STRUCTURAL ROBUSTNESS OF ELECTRIC MACHINE APPLICATIONS USING THE ICAD FRAMEWORK

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Abstract
Adequate control of three-phase machines (e.g., induction motors -IMs- and synchronous generators -SGs-) is of paramount importance for the electric power industry. These are multivariable, non-linear systems. In this paper, it is formally demonstrated using the ICAD framework that the electrical subsystems of the IM and of the permanent magnet SG, due to their inherent structural robustness, are the multivariable equivalent to stable, minimum-phase, single-input single-output systems. As a consequence, an adequate performance and robustness may be achieved through fixed, stable, minimum-phase, diagonal controllers –justifying the widespread use of control schemes based on fixed, classical linear controllers such as PI.

Key words
Decentralized control, frequency domain analysis and control, individual channel analysis and design, multivariable control, multivariable structure function.

1 Introduction
In a similar manner as the induction motor (IM) is the workhorse of the electric power industry when converting electrical into mechanical energy, the synchronous generator (SG) is the IM counterpart when transforming mechanical into electrical energy [Krause, Waszynzuk and Sudhoff, 2002; Ugalde-Loo, Ekanayake and Jenkins, 2013]. It is noteworthy that fixed linear controllers are able to provide an adequate, robust performance in practice. In line with this, the structural robustness of two types of electric machines is here investigated. Through the individual channel analysis and design (ICAD) framework [O’Reilly and Leithead, 1991], it is shown that the electrical subsystems of the IM and the permanent magnet SG (PMSG) share characteristics that make them the multiple-input multiple-output (MIMO, or multivariable) equivalent of stable, minimum-phase, uncertain, single-input single-output (SISO) systems. Such attributes allow the use of fixed, stable, minimum-phase, diagonal controllers –and shed light on how it is possible that simple PI controllers are sufficient to operate specific machines.

2 Individual Channel Analysis and Design
In order to define the existence of stabilizing controllers for any system it is of great significance to assess its zero-pole structure, which may be affected by parametric uncertainty. The interpretation of such structure for multivariable systems, in terms of control design, is more difficult. ICAD, a frequency domain multivariable control framework, allows bridging this gap [O’Reilly and Leithead, 1991]. ICAD makes possible to analyze the existence of stabilizing controllers...
The system (1)-(2) can be represented in terms of inputs, outputs, and the block diagram of the system. Its equivalent scalar individual channel configurations (3) are the following [O’Reilly and Leithead, 1991]:

1. The system open loop poles are stable.
2. The MSF has no unstable poles.
3. The limit of $\gamma(s)$ as $s \to \infty$ is equal to zero.
4. The Nyquist plot of $\gamma(s)$ does not encircle the point $(1,0)$.

In particular, conditions ii and iv are required for the transmission zeros to be minimum phase. However, this is not sufficient for the system to have integrity. If condition ii is not satisfied, the closed loop system may not possess integrity: compliance of condition i is also necessary for this. Condition iii is required so that arbitrary high-bandwidth control is possible.

If a system complies with all four conditions, the existence of a stabilizing controller (2) reduces to a controller that stabilizes simultaneously the individual channels (3) and the diagonal transfer functions (5) [O’Reilly and Leithead, 1991]. As $c_i(s)$ and $h_i(s)$ are stable and minimum phase, a system that complies with i-iv is the analogous of a stable and minimum phase SISO system, which may be controlled at an arbitrary bandwidth without incurring on unstable zero-pole cancellations. In addition, the resulting closed loop control system will also present integrity; i.e., stability if either controller $k_1(s)$ or $k_2(s)$ is deactivated. This gives the system basic fault tolerance properties. It is possible to extend the previous attributes to uncertain MIMO systems represented as individual channels (i.e., uncertain SISO systems).

Summarizing, it is possible to control a system complying with conditions i-iv, under parameter uncertainty and for realistic combinations of parameters, through fixed linear diagonal controllers. This is applicable, by extension, to more complex con-
3 Renewable Energy Technologies Application: the Permanent Magnet Synchronous Generator

Tidal stream and wind turbines (TSTs, WTs), along with other renewable energy technologies, are becoming considerably utilized in modern electrical power systems to mitigate climate change. They share some characteristics in terms of the electrical generators employed, system architecture and control strategies. In fact, both technologies aim to extract as much as possible energy from either the wind or the flow. Figure 3 shows the configuration of a turbine based on a PMSG and a full power converter applicable to wind and tidal stream turbines [Whitby and Ugalde-Loo, 2013].

Although the PMSG model is a multivariable, nonlinear system, the generator-side controller is normally designed using simplified SISO first order models. This practice may result in a control system with a limited performance that may require manual re-tuning. However, in this section it is formally shown that the PMSG has structural properties that allow the use of fixed, linear and low order controllers able to achieve system decoupling.

