

A CLASSICAL ANALOG OF LANDAU-ZENER TUNNELING AS A NEW TYPE OF THE MECHANICAL ENERGY TRAP

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Abstract

A detailed analytical study of irreversible energy transfer in an oscillatory system with time-dependent parameters has not been addressed thus far in the literature. This paper demonstrates a closed-form asymptotic solution of this problem for a system of two weakly coupled linear oscillators; the first oscillator with constant parameters is excited by an initial impulse, whereas the coupled oscillator with a time-dependent frequency is initially at rest but then acts as an energy trap. It is shown that in physically meaningful limiting cases the problem of irreversible energy transfer from the excited oscillator to the trap is reduced to a first-order equation with the solution in the form of the Fresnel integrals. In view of a mathematical analogy between energy transfer in a classical oscillatory system with variable parameters and non-adiabatic quantum Landau-Zener transition, the results of this paper, in addition to providing an analytical framework for understanding the transient dynamics of coupled oscillators, suggest an approximate procedure for solving the linear Landau-Zener problem with arbitrary initial conditions over a finite time-interval. A correctness of approximations is confirmed by numerical simulations.

Key words

Control of oscillations, control of time-varying systems

1. Introduction

The problem of energy transfer is currently a topic of intense research with a broad spectrum of applications, from multi-body systems and waves in fluids and plasmas, to semiconductors, and nano-crystals with graphene layers, among other novel applications; a rich variety of examples in diverse fields of applied mathematics, natural sciences, and engineering can be found in [Vakakis *et al.*, 2008]. However, most of the results reported in the literature are related to energy exchange in systems with

constant parameters. This work develops an analytical framework to investigate the dynamics of two weakly-coupled oscillators with time-varying frequencies, with special attention to an analogy between the energy transfer in this classical oscillatory system and quantum Landau-Zener tunneling.

The classic linear Landau-Zener problem [Landau, 1932; Zener, 1932] deals with a two-level system described by a Hamiltonian depending linearly on time. Due to its generality, the Landau-Zener scenario has been applied to numerous problems in various contexts, such as, e.g., laser physics [Sahakyan *et al.*, 2010], semiconductor superlattices [Rosam *et al.*, 2003], tunneling of optical [Trompeter *et al.*, 2006] or acoustic [Sanchis-Alepuz *et al.*, 2007; de Lima *et al.*, 2010] waves and quantum information processing [Saito, 2006], to name just a few examples.

Although a passage between two energy levels is an intrinsic feature of all above-mentioned processes, the demonstration of a direct connection between energy transfer in a classical oscillatory system with time-dependent parameters and non-adiabatic quantum Landau-Zener tunneling is a recent development. It has been shown that the equations of the slow passage through resonance in a system of two weakly coupled pendulums with a time-dependent frequency are asymptotically identical to the equations of the Landau-Zener tunneling problem, i.e., there exists a profound analogy between irreversible energy transfer in the oscillatory system and non-adiabatic quantum tunneling [Manevitch *et al.*, 2010; Kosevich, Manevitch and Manevitch, 2010; Kovaleva, Manevitch and Kosevich, 2011]. This phenomenon may be treated as an extension of the previously found analogy between adiabatic quantum tunneling and energy exchange in a chain of weakly coupled oscillators with constant parameters [Kosevich, Manevitch, and Savin, 2007, 2008].

While an exact solution to the Landau-Zener

equation is well-known [Zener, 1932], this equation is actually too complicated for any straightforward inferences about the system dynamics. After the seminal Landau paper [Landau, 1932], attention has focused on quasi-stationary solutions at infinitely large times (see, e.g., [Nakamura, 2002]). Recently, transient non-adiabatic tunneling has been studied asymptotically assuming quasi-stationary behavior of the system [Wittig, 2005; Berns, 2008].

