Suppressing noise-induced intensity pulsations in semiconductor lasers by means of time-delayed feedback

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***Abstract*— We investigate the possibility to suppress noise-induced intensity pulsations (relaxation oscillations) in semiconductor lasers by means of the Pyragas control scheme. In contrast to previous studies, where the control was used to enhance the correlation time and thus the coherence of the oscillations, we focus on the suppression of the oscillations and use the mean oscillation amplitude as a measure. We first consider a generic normal form model which is a paradigm for a system close to a Hopf bifurcation. Here, we find an analytic expression for the mean square amplitude of the oscillations. We then investigate the control scheme analytically and numerically in a laser model of Lang-Kobayashi type.**

I. Introduction

Time-delayed feedback control has been widely studied to stabilize unstable states [1], [2], [3], [4], [5]. Another direction of research on feedback has focused on the control of noise induced oscillations [6], [7], [8], [9], [10], [11], [12], [13]. Here the works have mainly studied the possibility to enhance the correlation time and thus to increase the regularity of the oscillations. In this paper we shift the attention towards the suppression of noise induced oscillations. This idea is first studied in a generic model consisting of a damped harmonic oscillator driven by noise. We then investigate a practically relevant example, namely a semiconductor laser subject to optical feedback by a Fabry-Perot resonator.

II. Generic model

The generic model we consider is a damped harmonic oscillator (stable focus) subject to noise (ξ) and feedback control

$$\begin{aligned} \dot{z}(t) &= (\lambda - i\omega_0)\, z(t) + D\xi(t) \qquad (z \in \mathbb{C}) \\ &\quad -K\,[z(t) - z(t-\tau)], \end{aligned} \tag{1}$$

where $\lambda < 0$ and ω_0 are the damping rate and the natural frequency of the oscillator respectively, D is the noise amplitude, K is the feedback strength and τ is the delay time of the control term. We consider uncorrelated white Gaussian noise

$$\begin{aligned} \xi(t) &= \xi_1(t) + i\xi_2(t), \qquad (\xi_i \in \mathbb{R}) \\ \langle \xi_i \rangle &= 0, \\ \langle \xi_i(t)\, \xi_j(t') \rangle &= \delta_{ij}\delta(t-t'). \end{aligned}$$

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Before proceeding, we transform (1) into a rotating frame $z(t) = u(t)\, e^{-i\omega_0 t}$

$$\begin{aligned} \dot{u}(t) &= (\lambda - K)\, u(t) + K\, e^{i\omega_0 \tau} u(t-\tau) + e^{i\omega_0 t} D\, \xi(t) \\ \Leftrightarrow \quad \dot{u}(t) &= a\, u(t) + b\, u(t-\tau) + D\, \tilde{\xi}(t), \end{aligned} \tag{2}$$

where $\tilde{\xi}(t) = e^{i\omega_0 t}\xi(t)$ is a noise term with the same properties as $\xi(t)$. The purpose of the transformation was to make a real, which will be necessary later. In [14] Küchler and Mensch analyzed equation (2) for real variables. We will here follow their ideas and adapt them to the complex case.

We will calculate the auto correlation function

$$G(t) := \langle u(s+t)\, \overline{u(s)} \rangle$$

in an interval $t \in [0, \tau]$. In particular, this gives the mean square radius $\langle r^2 \rangle = \langle |u|^2 \rangle = G(0)$ of the oscillations. With the Green's function $u_0(t)$ solving

$$\dot{u}_0(t) - a\, u_0(t) - b u_0(t-\tau) = \delta(t),$$

where $u_0(t) = 0$ for $t < 0$, we can formally find a solution of equation (1)

$$u(t) = \int_{-\infty}^{t} \mathrm{d}t_1\, u_0(t-t_1)\, D\, \tilde{\xi}(t_1). \tag{3}$$