![Figure 3. Wind/tidal stream turbine based on a PMSG [Whitby and Ugalde-Loo, 2013].](image)

3.1 Mathematical Model

The PMSG model used for applications on renewable energy generation is expressed in a $dq$ frame. It is described by [Krishnan, 2010]:

\[
\begin{align*}
\frac{d}{dt} i_d &= v_d - \frac{R_s}{L_d} i_d + \frac{1}{L_d} n_p \omega_{gen} i_q, \\
\frac{d}{dt} i_q &= v_q - \frac{R_s}{L_q} i_q - \frac{1}{L_q} n_p \omega_{gen} i_d - \frac{\psi_m n_p \omega_{gen}}{L_q}.
\end{align*}
\]

(7)

where $L_d$, $L_q$, are the self inductances of the stator; $R_s$ the stator resistance; $v_d$, $v_q$, the stator voltages; $i_d$, $i_q$, the stator currents; $\psi_m$ the flux linkage of the permanent magnet; $\omega_{gen}$ the generator mechanical speed; $\omega_r = n_p \omega_{gen}$ the electrical rotor speed; and $n_p$ the number of pole pairs. The model is completed by a suitable representation of the drive-train:

\[
\begin{align*}
\frac{d}{dt} \omega_{gen} &= \frac{1}{J} (\tau_r - \tau_{em}), \\
\tau_{em} &= \frac{3}{2} n_p [\psi_m i_q + (L_d - L_q) i_d i_q],
\end{align*}
\]

(8)

where $J$ is the combined inertia of the rotor and generator, $\tau_r$ the hydro or aerodynamic torque developed by the rotor, and $\tau_{em}$ the electromagnetic torque.

Although system (7)-(8) is nonlinear, $\omega_{gen}$ varies at speeds well below the closed loop currents subsystem. This bandwidth separation allows considering $\omega_{gen}$ as an uncertain constant parameter when analyzing the currents subsystem. This is a well-known and accepted property of some nonlinear systems.

3.2 State-Space Representation

Let the PMSG be represented by (7) and (8). In vector control (or field oriented control) schemes, the generator-side converter controls the operation of the electric machine by effectively regulating the stator currents $i_d$, $i_q$, through the stator voltages $v_d$, $v_q$, in (7). The system has a state-space form

\[
\dot{x} = Ax + Bu,
\]

\[
y = Cx + Du,
\]

(9)

where

\[
x = \begin{bmatrix} i_d & i_q \end{bmatrix}^T, \quad u = \begin{bmatrix} v_d & v_q \end{bmatrix}^T, \quad y = \begin{bmatrix} i_d & i_q \end{bmatrix}^T,
\]

(10)

and

\[
A = \begin{bmatrix} -\frac{R_s}{L_d} & \frac{1}{L_d} n_p \omega_{gen} \\ \frac{L_d}{L_q} & -\frac{R_s}{L_q} \end{bmatrix},
\]

(11)

\[
B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = 0_{2 \times 2}.
\]

3.3 Transfer Matrix Representation

Although the system described by (7) and (8) is nonlinear, the generator mechanical speed $\omega_{gen}$ varies at
speeds considerably below the closed loop of the current subsystem. Due to such bandwidth separation, it is possible to consider (9)-(11) as linear time-invariant (LTI) and thus design a linear controller robust to parametric variations, with $\omega_{\text{gen}} \in [\omega_{\text{gen,min}},\omega_{\text{gen,max}}]$ being the uncertain parameter. Thus, for a particular value of $\omega_{\text{gen}}$, system (9) has a representation in the frequency domain as

$$
y(s) = G_{\omega}(s)u(s),
$$

$$
\begin{bmatrix}
  i_d(s) \\
  i_q(s)
\end{bmatrix} =
\begin{bmatrix}
  g_{11}(s) & g_{12}(s) \\
  g_{21}(s) & g_{22}(s)
\end{bmatrix}
\begin{bmatrix}
  v_d(s) \\
  v_q(s)
\end{bmatrix},
$$

(12)

where $G_{\omega}(s) = C(sI - A)^{-1}B$ is the transfer matrix. The elements of the transfer matrix are, explicitly,

$$
G_{\omega}(s) = \begin{bmatrix}
  L_q s + R_s & L_q n_q \omega_{\text{gen}} \\
  -L_d n_r \omega_{\text{gen}} & L_d s + R_s
\end{bmatrix},
$$

(13)

with

$$
d_{\omega}(s) = L_d L_q s^2 + (L_d R_s + L_q R_s)s + R_s^2 + L_d L_q n_q^2 \omega_{\text{gen}}^2,
$$

(14)

$$
d_{\omega}(s) = d_1 s^2 + d_2 s + d_3.
$$

3.4 Individual Channel Analysis

System (12) conforms to the structure of a classical 2×2 ICAD system, and thus, the standard analysis and results from the ICAD framework apply directly. Conditions i-iv from Section 2 are proved to define the existence of stabilizing controllers for (13).

