t \in [t_0, T]. \qquad (3)$$

Here matrices $A(t)$ and $G(t)$ (of dimensions $n \times n$ and $n \times m$, respectively) are assumed to be continuous on $t \in [t_0, T]$, X_0 and U are compact and convex. The nonlinear n-vector function $f(x)$ in (1) is assumed to be of quadratic type

$$f(x) = (f_1(x), \ldots, f_n(x)),$$
$$f_i(x) = x' B_i x, \; i = 1, \ldots, n, \qquad (4)$$

where B_i is a constant $n \times n$ - matrix ($i = 1, \ldots, n$).

Consider the following differential inclusion [8] related to (1)–(3)

$$\dot{x}(t) \in A(t)x(t) + f(x(t)) + P(t), \quad \text{for a.e. } t \in [t_0, T], \qquad (5)$$
$$x(t_0) = x_0 \in X_0,$$

where $P(t) = G(t)U$.

Let absolutely continuous function $x(t) = x(t, t_0, x_0)$ be a solution to (5) with initial state x_0 satisfying (2). The differential system (1)–(3) (or equivalently, (5)) is studied here in the framework of the theory of uncertain dynamical systems (differential inclusions) through the techniques of trajectory tubes [4]

$$X(\cdot, t_0, X_0) = \{x(\cdot) = x(\cdot, t_0, x_0) \mid x_0 \in X_0\} \qquad (6)$$

of solutions to (1)–(3) with their t-cross-sections $X(t) = X(t, t_0, X_0)$ being the reachable sets at instant t for control system (1)–(3).

One of the principal points of interest of the theory of control under uncertainty conditions is to study the set of all solutions $x(t) = x(t, t_0, x_0)$ to (1)–(3) (respectively, (5)) with the additional constraint (the ”viability” constraint [4], [9])

$$x(s) \in Y(s), \quad s \in [t_0, t] \qquad (7)$$

where $Y(\cdot)$ is a convex compact valued multifunction.

The viability constraint (7) may be induced by state constraints defined for a given plant model or by the so-called measurement equation

$$y(t) = C(t)x + w, \qquad (8)$$

where y is the measurement, $C(t)$ is a $p \times n$-matrix function, w is the unknown but bounded noise and

$$w \in Q(t), \quad Q(t) \subset R^p.$$

The problem consists in describing the set $X(\cdot) = \cup_{x_0 \in X_0}\{x(\cdot) = x(\cdot, t_0, x_0)\}$ of solutions to the differential inclusion (5) under constraint (7) (the viable trajectory tube [4]). The point of special interest is to describe the t – cross-section $X(t)$ of this map that is actually the attainability domain of this system at the instant t.

Basing on results of ellipsoidal calculus ([5], [6]) developed for linear uncertain systems we present here the modified state estimation approaches which use the special structure of nonlinearity of studied control system (1)–(4) and combine advantages of estimating tools mentioned above.

III. External Estimates of Reachable Sets and Trajectory Tubes

A. Basic Notations

We introduce here the following notations. Let R^n be the n–dimensional Euclidean space and $(x, y) = x'y$ be the usual inner product of $x, y \in R^n$ with prime as a transpose, $\| x \| = (x'x)^{1/2}$. We denote as $B(a, r)$ the ball in R^n, $B(a, r) = \{x \in R^n : \| x - a \| \leq r\}$, I is the identity $n \times n$-matrix. Denote by $E(a, Q)$ the ellipsoid in R^n, $E(a, Q) = \{x \in R^n : (Q^{-1}(x - a), (x - a)) \leq 1\}$ with center $a \in R^n$ and symmetric positive definite $n \times n$–matrix Q. For any $n \times n$-matrix Q denote its track as Tr Q and its determinant as $|Q|$.