In this paper, we describe a model of two weakly coupled oscillators with the time-dependent frequency detuning. We transform the system of two differential equations into a single integro-differential equation for the coupled oscillator (the energy trap) and derive the evolutionary equations describing the slowly-varying envelopes of near-resonance motion for both oscillators. We demonstrate that the second-order equation for the slow envelope of the trap oscillations is identical to that of the Landau-Zener problem. Then we show that the latter equation can be reduced to the first-order equation in two special cases; in the first case, the mass of the excited oscillator far exceeds the mass of the coupled trap; in the second case, the coefficients of weak coupling are lesser than the detuning rate. In both cases, we find an explicit asymptotic solution in the form of the Fresnel integrals. The theory is illustrated by numerical simulations.

2. Equations of motion

For brevity, we investigate resonant energy transfer in a system of two weakly coupled linear oscillators. We suppose that the first oscillator with mass m_1 and stiffness c_1 is excited by an initial impulse V ; the coupled oscillator with mass m_2 and time-dependent stiffness $C_2(t) = c_2 - (k_1 - k_2)t$, $k_{1,2} > 0$ is initially at rest; the oscillators are connected by linear coupling of stiffness c_{12} . The displacements and velocities of the oscillators are denoted by u_i and $V_i = du_i/dt$, $i = 1, 2$. We will prove that the second oscillator with a time-dependent frequency acts as an energy trap and ensures a visible reduction of oscillations of the excited mass.

The quasi-resonance interaction between the oscillators implies that $(c_1/m_1)^{1/2} = (c_2/m_2)^{1/2} = \omega$, a likely small detuning may be included in the coefficient k_1 . Assuming weak coupling, we define the small parameter of the problem as $c_{12}/c_2 = 2\varepsilon \ll 1$. Then we introduce the dimensionless parameters $c_{12}/c_r = 2\varepsilon\lambda_r$, $r = 1, 2$; $\lambda_2 = 1$; $k_1/c_2 = 2\varepsilon\sigma$, $k_2/(c_2\omega) = 2\varepsilon^2\beta^2$ and the dimensionless time-scales $\tau_0 = \omega t$, $\tau_1 = \varepsilon\tau_0$. Using these notations, the equations of motion can be written in the dimensionless form

$$\begin{aligned} \frac{d^2 u_1}{d\tau_0^2} + u_1 + 2\varepsilon\lambda_1(u_1 - u_2) &= 0, \\ \frac{d^2 u_2}{d\tau_0^2} + u_2 + 2\varepsilon\lambda_2(u_2 - u_1) - 2\varepsilon\zeta(\tau_1)u_2 &= 0, \end{aligned} \quad (1)$$

with the initial conditions $\tau_0 = 0$, $u_1 = u_2 = 0$; $v_1 = V/\omega = V_0$, $v_2 = 0$, $v_i = du_i/d\tau_0$. Here and below the coefficient $\zeta(\tau_1) = \sigma - 2\beta^2\tau_1$ defines the detuning modulation.

In order to develop an effective asymptotic procedure, we express the solution u_1 of the first equation in (2.3) as

$$u_1(\tau_0) = \omega_\varepsilon^{-1} [V_0 \sin \omega_\varepsilon \tau_0 + 2\varepsilon\lambda_1 \int_0^{\tau_0} u_2(s) \sin \omega_\varepsilon(\tau_0 - s) ds], \quad (2)$$

where $\omega_\varepsilon = (1 + 2\varepsilon\lambda_1)^{1/2}$. The substitution of (2) transforms the second equation of (1) into the integro-differential equation

$$\begin{aligned} \frac{d^2 u_2}{d\tau_0^2} + (1 + 2\varepsilon\lambda_2)u_2 - 2\varepsilon\zeta(\tau_1)u_2 &= \\ = 2\varepsilon\omega_\varepsilon^{-1} \lambda_2 [V_0 \sin \omega_\varepsilon \tau_0 + 2\varepsilon\lambda_1 \int_0^{\tau_0} u_2(s) \sin \omega_\varepsilon(\tau_0 - s) ds], \end{aligned} \quad (3)$$

with the initial conditions $\tau_0 = 0$: $u_1 = u_2 = 0$; $v_1 = V_0$, $v_2 = 0$. Hence, instead of two coupled second-order equations (1), we consider a single integro-differential equation (3) for u_2 . The process u_1 is then found from Eq. (1).