3.4.1 Condition i: the system is open loop stable:

Let

$$
A = \{ G_{\omega}(s) : \omega_{\text{gen}} \in \mathbb{R} \}
$$

be the set of transfer functions for every shaft speed $\omega_{\text{gen}}$ as defined by (13). The elements of set $A$ are stable if and only if the poles of $G_{\omega}(s)$ are stable; i.e., $d_{\omega}(s)$ satisfies the Routh-Hurwitz stability criterion:

$$
\Re \{ \text{poles} \{d_{\omega}(s)\} \} < 0 \Leftrightarrow \{ d_1 > 0, d_2 > 0, d_3 > 0 \}.
$$

(15)

It is apparent from (14) that condition (16) is satisfied as $d_1 > 0$, $d_2 > 0$ and $d_3 > 0$ for realistic combinations of machine parameters (positive inductances and resistances). Therefore, $A$ is open loop stable $\forall \omega_{\text{gen}}$.

3.4.2 Condition ii: the MSF is stable:

Let the individual channels be defined as

$$
c_1(s) : v_d(s) \rightarrow i_d(s)
$$

$$
c_2(s) : v_q(s) \rightarrow i_q(s).
$$

(17)

The MSF is obtained according to (4) as follows:

$$
\gamma(s) = \frac{g_{12}(s) g_{21}(s)}{g_{11}(s) g_{22}(s)} = -\frac{L_d L_q n_q^2 \omega_{\text{gen}}^2}{(L_q s + R_s)(L_d s + R_s)}.
$$

(18)

Let

$$
B = \{ \gamma(s) : \omega_{\text{gen}} \in \mathbb{R} \}
$$

be the resulting set of MSFs in (18). It is immediate that the elements of $B$ are stable $\forall \omega_{\text{gen}}$ since the poles of $\gamma(s)$ are given by $\{-R_s/L_q, -R_s/L_d\}$ for any realistic combination of machine parameters.

3.4.3 Condition iii: the limit of $\gamma(s)$ as $s \rightarrow \infty$ is zero:

Since the relative degree of (18) is 2, it is immediate that

$$
\lim_{s \rightarrow \infty} \gamma(s) = 0.
$$

(19)

3.4.4 Condition iv: the Nyquist plot of $\gamma(s)$ does not encircle (1, 0):

Since condition iii is fulfilled, the Nyquist plot of $\gamma(s)$ does not encircle point (1, 0) if

$$
\Re \{ \gamma(j\omega) \} < 1, \forall \omega \in E,
$$

(20)

with

$$
E = \{ \omega : \arg \{ \gamma(j\omega) \} = 0, \omega \in \mathbb{R} \}.
$$

(21)

Thus, to satisfy condition iv, all the intersections of the Nyquist trajectory of $\gamma(s)$ with the real axis, represented by set $E$, should be to the left of (1, 0). Evaluating the MSF (18) at $s = j\omega$ yields:

$$
\gamma(j\omega) = \frac{-L_d L_q n_q^2 \omega_{\text{gen}}^2}{R_s^2 - L_d L_q \omega^2 + j(\omega L_d R_s + \omega L_q R_s)},
$$

(22)

which can be rewritten as:

$$
\gamma(j\omega) = \frac{n_\gamma}{r_\gamma + j(\im_\gamma)} = \frac{\re_\gamma n_\gamma - j(\im_\gamma n_\gamma)}{\re_\gamma^2 + \im_\gamma^2},
$$

(23)

with $r_\gamma, \im_\gamma \in \mathbb{R}$, and

$$
n_\gamma = -L_d L_q n_q^2 \omega_{\text{gen}}^2, \quad r_\gamma = R_s^2 = L_d L_q \omega^2, \quad \im_\gamma = \omega L_d R_s + \omega L_q R_s.
$$

(24)

The real and imaginary parts of $\gamma(j\omega)$ are given by:

$$
\Re[\gamma(j\omega)] = \frac{r_\gamma n_\gamma}{\re_\gamma^2 + \im_\gamma^2},
$$

(25)

$$
\Im[\gamma(j\omega)] = -\frac{j \im_\gamma n_\gamma}{\re_\gamma^2 + \im_\gamma^2}.
$$

Set $E$ is obtained by calculating the frequency values where the argument of $\gamma(j\omega)$ is equal to zero; i.e.,
arg \left\{ \gamma_0(j\omega) \right\} = 0 
\iff 
\frac{\text{Im} \left\{ \gamma_0(j\omega) \right\}}{\text{Re} \left\{ \gamma_0(j\omega) \right\}} = \frac{-i_m n_{\gamma}}{i_m n_{\gamma}} = 0, \tag{26}
which is true if and only if \( re, n_{\gamma} \to \pm \infty \) or \( im, n_{\gamma} = 0 \). Notice from (24) that
\[
re, n_{\gamma} \to \infty \iff \omega \to \pm \infty \quad \text{and} \quad \omega \to \pm \infty \Rightarrow \gamma_0(j\omega) \to 0.
\]
Therefore, the elements of \( E \) are obtained by solving \( im, n_{\gamma} = 0 \) for \( \omega \). Thus condition (22) is rewritten as
\[
E = \left\{ \omega : im, n_{\gamma} = 0, \omega \in \mathbb{R} \right\}. \tag{27}
\]
Elements of (27) are found using (24) as follows:
\[
im, n_{\gamma} = -L_d L_q R_{n_r^2} \omega^2_{gen} (L_d + L_q) \omega = 0, \tag{28}
\]
from where it is obvious that \( E = 0 \), meaning that the only intersection of the Nyquist plot of \( \gamma_0(j\omega) \) with the real axis (besides the origin) occurs at \( \omega = 0 \).