Denote comp R^n to be the variety of all compact subsets $A \subseteq R^n$ and conv R^n to be the variety of all compact convex subsets $A \subseteq R^n$. Denote as $h(A, B)$ the Hausdorff distance for A, $B \subseteq R^n$, $h(A, B) = \max\{h^+(A, B),\ h^-(A, B)\}$, with $h^+(A, B)$ and $h^-(A, B)$ being the Hausdorff semidistances between A and B, $h^+(A, B) = \sup\{d(x, B) \mid x \in A\}$, $h^-(A, B) = h^+(B, A)$, $d(x, A) = \inf\ \{\|x - y\| \ \mid y \in A\}$.

B. Results

The approach presented here uses the techniques of ellipsoidal calculus developed for linear control systems. It should be noted that external ellipsoidal approximations of trajectory tubes may be chosen in various ways and several minimization criteria are well-known. We consider here the ellipsoidal techniques related to construction of external estimates with minimal volume (details of this approach and motivations for linear control systems may be found in [6], [5]).

Assume here that $P(t) = E(a, Q)$ in (5), matrices B_i $(i = 1, ..., n)$ are symmetric and positive definite, $A(t) \equiv A$. We may assume that all trajectories of the system (5)-(2) belong to a bounded domain $D = \{x \in R^n : \| x \| \leq K\}$ where the existence of such constant $K > 0$ follows from classical theorems of the theory of differential equations and differential inclusions [8].

From the structure (4) of the function f we have two auxiliary results. Their proofs are based on the algebraic properties of quadratic forms and are omitted here.

Lemma 1: The following estimate is true

$$\| f(x) \| \leq N, \qquad N = K^2 (\sum_{i=1}^{n} \lambda_i^2)^{1/2},$$

where λ_i is the maximal eigenvalue for matrix B_i $(i = 1, ..., n)$.

Lemma 2: For all $t \in [t_0, T]$ the inclusion

$$X(t) \subset X^*(t)$$

holds where $X^*(\cdot)$ is a trajectory tube of the linear differential inclusion

$$\dot{x} \in Ax + B(c, \sqrt{n}N/2), \quad x_0 \in X_0, \tag{9}$$

where $c = \{N/2, \ldots, N/2\} \in R^n$.

The following theorem gives the external estimate of the trajectory tube $X(t)$ of the differential inclusion (5).

Theorem 1: Let $X_0 = B(0, r)$, $r \leq K$ and

$$t_* = \min\ \{\frac{K - r}{\sqrt{2}M}\ ;\ \frac{1}{L}\ ;\ T\}.$$

Then for all $t \in [t_0, t_*]$ the following inclusion is true

$$X(t, t_0, X_0) \subset E(a^+(t), Q^+(t)), \tag{10}$$

where

$$M = K\sqrt{\lambda} + N + P, \quad P = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \sqrt{\tilde{\lambda}},$$

$$L = \sqrt{\lambda} + 2K \left(\sum_{i=1}^{n} \lambda_i^2\right)^{1/2},$$

with λ, λ_i and $\tilde{\lambda}$ being the maximal eigenvalues of matrices AA', B_i $(i = 1, ..., n)$ and Q respectively, and vector function $a^+(t)$ and matrix function $Q^+(t)$ satisfy the equations

$$\dot{a}^+ = Aa^+ + a + c, \quad a^+(t_0) = 0 \tag{11}$$

$$\begin{gathered} \dot{Q^+} = AQ^+ + Q^+A^T + qQ^+ + q^{-1}Q^*, \\ q = \{n^{-1}\mathrm{Tr}((Q^+)^{-1}Q^*)\}^{1/2}, \\ Q^+(t_0) = Q_0 = r^2 I. \end{gathered} \tag{12}$$

Here

$$Q^* = (p^{-1} + 1)\tilde{Q} + (p + 1)Q, \quad \tilde{Q} = \frac{nN^2}{2}I, \tag{13}$$

and p is the unique positive solution of the equation

$$\sum_{i=1}^{n} \frac{1}{p + \alpha_i} = \frac{n}{p(p + 1)}, \tag{14}$$

with $\alpha_i \geq 0$ $(i = 1, ..., n)$ being the roots of the following characteristic equation

$$|\ \tilde{Q} - \alpha Q\ | = 0. \tag{15}$$

Proof: Applying Lemmas 1-2 and the ellipsoidal techniques [6], [5], and comparing the inclusions (5) and (9) we come to the relation (10). ∎