Using (3) we obtain

$$\begin{aligned} G(t) &= \langle u(\tilde{t}+t)\, \overline{u(\tilde{t})} \rangle \\ &= \int_{-\infty}^{\tilde{t}+t} dt_1 \int_{-\infty}^{\tilde{t}} dt_2\, u_0(\tilde{t}+t-t_1)\, \overline{u_0(\tilde{t}-t_2)} \\ &\quad \cdot D^2\, \langle \tilde{\xi}(t_1)\, \overline{\tilde{\xi}(t_2)} \rangle \\ &\overset{s=\tilde{t}-t_1}{=} 2D^2 \int_0^{\infty} ds\, u_0(s+t)\, \overline{u_0(s)} \\ &\equiv 2D^2\, C(t). \end{aligned}$$

The Green's function u_0 can be calculated [15], [14] by iteratively integrating equation (2) on intervals $[k\,\tau,\, (k+1)\,\tau)$

$$u_0(t) = \sum_{k=0}^{\lfloor t/\tau \rfloor} \frac{b^k}{k!}\, (t - k\,\tau)^k\, e^{a\,(t-\tau k)}.$$

From the definition of C and u_0 it follows, that C satisfies the following equations

$$\begin{aligned} C(t) &= \overline{C(-t)} && && (4)\\ \dot{C}(t) &= a\,C(t) + b\,C(t-\tau) && (t>0) && (5)\\ \dot{C}(t) &= a\,C(t) + b\,\overline{C(\tau-t)} && (t>0). && (6) \end{aligned}$$

Using these three equations, we can find an ODE for C

$$\begin{aligned} \frac{d^2}{dt^2}C(t) &\overset{(6)}{=} a\,C'(t) - b\,\overline{C'(\tau-t)}\\ &\overset{(5)}{=} a\,[a\,C(t) + b\,C(t-\tau)]\\ &\quad -b\,\overline{[a\,C(\tau-t) + b\,C(-t)]}\\ &= a^2C(t) + a\,b\,C(t-\tau)\\ &\quad -a\,b\,\overline{C(t-\tau)} - |b|^2\overline{C(-t)}\\ &= (a^2-|b|^2)\,C(t). \end{aligned}$$

Here it was necessary to have a real a, in order for the delay terms to cancel. Thus C is of the form

$$C(t) = A\,e^{\Lambda t} + B\,e^{-\Lambda t},$$

with

$$\Lambda = \sqrt{a^2+b^2} = \sqrt{(\lambda-K)^2-K^2}.$$

The complex coefficients A and B can be found from the equations

$$\begin{aligned} C(0) &= \overline{C(0)} \in \mathbb{R}, && (7)\\ \dot{C}(0) &= a\,C(0) + b\,\overline{C(\tau)} && (8) \end{aligned}$$

and

$$\begin{aligned} -1 &= \int_0^\infty ds\,\frac{d}{ds}\left[u_0(s)\,\overline{u_0(s)}\right] && (9)\\ &= \int_0^\infty ds\,\left[\dot{u}_0(s)\,\overline{u_0(s)} + u_0(s)\,\overline{\dot{u}_0(s)}\right]\\ &= a\,C(0) + b\,\overline{C(\tau)} + a\,C(0) + \overline{b}\,C(\tau). \end{aligned}$$

Solving equations (7),(8) and (9) for A and B gives the mean square oscillation radius. See eq. (11) at the top of the next page.

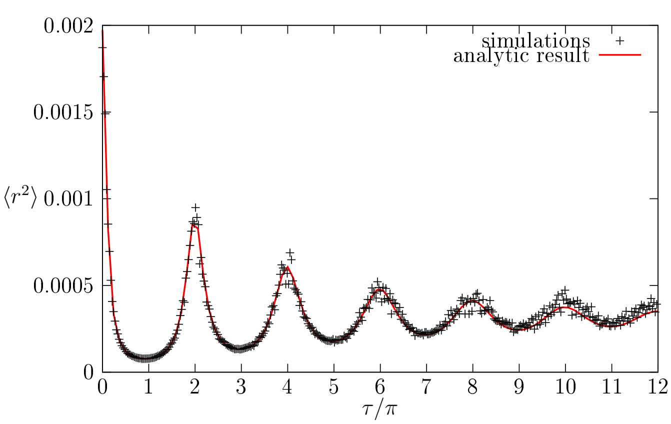

Fig. 1. Mean square radius $\langle r^2\rangle$ of oscillations as a function of the delay time τ. Parameters: $\lambda = -0.01$, $\omega_0 = 1$, $D = 0.001/2\pi$, $K = 0.2$