Using (25), condition (21) is rewritten as:
\[
\text{Re} \{ \gamma_0(j\omega) \} < 0 \iff \frac{re, n_{\gamma}}{re, \gamma} + im, \gamma < 1. \tag{29}
\]
Since \( re, n_{\gamma}, im, \gamma \in \mathbb{R} \),
\[
\frac{re, n_{\gamma}}{re, \gamma} + im, \gamma < 1 \iff re, n_{\gamma} - \left( \frac{re, \gamma}{im, \gamma} \right) < 0. \tag{30}
\]
Evaluating (30) for \( \omega = 0 \) yields
\[
re, n_{\gamma} - \left( \frac{re, \gamma}{im, \gamma} \right) \bigg|_{\omega=0} = -L_d L_q R_{n_r^2} \omega^2_{gen} - R^4_s. \tag{31}
\]
It can be seen that
\[
\text{Re} \{ \gamma_0(j0) \} < 1 \iff -L_d L_q R_{n_r^2} \omega^2_{gen} - R^4_s < 0, \tag{32}
\]
which proves condition (21); the Nyquist plot of \( \gamma_0(s) \) does not encircle the point \((1,0)\) for any realistic combination of parameters \( \forall \omega_{gen} \in \mathbb{R} \).

4 High Performance Induction Motor Applications

IMs are widely used on industrial applications due to their attractive cost-effect attributes. However, for high performance applications such as high precision positioning, the operation of IMs is more complex than that of traditional DC motors. Within this context, the most successful control scheme is the rotor-flux indirect field oriented control (RIFOC) [Rodríguez, Kennel, Espinoza, Trincado, Silva and Rojas, 2012]. This is based on the introduction of torque- and flux-producing virtual stator currents. In this manner the IM can be operated as a DC motor. The scheme, shown in Figure 4, requires an internal controller which decouples the stator currents (or the electrical subsystem) [Amézquita-Brooks, Licéaga-Castro and Licéaga-Castro, 2013].

Although the IM is a MIMO non-linear system, the stator currents controller is normally designed using simplified SISO first order models as in the case of the PMSG. Similarly, this results in control systems with limited performance requiring extensive manual tuning. In a similar fashion as in Section 3, it is formally demonstrated in this section that the IM has structural properties amenable to using fixed, linear, low order controllers for system decoupling.

![Figure 4. Traditional RIFOC IM control scheme.](image)

4.1 Mathematical Model

The IM model is described by the following differential equations [Krishnan, 2001]:
\[
\begin{aligned}
d &\frac{d}{dt} i_{\alpha s} = a_{11} i_{\alpha s} + \frac{L_m R_{r}}{\sigma L_s L_r} \psi_{\alpha s} + \frac{L_m \omega_{r}}{\sigma L_s L_r} \psi_{\beta s} + \frac{\psi_{\alpha s}}{\sigma L_s}, \\
d &\frac{d}{dt} i_{\beta s} = a_{22} i_{\beta s} - \frac{L_m \omega_{r}}{\sigma L_s L_r} \psi_{\alpha s} + \frac{L_m R_{r}}{\sigma L_s L_r} \psi_{\beta s} + \frac{\psi_{\beta s}}{\sigma L_s}, \\
d &\frac{d}{dt} \psi_{\alpha r} = \frac{L_m R_{r}}{L_r} i_{\beta s} + \omega_{r} \psi_{\alpha r} - \frac{R_e}{L_r} \psi_{\beta r}, \\
d &\frac{d}{dt} \psi_{\beta r} = \frac{L_m R_{r}}{L_r} i_{\alpha s} + \omega_{r} \psi_{\beta r} - \frac{R_e}{L_r} \psi_{\alpha r},
\end{aligned}
\tag{33}
\]

where \( L_s, L_r, L_m \), are the stator, rotor and mutual inductances; \( R_s, R_r, \) the stator and rotor resistances; \( \psi_{\alpha s}, \psi_{\beta s} \), the stator voltages; \( i_{\alpha s}, i_{\beta s} \), the stator currents; \( \psi_{\alpha r}, \psi_{\beta r} \), the rotor fluxes; \( \omega_r \), the electrical rotor speed; and
\[
a_{11} = a_{22} = \frac{L^2_m R_s L_r}{\sigma L_s L_r^2}.
\]
The dispersion coefficient \( \sigma \) is defined as:
\[
\sigma = 1 - \frac{L^2_m}{L_s L_r}. \tag{34}
\]