C. Example

Consider the following control system

$$\begin{cases} \dot{x}_1 &= 6x_1 + u_1, \\ \dot{x}_2 &= x_1^2 + x_2^2 + u_2, \end{cases} \quad 0 \le t \le T, \tag{16}$$

$$X_0 = B(0,1),\ P(t) = B(0,1),\ T = 0.15,\ K = 2.6. \tag{17}$$

Results of computer simulations based on the above theorem for this system are given at Fig. 1.

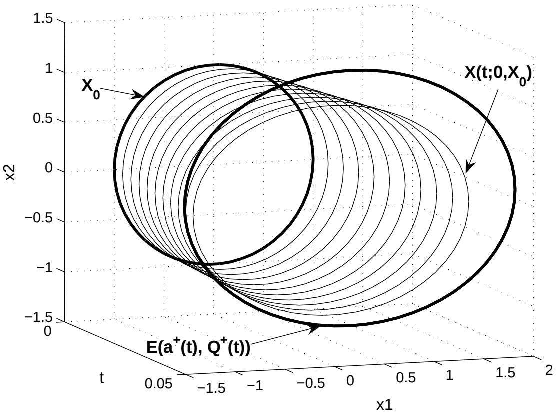

Fig. 1. Reachable sets $X(t, t_0, X_0)$ and their estimates $E(a^+(t), Q^+(t))$ (here $t_* = \sqrt{2}/29.2$).

IV. Upper Bounds of Viable Trajectory Tubes

A. Preliminaries from Funnel Equations Theory

Consider a control system described by ordinary differential equations

$$\dot{x} = f(t, x, u), \tag{18}$$

$$u \in Q(t, x) \tag{19}$$

with a function $f : T \times R^n \times R^m \to R^n$ measurable in t and continuous in other variables and with $Q(t, x)$ being a set-valued map ($Q : T \times R^n \to \text{comp}R^m$) which is measurable in t and continuous in x. We assume that notions of continuity and measurability of set-valued maps are taken in the sense of [9].

The given data allows to consider a set-valued function

$$\mathcal{F}(t, x) = \bigcup \{\ f(t, x, u) \ \mid\ u \in Q(t, x)\ \} \tag{20}$$

and further on, a differential inclusion

$$\dot{x} \in \mathcal{F}(t, x). \tag{21}$$

Assume as in previous sections that the initial condition to the system (18) (or to the differential inclusion (21)) is unknown also but bounded

$$x(t_0) = x_0, \quad x_0 \in X_0 \in \text{comp}R^n. \tag{22}$$

Denote $x(t) = x(t, t_0, x_0)$ ($t \in [t_0, T]$) to be a solution to (21) that starts at point $x(t_0) = x_0 \in X_0$. We take here the Caratheodory–type trajectory $x(\cdot)$, i.e. as an absolutely continuous function $x(t)$ $(t \in [t_0, T])$ that satisfies the inclusion

$$\frac{d}{dt}\, x(t) = \dot{x}(t) \in \mathcal{F}(t, x(t)) \tag{23}$$

for almost all $t \in [t_0, T]$.

We require that all solutions $\{\ x(t) = x(t, t_0, x_0) \ \mid\ x_0 \in X_0\ \}$ exist and are extendable up to the instant T that is possible under some additional assumptions [8]. So we may define a trajectory tube $X(\cdot, t_0, X_0)$ similar to (6).

The approach discussed here is related to evolution equations of the funnel type that describe the dynamics of set–valued system states $X(t, t_0, X_0)$. The basic assumptions on set–valued map $\mathcal{F}(t, x)$ for the following results to be true may be found in [4], [10], [11].