3. Complex envelope

The asymptotic analysis of Eqs (1) and (3) is performed with help of the so-called complexification-averaging technique, based on the complexification of the dynamics and the separation of the fast and slow time-scales (Manevich and Manevitch, 2005).

We introduce a pair of the complex-valued variables ψ and ψ^* by the following formulas

$$\psi = v_2 + iu_2, \quad \psi^* = v_2 - iu_2 \quad (4)$$

From (3), (4), we derive the equation for $\psi(\tau_0, \varepsilon)$

$$\begin{aligned} \frac{d\psi}{d\tau_0} - i\psi - i\varepsilon(\lambda_2 - \zeta(\tau_1))(\psi - \psi^*) &= \\ = 2\varepsilon\lambda_2\omega_\varepsilon^{-1} [V_0 \sin \omega_\varepsilon \tau_0 - \\ - i\varepsilon\lambda_1 \int_0^{\tau_0} (\psi(s, \varepsilon) - \psi^*(s, \varepsilon)) \sin \omega_\varepsilon(\tau_0 - s) ds], \quad \psi(0) = 0. \end{aligned} \quad (5)$$

In order to separate resonance harmonics, the solution of Eq. (5) is written as

$$\psi(\tau_0, \varepsilon) = \varphi(\tau_0, \varepsilon) e^{i\omega_\varepsilon \tau_0}, \quad (6)$$

where the slow envelope $\varphi(\tau_0, \varepsilon)$ is constructed in the form of the multiple-scales expansion [Nayfeh, 2000]

$$\varphi(\tau_0, \varepsilon) = \varphi_0(\tau_1) + \varepsilon\varphi_1(\tau_0, \tau_1) + \dots \quad (7)$$

A series of standard transformations [Kovaleva, Manevitch and Kosevich, 2011] yields the following equation for the leading-order term $\varphi_0(\tau_1)$:

$$\frac{d\varphi_0}{d\tau_1} - i(\rho + 2\beta^2\tau_1)\varphi_0 = -i\lambda_2 V_0 - \lambda_1\lambda_2 \int_0^{\tau_1} \varphi_0(r)dr, \quad (8)$$

$$\varphi_0(0) = 0,$$

where $\rho = \lambda_2 - \lambda_1 - \sigma$. Equation (8) is equivalent to the second-order differential equation

$$\frac{d^2\varphi_0}{d\tau_1^2} - i(\rho + 2\beta^2\tau_1)\frac{d\varphi_0}{d\tau_1} + (\lambda_1\lambda_2 - i\beta^2)\varphi_0 = 0. \quad (9)$$

with the initial conditions $\tau_1 = 0$: $\varphi_0 = 0$, $d\varphi_0/d\tau_1 = -i\lambda_2 V_0$. The equivalence of Eq. (9) and the equation of the Landau-Zener transient tunneling problem has been demonstrated in [Kovaleva, Manevitch and Kosevich, 2011].

Once the slow envelope $\varphi_0(\tau_1)$ is found, the leading-order approximations for u_2 and v_2 can be derived from (6), (7). We obtain

$$\begin{aligned} u_{20}(\tau_0, \tau_1) &= |\varphi_0(\tau_1)|\sin(\omega_\varepsilon\tau_0 + \alpha(\tau_1)), \\ v_{20}(\tau_0, \tau_1) &= |\varphi_0(\tau_1)|\cos(\omega_\varepsilon\tau_0 + \alpha(\tau_1)), \\ \alpha(\tau_1) &= \arg\varphi_0(\tau_1). \end{aligned} \quad (10)$$

Partial energy of the oscillator is calculated as

$$e_{20}(\tau_1) = \frac{1}{2}(\langle u_{20}^2 \rangle + \langle v_{20}^2 \rangle) = \frac{1}{2}|\varphi_0(\tau_1)|^2, \quad (11)$$

$\langle \cdot \rangle$ denotes is the averaging over the ‘‘fast’’ period $T = 2\pi/\omega_\varepsilon$.