Equations in (33) represent the electrical subsystem of the IM. The model is completed by
\[
\begin{aligned}
d &\frac{d}{dt} \omega_r = \frac{P}{2J} (\tau_s - \tau_L), \\
\tau_s &= \frac{3}{2} \left( \frac{P}{2} \right) \frac{L_m}{L_r} (\psi_{\alpha r} i_{\beta s} - \psi_{\beta r} i_{\alpha s}),
\end{aligned}
\tag{35}
\]
where \( J \) is is the rotor inertia, \( \tau_s \) the load torque, \( \tau_e \) the electromagnetic torque, and \( P \) the number of poles.

As in the case of the PMSG, although system (33)-(35) is nonlinear, \( \omega_r \) varies at speeds well below the closed loop currents subsystem. This bandwidth separation allows considering \( \omega_r \) as an uncertain constant parameter when analyzing the currents subsystem.
4.2 State-Space Representation

Let an IM be represented by (33) and (35). Vector control schemes require the control of the stator currents $i_{αs}, i_{βs}$, by driving the stator voltages $v_{αs}, v_{βs}$, using a voltage source inverter. The system has a state-space representation (9), where

$$\mathbf{x} = \begin{bmatrix} i_{αs} & i_{βs} & ω_r & ψ_r^β \end{bmatrix}^T,$$

$$\mathbf{u} = \begin{bmatrix} v_{αs} & v_{βs} \end{bmatrix}^T,$$

$$\mathbf{y} = \begin{bmatrix} i_{αs} & i_{βs} \end{bmatrix}^T,$$

and

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & L_m R_r & \frac{L_m ω_r}{L_s L_r} \\ 0 & a_{22} & -L_m R_r & \frac{L_m ω_r}{L_s L_r} \\ \frac{L_m R_r}{L_r} & 0 & -R_r & -ω_r \\ 0 & \frac{L_m R_r}{L_r} & ω_r & -R_r \end{bmatrix},$$

$$\mathbf{B} = \begin{bmatrix} \frac{1}{L_s} & 0 & 0 & 0 \\ 0 & \frac{1}{L_s} & 0 & 0 \end{bmatrix}^T,$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{D} = \mathbf{0}_{2×2}.$$  

(36)  

(37)

4.3 Transfer Matrix Representation

The stator currents subsystem of the IM is a nonlinear plant. However, it is possible to consider the realization (9), (36), (37) as LTI since the rotor speed $ω_r$ varies at speeds considerably below the closed loop of the current subsystem. Such a bandwidth separation allows the design of a linear controller robust to parametric variations, with $ω_r ∈ [ω_r, min, ω_r, max]$ being the uncertain parameter.

Following the same procedure as in Section 3, the system is represented in the frequency domain for particular values of $ω_r$ as

$$\mathbf{y}(s) = \mathbf{G}_{ω_r}(s)\mathbf{u}(s),$$

$$\begin{bmatrix} i_{αs}(s) \\ i_{βs}(s) \end{bmatrix} = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \begin{bmatrix} v_{αs}(s) \\ v_{βs}(s) \end{bmatrix},$$

$$\mathbf{G}_{ω_r}(s) = \begin{bmatrix} n_{ωr,11}(s) & n_{ωr,12}(s) \\ n_{ωr,21}(s) & n_{ωr,22}(s) \end{bmatrix} d_{ωr}(s),$$

(38)  

(39)

Elements of (39) are given as

$$n_{ωr,11}(s) = n_{ωr,22}(s) = \frac{1}{σ L_s},$$

$$\left( s^3 + \frac{2σ L_s L_r^2 R_r + L_m^2 R_r L_r + L_r^3 R_s}{σ L_s^3} \right) +$$

$$\left( \frac{σ L_s L_r (L_r^2 ω_r^2 + R_r^2) + 2L_r^2 R_s R_r + L_m^2 R_r^2}{σ L_s^3} \right) +$$

$$\left( \frac{L_r^2 R_s ω_r^2 + L_r R_s R_r^2}{σ L_s^3} \right),$$

$$n_{ωr,12}(s) = -n_{ωr,21}(s) = -\frac{L_m^2 R_r ω_r}{σ^2 L_s^2 L_r},$$

$$d_{ωr}(s) = s^4 + d_{ω1}s^3 + d_{ω2}s^2 + d_{ω3}s + d_{ω4}.$$  

(40)

4.4 Individual Channel Analysis

System (38) also conforms to the structure of a classical 2×2 ICAD system. Conditions i-iv from Section 2 are proved in a similar way as in Section 3.