Let us consider the "equation"

$$\lim_{\sigma \to +0} \sigma^{-1} h(\ X(t+\sigma),\ \bigcup_{x \in X(t)} (x + \sigma\ \mathcal{F}(t, x))\ ) = 0, \tag{24}$$

with

$$X(t_0) = X_0, \quad t \in [t_0, T]. \tag{25}$$

Theorem 2: ([10], [11]) The set-valued function $X(t) = X(t, t_0, X_0)$ is the unique solution to the evolution equation (24)-(25).

Consider the analogy of the funnel equation (24)-(25) but now for the viable trajectory tubes $X(t) = X(t, t_0, X_0)$ found under the viability constraint (7):

$$\lim_{\sigma \to +0} \sigma^{-1} h(X(t+\sigma),\ \bigcup_{x \in X(t)} (x + \sigma\mathcal{F}(t, x)) \cap Y(t+\sigma)) = 0, \tag{26}$$

$$X(t_0) = X_0, \quad t \in [t_0, T]. \tag{27}$$

The following result provides that this relation describes the viable trajectory tube.

Theorem 3: ([4]) The viable trajectory tube $X(t) = X(t, t_0, X_0)$ is the unique solution to the evolution equation (26)-(27).

Other versions of funnel equations may be considered by substituting the Hausdorff distance h for a semidistance h^+ [5]. The solution to the h^+-versions of the evolution equation may be not unique and the "maximal" one (with respect to inclusion) is studied in this case. Mention here also the second order analogies of funnel equations for differential inclusions and control systems based on ideas of Runge-Kutta scheme [12], [13], [14]. Discrete approximations for differential inclusions based on set-valued Euler's method were developed in [12], [15].

B. Results

Let us discuss the estimation approach based on techniques of evolution funnel equations. Consider the following system

$$\dot{x} = Ax + \tilde{f}(x)d,\ x_0 \in X_0,\ t_0 \le t \le T, \tag{28}$$

where $x \in R^n$, $\|x\| \le K$, d is a given n-vector and a scalar function $\tilde{f}(x)$ has a form $\tilde{f}(x) = x'Bx$ with a symmetric and positive definite matrix B.

Note that the direct application of funnel equations for finding trajectory tubes $X(t)$ is very difficult because it takes a huge amount of computations based on grid techniques. The following theorem related to our special case of nonlinearity presents an easy computational tool to find estimates of $X(t)$ by step-by-step procedures. For a simpler case of system nonlinearities the approach was presented in [16].

Theorem 4: Let $X_0 = E(a, k^2 B^{-1})$ with $k \neq 0$. Then for all $\sigma > 0$ the following inclusion holds

$$X(t_0 + \sigma, t_0, X_0) \subseteq E(a(\sigma), Q(\sigma)) + o(\sigma)B(0, 1), \quad (29)$$

where

$$a(\sigma) = a + \sigma(Aa + a'Ba \cdot d + k^2 d), \quad (30)$$

$$Q(\sigma) = k^2(I + \sigma R)B^{-1}(I + \sigma R)', \; R = A + 2da'B \quad (31)$$

and $\lim_{\sigma\to+0} \sigma^{-1}o(\sigma) = 0$.

Proof: The funnel equation for (28) is

$$\lim_{\sigma\to+0} \sigma^{-1}h(X(t+\sigma, t_0, X_0), \bigcup_{x\in X(t,t_0,X_0)} \{x + \sigma(Ax+$$

$$+\tilde{f}(x)d)\} = 0, \; t \in [t_0, T], \quad X(t_0, t_0, X_0) = X_0. \quad (32)$$

If $x_0 \in \partial X_0$ where ∂X_0 means the boundary of X_0, we have

$$\tilde{f}(x_0) = k^2 + 2a'Bx - a'Ba$$

and from (32) we have also

$$\bigcup_{x_0\in\partial X_0} \{(I + \sigma A)x_0 + \sigma \tilde{f}(x_0)d\} =$$

$$= \bigcup_{x_0\in\partial X_0} \{(I + \sigma R)x_0 + \sigma(k^2 - a'Ba)d\}. \quad (33)$$

Note that if the ellipsoid in (29) gives the tube estimate for the system with ∂X_0 as starting set, then also for the system with X_0 as starting set. Applying Theorem 2 and taking into account the equality (33) and the above remark we come to the estimate (29). ∎

Based on this result we may formulate the following scheme that gives the external estimate of trajectory tube $X(t)$ of the system (28) with given accuracy.