3. Calculation of u_1

Once u_{20} is determined, u_1 can be directly found from (1). However, in order to demonstrate an analogy between the classical model (1) and the Landau-Zener equations, we approximately calculate u_1 using the following representation

$$y = v_1 + iu_1, y^* = v_1 - iu_1, \quad (12)$$

in which

$$\begin{aligned} y(\tau_0, \varepsilon) &= \eta(\tau_0, \varepsilon) e^{i\omega_\varepsilon\tau_0}, \\ \eta(\tau_0, \varepsilon) &= \eta_0(\tau_1) + \varepsilon\eta_1(\tau_0, \tau_1) + \dots \end{aligned} \quad (13)$$

After a series of transformations [Kovaleva, Manevitch, and Kosevich, 2011], we obtain the resulting system for the leading-order approximations $\varphi_0(\tau_1)$, $\eta_0(\tau_1)$

$$\frac{d\eta_0}{d\tau_1} = -i\lambda_1\varphi_0(\tau_1), \quad \eta_0(0) = V_0, \quad (14)$$

$$\begin{aligned} \frac{d\varphi_0}{d\tau_1} - i(\rho + 2\beta^2\tau_1)\varphi_0 &= -\lambda_2[iV_0 + \lambda_1 \int_0^{\tau_1} \varphi_0(r)dr], \\ \varphi_0(0) &= 0. \end{aligned}$$

It is easy to obtain the main approximation of the solution u_1, v_1 in the following form

$$\begin{aligned} u_{10}(\tau_0, \tau_1) &= |\eta_0(\tau_1)|\sin(\omega_\varepsilon\tau_0 + \alpha(\tau_1)) \\ v_{10}(\tau_0, \tau_1) &= |\eta_0(\tau_1)|\cos(\omega_\varepsilon\tau_0 + \alpha(\tau_1)), \\ \alpha(\tau_1) &= \arg(\eta_0(\tau_1)). \end{aligned} \quad (15)$$

The partial energy of the oscillator is calculated as

$$e_{10}(\tau_1) = \frac{1}{2}(\langle u_{10}^2 \rangle + \langle v_{10}^2 \rangle) = \frac{1}{2}|\eta_0(\tau_1)|^2. \quad (16)$$

4. Approximate analysis of energy transfer

The analysis of the full system (14) can be significantly simplified if the integral terms in the second equation can be omitted. It is easy to prove that this term may be ignored in two cases, $2\beta^2 \gg \lambda_1\lambda_2$ or/and $m_1 \gg m_2$.

If $2\beta^2 \gg \lambda_1\lambda_2$, then the truncated equation for the slowly-varying envelope $\varphi_0(\tau_1)$ is written as

$$\frac{d\varphi_0}{d\tau_1} - i(-\sigma + 2\beta^2\tau_1)\varphi_0 = -i\lambda V_0, \quad \varphi_0(0) = 0. \quad (17)$$

Equation (17) possesses the precise solution

$$\varphi_0(\tau_1) = -i\frac{\lambda V_0}{\beta} F(\tau_1, \theta_0) e^{i(\beta\tau_1 + \theta_0)^2}, \quad (18)$$

where $\theta_0 = -\sigma/2\beta$ and

$$\begin{aligned} F(\tau_1, \theta_0) &= \int_{\theta_0}^{\beta\tau_1 + \theta_0} e^{-ih^2} dh = \\ &= [C(\beta\tau_1 + \theta_0) - C(\theta_0)] - i[S(\beta\tau_1 + \theta_0) - S(\theta_0)], \end{aligned} \quad (19)$$

$C(x)$ and $S(x)$ are the cos- and sin-Fresnel integrals.