4.4.1 Condition i: the system is open loop stable:

Let

$$C = \{ \mathbf{G}_{ω_r}(s) : ω_r ∈ \mathbb{R} \}$$

be the set of transfer functions for every mechanical speed $ω_r$ as defined by (39). The elements of $C$ are stable if and only if the poles of $\mathbf{G}_{ω_r}(s)$ are stable. This requires that the real part of the roots of $d_{ωr}(s)$ in (40) satisfy the Routh-Hurwitz stability criterion; i.e.,

$$\text{Re}\{\text{poles}\{d_{ωr}(s)\}\} < 0 \Leftrightarrow$$

$$\begin{cases}
    d_{ω1} > 0, d_{ω2} > 0, d_{ω3} > 0, d_{ω4} > 0, \\
    d_{ω1} > 0, d_{ω2} - d_{ω3} > 0, d_{ω4} > 0, \\
    d_{ω1}d_{ω2}d_{ω3} - d_{ω1}^2d_{ω4} - d_{ω2}^2d_{ω3} > 0.
\end{cases}$$

(43)

From (34) it can be seen that $σ > 0$ for any realistic combination of inductances (positive values), since $L_s = L_{ls} + L_m$ and $L_r = L_{lr} + L_m$, where $L_{ls}$ and $L_{lr}$ are the stator and rotor leakage inductances. Thus, it is clear from (41) that $d_{ω1} > 0$, $d_{ω2} > 0$, $d_{ω3} > 0$, and $d_{ω4} > 0$ for realistic combinations of machine parameters (positive inductances and resistances). It can be shown that
Further algebraic manipulation shows that
\[ \begin{align*}
\eta_1 \eta_2 - \eta_1 & = \beta(3L_3^2L_2^2R_sR_2^2 + L_m^4R_c^2) + \\
& + \beta(4\sigma L_3^3R_sR_2^2 + 3\sigma L_sL_rL_m^3R_2^3 + 2L_s^2R_sR_2) + \\
& + \beta[\mu^2L_3^3R_3^3(2\sigma L_sL_r + L_m^2) + 2\mu^2L_2^2L_2^3R_2^3],
\end{align*} \]
where \( \beta = (s^4L^4L_0^2)^{-1} \). It is clear that
\[ \eta_1 \eta_2 - \eta_1 > 0 \] \hspace{1cm} (53)
\[ \forall \omega_r. \] Therefore, set \( D \) is stable \( \forall \omega_r \) for any realistic combination of machine parameters.

\[ \square \]

### 4.4.3 Condition iii: the limit of \( \gamma_{\omega_r}(s) \) as \( s \to \infty \) is zero

As in the case of a PMSG, the proof for an IM is immediate, since it can be seen from (40) and (47) that the relative degree of \( \gamma_{\omega_r}(s) \) is 4.

\[ \square \]

### 4.4.4 Condition iv: the Nyquist plot of \( \gamma_{\omega_r}(s) \) does not encircle \((1, 0)\): Since condition iii is fulfilled, the Nyquist plot of \( \gamma_{\omega_r}(s) \) does not encircle the point \((1, 0)\) if
\[ \text{Re}\left\{ \gamma_{\omega_r}(j\omega) \right\} < 1, \forall \omega \in F \] \hspace{1cm} (54)
with
\[ F = \{ \omega: \ \arg\left[ \gamma_{\omega_r}(j\omega) \right] = 0, \omega \in \mathbb{R}\}. \] \hspace{1cm} (55)
This requires that all the intersections of the Nyquist trajectory of \( \gamma_{\omega_r}(s) \) with the real axis, represented by \( F \), should be to the left of \((1, 0)\). Following some algebraic manipulation and by using (40), the MSF (47), evaluated at \( s = j\omega \), is given as:
\[ \gamma_{\omega_r}(j\omega) = \frac{L_2^2L_4^2R_3^2\omega^2}{d_2^2}, \] \hspace{1cm} (56)
where
\[ d_2(j\omega) = -j\omega^3L_2L_4^2\omega^2 - L_3^2R_s\omega^2 + j\omega\beta L_3^2L_2^2 + \\
+ 2\omega^2\sigma L_sL_3^2R_2 - j\omega L_sL_rR_2^2 + \omega^2L_3^2R_s + \\
-2j\omega L_2^2R_sR_2 - L_3R_sR_2^2 + \omega^2L_3^2L_4R_s + \\
- j\omega L_2^2L_4R_s, \]
which in turn is rewritten as
\[ d_2 = re_{d_2} + j(im_{d_2}) \]

\[ \text{to facilitate the analysis, with } re_{d_2}, im_{d_2} \in \mathbb{R}, \] and
\[ re_{d_2} = -L_3^2R_s\omega^2 + 2\omega^2\sigma L_sL_3^2R_2 + \omega^2L_3^2R_s + \\
- L_3R_sR_2^2 + \omega^2L_sL_2^2R_2, \]
\[ im_{d_2} = -\omega L_sL_3^2\omega^2 + \omega^3\sigma L_sL_3^2 - \omega L_sL_rR_2^2 + \\
-2\omega L_2^2R_sR_2 - \omega L_3^2R_2. \] \hspace{1cm} (57)
Therefore \( \gamma_{\omega_r}(j\omega) \) in (56) can be expressed as:
\[ \gamma_{\omega r}(j\omega) = \frac{L_1^4 L_2^4 R_r^2 \omega^2 \omega^2}{(\text{re}_{\omega r}^2 - \text{im}_{\omega r}^2)^2 + (2\text{re}_{\omega r} \text{im}_{\omega r})^2}, \]  
(58)

which was obtained after realizing the denominator, where

\[ \zeta = \left( \text{re}_{\omega r}^2 - \text{im}_{\omega r}^2 \right) - 2j\left( \text{re}_{\omega r} \text{im}_{\omega r} \right). \]