Algorithm 1: Subdivide the time segment $[t_0, T]$ into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + ih$ $(i = 1, \ldots, m)$, $h = (T - t_0)/m$, $t_m = T$.

Step 1. Given $X_0 = E(a, k_0^2 B^{-1})$ with $k_0 \neq 0$, define $X_1 = E(a_1, Q_1)$ from Theorem 4 for $a_1 = a(\sigma)$, $Q_1 = Q(\sigma)$, $\sigma = h$.

Step 2. Find the smallest constant k_1 such that

$$E(a_1, Q_1) \subset \tilde{X}_1 = E(a_1, k_1^2 B^{-1}),$$

and it is not difficult to prove that k_1^2 is the maximal eigenvalue of the matrix $B^{1/2}Q_1B^{1/2}$.

Step 3. Consider the system on the next subsegment $[t_1, t_2]$ with $E(a_1, k_1^2 B^{-1})$ as the initial ellipsoid at instant t_1.

Next steps continue iterations 1-3. At the end of the process we will get the external estimate $E(a(t), Q(t))$ of the tube $X(t)$ with accuracy tending to zero when $m \to \infty$.

Consider the estimation of the viable trajectory tube $X(t)$ of the system (28) under constraint (7). We modify Algorithm 1 taking into account the viability constraint (7) where we take $Y(t) = Y = E(y_0, D)$.

In this case from Theorems 3-4 we have the main inclusion

$$X(t_0 + \sigma, t_0, X_0) \subseteq E(a(\sigma), Q(\sigma)) \cap E(y_0, D) +$$

$$+ o(\sigma)B(0, 1), \quad X_0 = E(a, k^2 B^{-1}), \quad (34)$$

which allows to formulate the modified algorithm which is more complicated now (all notations in (34) are taken from Theorem 4).

Algorithm 2: Subdivide the time segment $[t_0, T]$ into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + ih$ $(i = 1, \ldots, m)$, $h = (T - t_0)/m$, $t_m = T$.

Step 1. Given $X_0 = E(a, k_0^2 B^{-1})$ with $k_0 \neq 0$, define $X_1 = E(a_1, Q_1)$ from (34) (as at Step 1 of Algorithm 1) for $a_1 = a(\sigma)$, $Q_1 = Q(\sigma)$, $\sigma = h$.

Step 2. Consider the intersection of ellipsoids $X_1 = E(a_1, Q_1)$ and $Y(t) = Y = E(y_0, D)$ and find the smallest (with respect to some criterion, e.g. as in [17]) ellipsoid $X_1^* = E(a_1^*, Q_1^*)$ such that

$$E(a_1, Q_1) \cap E(y_0, D) \subset E(a_1^*, Q_1^*).$$

Step 3. Find the smallest constant k_1 such that

$$E(a_1^*, Q_1^*) \subset \tilde{X}_1 = E(a_1^*, k_1^2 B^{-1}),$$

k_1^2 is the maximal eigenvalue of the matrix $B^{1/2}Q_1^*B^{1/2}$.

Step 4. Consider the system on the next interval $[t_1, t_2]$ with $E(a_1^*, k_1^2 B^{-1})$ as the initial ellipsoid taken at initial instant t_1.

Next steps continue iterations 1-4. At the end of the process we will get the external estimating tube $E(a^*(t), Q^*(t))$ of the tube $X(t)$ with accuracy tending to zero when $m \to \infty$.

C. Examples

Consider the following system

$$\begin{cases} \dot{x_1} &= -x_1, \\ \dot{x_2} &= \frac{1}{2}x_2 + 3(\frac{x_1^2}{4} + x_2^2), \end{cases} \quad (35)$$

$$X_0 = E(0, Q_0), \; Q_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}. \quad (36)$$

Here $d = \{0, 3\}$, $\tilde{f}(x) = x'Bx$ with $B = Q_0^{-1}$, $T = 0.15$. Results of computer simulations based on Theorem 4 are shown at Fig. 2.