The amplitude of oscillations can be easily evaluated in two limiting cases:

1. If $\beta\tau_1 \ll \sqrt{2}$, then $\varphi_0(\tau_1) \approx \lambda V_0 \tau_1$.
2. If $\beta\tau_1 \gg \sqrt{2}$, then the following asymptotic representations holds [Gradshteyn and Ryzhik, 2000]

$$\begin{aligned} \varphi_0(\tau_1) \rightarrow \bar{\varphi}_0 &= -i\frac{\lambda V_0}{\beta} \left\{ \left[\sqrt{\frac{\pi}{8}} - C(\theta_0) \right] - \right. \\ &\left. - i \left[\sqrt{\frac{\pi}{8}} - S(\theta_0) \right] \right\}, \text{ as } \tau \rightarrow \infty. \end{aligned} \quad (20)$$

The energy of quasi-stationary oscillations is calculated as $\bar{e}_{20} = \frac{1}{2}|\bar{\varphi}_0|^2$; the energy \bar{e}_{10} can be found from (14) – (16). Note that an analysis of the approximate solution as $\tau_1 \rightarrow \infty$ is formally incorrect but expression (20) can be considered as an illustration of a transition from the initial rest state to quasi-stationary oscillations.

Figure 1 illustrates the occurrence of targeted energy transfer in the system with the parameters $\varepsilon = 0.136$; $V_0 = 0.368$; $\varepsilon\sigma = 0.1125$; $(\varepsilon\beta)^2 = 0.025$; $\lambda = 1$.

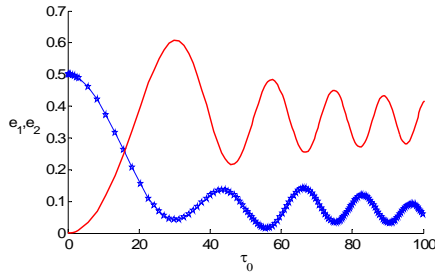


Figure 1. Energy of the oscillator (blue) and the trap (red)

If $m_1 \gg m_2$, then the truncated equation for the slowly-varying envelope $\varphi_0(\tau_1)$ takes the form

$$\frac{d\varphi_0}{d\tau_1} - i(\rho_1 + 2\beta^2 \tau_1)\varphi_0 = -i\lambda_2 V_0, \varphi_0(0) = 0, \quad (21)$$

where $\rho_1 = \lambda_2 - \sigma$. It is obvious that the solution of Eq. (21) is similar to (18), namely,

$$\varphi_0(\tau_1) = -i \frac{\lambda V_0}{\beta} F(\tau_1, \theta_1) e^{i(\beta\tau_1 + \theta_1)^2}, \quad (22)$$

where $\theta_1 = \rho_1/\sigma$. The asymptotic behaviour of the complex envelope $\varphi_0(\tau_1)$ is akin to (20).

Figure 2 demonstrates irreversible energy transfer from the first oscillator to the trap in the system with the parameters $m_1 = 5m_2$, $\varepsilon = 0.05$; $V_0 = 1$, $\varepsilon\sigma = 0.1125$, $\varepsilon\beta = 0.1$, $\lambda = 1$.

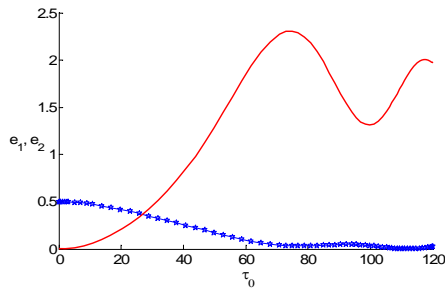


Figure 1. Energy of the oscillator (blue) and the trap (red)

5. Conclusion

This paper demonstrates that in significant limiting cases the problem of irreversible energy transfer in an oscillatory system with time-dependent parameters can be efficiently solved in terms of the Fresnel integrals. It is shown that the evolution equations of the slow passage through resonance are identical to the equations of the Landau-Zener tunneling problem, and, therefore, the suggested asymptotic solution of the classical problem provides a simple analytic description of the quantum Landau-Zener tunneling with arbitrary initial conditions over a finite time-interval. A correctness of approximations is confirmed by numerical simulations.

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