Separating (58) into real and imaginary components yields

\[ \text{Re}\left[ \gamma_{\omega r}(j\omega) \right] = \frac{L_1^2 L_m^4 R_r^2 \omega^2 \omega^2 (\text{re}_{\omega r}^2 - \text{im}_{\omega r}^2)}{(\text{re}_{\omega r}^2 - \text{im}_{\omega r}^2)^2 + (2\text{re}_{\omega r} \text{im}_{\omega r})^2}, \]
\[ \text{Im}\left[ \gamma_{\omega r}(j\omega) \right] = \frac{-L_1^2 L_m^4 R_r^2 \omega^2 \omega^2 (\text{re}_{\omega r}^2 - \text{im}_{\omega r}^2)}{(\text{re}_{\omega r}^2 - \text{im}_{\omega r}^2)^2 + (2\text{re}_{\omega r} \text{im}_{\omega r})^2}. \]
\[ \text{(59)} \]

For simplicity, let

\[ c_1 = \text{re}_{\omega r}^2 - \text{im}_{\omega r}^2, \quad c_2 = 2\text{re}_{\omega r} \text{im}_{\omega r}, \]
\[ \text{Re}\left[ \gamma_{\omega r}(j\omega) \right] = \frac{L_1^2 L_m^4 R_r^2 \omega^2 \omega^2 c_1}{c_1^2 + c_2^2}, \]
\[ \text{Im}\left[ \gamma_{\omega r}(j\omega) \right] = \frac{-L_1^2 L_m^4 R_r^2 \omega^2 \omega^2 c_2}{c_1^2 + c_2^2}. \]
\[ \text{(60)} \]

Using (62), condition (54) may be rewritten as

\[ \frac{L_1^2 L_m^4 R_r^2 \omega^2 \omega^2 c_1}{c_1^2 + c_2^2} < 1, \forall \omega \in F. \]
\[ \text{(61)} \]

Since \( c_1, c_2 \in \mathbb{R} \),

\[ \frac{L_1^2 L_m^4 R_r^2 \omega^2 \omega^2 c_1}{c_1^2 + c_2^2} < 1 \iff L_1^2 L_m^4 R_r^2 \omega^2 \omega^2 c_1 < c_1^2 + c_2^2 \]
\[ \text{(65)} \]

It can be seen that for \( c_1 < 0 \), inequality (65) holds \( \forall \omega \in F \). This is sufficient to prove condition \( iv \). However, if \( c_1 > 0 \), then (65) is equivalent to

\[ \frac{L_1^2 L_m^4 R_r^2 \omega^2 \omega^2 c_1}{c_1^2 + c_2^2} < 1 \iff c_1 + \frac{c_2}{c_1} - L_1^2 L_m^4 R_r^2 \omega^2 \omega^2 > 0 \]
\[ \text{(66)} \]

In order to prove (65) and (66) it is necessary to calculate set \( F \) defined by (55); i.e., the frequency values \( \omega \) at which the argument of \( \gamma_{\omega r}(s) \) is equal to zero. From (62) and (63),

\[ \arg \left[ \gamma_{\omega r}(j\omega) \right] = 0 \iff \text{Im}\left[ \gamma_{\omega r}(j\omega) \right] = \frac{c_2}{c_1} = 0, \]
\[ \text{(67)} \]

which is true if and only if \( c_1 \rightarrow \pm \infty \) and/or \( c_2 = 0 \). Considering, by condition \( iii \), that

\[ c_1 \rightarrow \pm \infty \iff \omega \rightarrow \pm \infty \] and \( \omega \rightarrow \pm \infty \iff \gamma_{\omega r}(j\omega) \rightarrow 0, \]

then the elements of \( F \) are the roots of \( c_2 \). That is,

\[ F = \{ \omega: c_2 = 0, \omega \in \mathbb{R} \}. \]
\[ \text{(68)} \]

Elements of (68) are found using (61) and (57) as

\[ F = \{ F(1), F(2), F(3) \}, \]
\[ \text{(69)} \]

with

\[ F(1) = 0, \quad F(2) = \pm \sqrt{\frac{\Omega (R_r^2 + \omega_r^2)}{\Omega}}, \]
\[ \text{(70)} \]

and

\[ \Omega = 2\sigma L_s L_r R_r + L_s^2 R_r + L_r^2 R_r, \]
\[ \Psi = \sigma L_s L_r, \]
\[ \Upsilon = 2L_s L_r R_r + L_r^2 R_r. \]
\[ \text{(71)} \]