Assume now that state constraint (7) is also present for the system (35) with $Y = B(0, r)$, $r = 2.4$.

Applying Algorithm 2 we discover that the viability constraint becomes important in estimation only after 10th iteration so we may use there the simpler Algorithm 1. After that beginning with the 11th iteration the whole four-steps procedure works. Fig. 3-4 illustrate this estimation process.

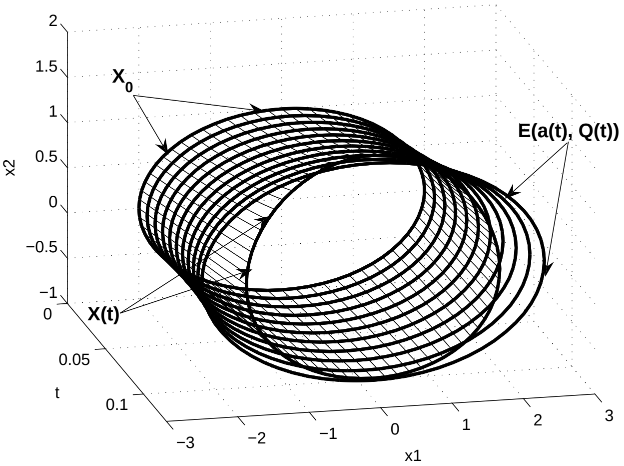

Fig. 2. Trajectory tube $X(t, t_0, X_0)$ and its external ellipsoidal tube $E(a(t), Q(t))$.

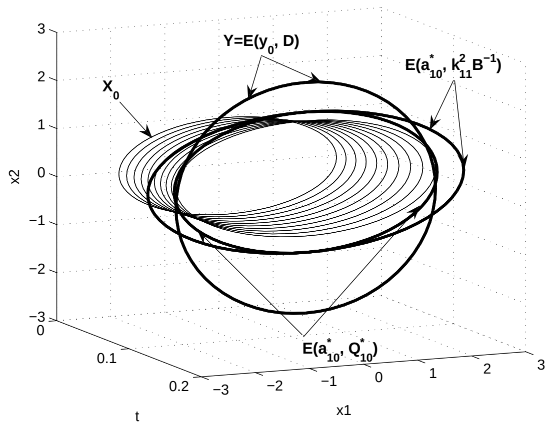

Fig. 3. Viable trajectory tube $X(t, t_0, X_0)$ and its external ellipsoidal tube $E(a^*(t), Q^*(t))$.

V. Conclusions

The paper deals with the problems of control and state estimation for a dynamical control system described by differential inclusions with unknown but bounded initial state. The solution to the differential system is studied through the techniques of trajectory tubes with their cross-sections $X(t)$ being the reachable sets at instant t to control system.

Basing on the well-known results of ellipsoidal calculus developed for linear uncertain systems we present the modified state estimation approaches which use the special nonlinear structure of the control system and simplify calculations. Examples and numerical results related to procedures of set-valued approximations of trajectory tubes and reachable sets were also presented.

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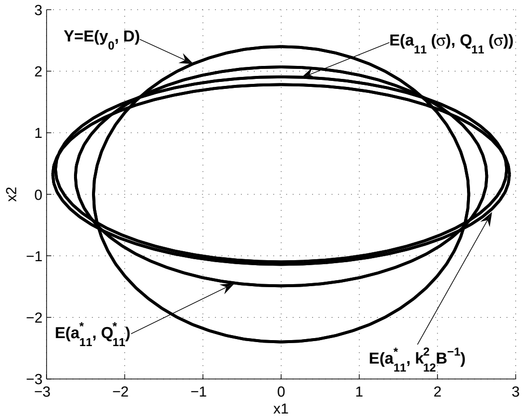

Fig. 4. Algorithm 2: calculations at steps 1-4 of 11th iteration.

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