Since

\[ \text{Re}\left[ \gamma_{\omega r}(j\omega) \right] = \text{Re}\left[ \gamma_{\omega r}(-j\omega) \right], \forall \omega \in \mathbb{R}^+, \]

only positive values of \( \omega \) in set \( F \) described by (69) are tested in condition (54). If \( \omega = F(1) \), it follows directly from (56) that

\[ \gamma_{\omega r}(j\omega) = 0 \Rightarrow \text{Re}\left[ \gamma_{\omega r}(jF(1)) \right] < 1. \]
\[ \text{(72)} \]

If \( \omega = F(2) \), \( c_1 \) is calculated using (57) and (61) as:

\[ c_1 = -R_r \left[ R_r^4 + (R_r \omega_r L_r)^2 \right] \cdot c_\alpha, \]
\[ \text{(73)} \]

where

\[ c_\alpha = \left[ L_1^4 R_r^2 + 3\sigma L_s L_r L_r^2 R_r^2 + 3L_s^2 L_r^2 R_r R_r + +L_s^2 L_r^2 \sigma \omega_r^2 + 2(\sigma L_s L_r R_r)^2 + 2L_s^4 R_r^2 + ++4\sigma L_s L_r R_r R_r + 2L_s^4 (\sigma \omega_\alpha)^2 \right]^2. \]
\[ \text{(74)} \]

It is obvious that \( c_\alpha > 0 \), and therefore, \( c_1 < 0 \) for any realistic combination of machine parameters. From (65), it follows that

\[ c_1 < 0 \Rightarrow \text{Re}\left[ \gamma_{\omega r}(jF(2)) \right] < 1. \]
\[ \text{(75)} \]

If \( \omega = F(3) \), \( c_1 \) is obtained as

\[ c_1 = \frac{R_r^2}{\sigma^2 L_s^2 L_r^2} \cdot c_\alpha. \]
\[ \text{(76)} \]

Since \( c_\alpha > 0 \), from (74) it follows that \( c_1 > 0 \). In this case, the inequality defined in (66) is checked; that is,

\[ \text{Re}\left[ \gamma_{\omega r}(jF(3)) \right] < 1 \iff c_1 + \frac{c_2}{c_1} - L_1^2 L_m^4 R_r^2 \omega^2 \omega^2 > 0 \]
\[ \text{(77)} \]

Algebraic calculation for \( \omega = F(3) \) gives
\[ c_1 + \frac{c_2}{c_1} - L_i^2 l_m r^2 s^2 \omega_\text{c}^2 = \frac{R_i^2}{s^2 L_i^2 l_m} \cdot \left[ L_i^2 R_i^2 + 2L_i^2 l_m R_i R_r + L_m^2 R_r^2 + \ldots \right. \\
+ \sigma L_i L_r (L_i^2 l_m^2 \omega_\text{c}^2 + 2L_i^2 l_m R_i R_r + 2L_m^2 R_r^2) + \ldots \\\n+ L_i^2 (\sigma L_i \omega_\text{c})^2 + (\sigma L_i L_r R_r)^2 \right]. \\
\cdot \left[ 4L_i^2 R_i^2 + 4L_i^2 l_m R_i R_r + L_m^2 R_r^2 + \ldots \right. \\
+ 8\sigma L_i L_m R_r R_r + 4\sigma L_i L_r (L_i^2 l_m^2 R_r^2) + \ldots \\\n+ 4L_i^2 (\sigma L_i \omega_\text{c})^2 + 4(\sigma L_i L_r R_r)^2 \right]
\]

which shows that (75) is fulfilled.

Since (70), (73) and (75) are true, condition (54) has been proven: the Nyquist plot of \( \gamma_{\omega r}(s) \) does not encircle the point \((1,0) \) \( \forall \omega_r \in \mathbb{R} \) for any realistic combination of IM machine parameters.

5 Discussion and Concluding Remarks

The mathematical proofs presented in Sections 3 and 4, namely that the PMSG and the IM comply with conditions i-iv, show that the existence of a diagonal stabilizing controller for either machine reduces to the existence of a controller which simultaneously stabilizes the individual channels and the diagonal transfer functions. As in both cases the channels and the diagonal transfer functions are stable and minimum phase, the systems may be controlled at an arbitrary bandwidth without incurring on unstable zero/pole cancellations.

Compliance of conditions i-iv for all realistic combinations of IM machine parameters is a consequence of the inherent structural robustness of the PMSG and the IM. These attributes shed some light as to why simple diagonal stabilizing controllers are able to achieve an adequate system performance under parametric variations—in both cases, the rotor speed.

Although the studies here presented are based on a simple diagonal control structure, the results can be generalized to more complex structures by extension.

References