Figure 1 displays $\langle r^2\rangle$ as a function of the delay time τ. The mean square oscillation radius is modulated over τ with a period $T_0 = 2\pi/\omega_0$. The maxima and minima occur at

$$\tau_+ = n\,T_0 \quad \text{and} \quad \tau_- = \frac{2n+1}{2}\,T_0$$

respectively. The smallest oscillation radius is reached at

$$\tau_{opt} = T_0/2.$$

III. Laser model

In this section we investigate the effects of feedback and noise in a semiconductor laser. A laser with feedback from a conventional mirror can be described by the Lang-Kobayashi equations [16].

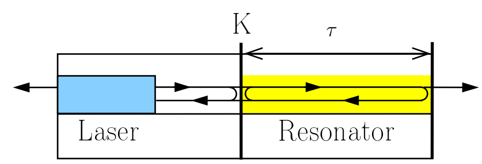

Fig. 2. Setup of a laser coupled to a Fabry-Perot resonator realizing the time-delayed feedback control

Feedback from a Fabry-Perot resonator is described by a modified set of equations [17], [18]

$$\begin{aligned} \frac{d}{dt}E &= \frac{1}{2}(1+i\,\alpha)\,n\,E - e^{i\varphi}K\,[E(t) - e^{i\psi}E(t-\tau)] + F_E(t),\\ T\,\frac{d}{dt}n &= p - n - (1+n)\,|E|^2, \end{aligned}$$

where

E	is the complex field amplitude,
n	is the carrier density,
α	is the linewidth enhancement factor,
K	is the feedback strength,
τ	is the roundtrip time in the Fabry-Perot,
p	is the pumping current,
T	is a timescale parameter,
F_E	is a noise term describing spontaneous emission,
φ, ψ	are phases dependening on the mirror positions.

We assume Gaussian white noise

$$\langle F_E\rangle = 0, \qquad \langle F_E(t)\,\overline{F_E(t')}\rangle = R_{sp}\delta(t-t'),$$

with the spontaneous emission rate

$$R_{sp} = \beta(n+n_0).$$

Transforming these equations into equations for intensity and phase $E = \sqrt{I}\,e^{i\phi}$ yields

$$\begin{aligned} \frac{d}{dt}I &= n\,I - 2K\,[I - \sqrt{I}\sqrt{I_\tau}\,\cos(\phi_\tau-\phi)] + R_{sp} + F_I(t),\\ \frac{d}{dt}\phi &= \frac{1}{2}\alpha\,n + K\,\frac{\sqrt{I_\tau}}{\sqrt{I}}\,\sin(\phi_\tau-\phi) + F_\phi(t), && (10)\\ T\,\frac{d}{dt}n &= p - n - (1+n)\,I, \end{aligned}$$

$$
\begin{aligned}
C(0) &= \langle r^2 \rangle = \mathrm{Re}(A) + \mathrm{Re}(B) \\
&= -\frac{1}{4\Lambda} \cdot \frac{K^2 + 2\Lambda^2 - K^2 \cosh(2\Lambda\tau)}{K\,\cosh(\Lambda\tau)\,[\Lambda\cos(\omega_0\tau) + K\,\sinh(\Lambda\tau)] + a\,[\Lambda + K\cos(\omega_0\tau)\,\sinh(\Lambda\tau)]}.
\end{aligned} \tag{11}
$$

with

$$
\begin{aligned}
\langle F_I \rangle = 0, \qquad & \langle F_\phi \rangle = 0, \\
\langle F_I(t)\, F_I(t') \rangle &= 2R_{sp}\,\delta(t-t') \\
\langle F_\phi(t)\, F_\phi(t') \rangle &= \frac{R_{sp}}{2I}\,\delta(t-t').
\end{aligned}
$$

Setting $\frac{d}{dt}I = 0$, $\frac{d}{dt}n = 0$, $\frac{d}{dt}\phi = \text{const}$, $K = 0$ and replacing the noise terms by there mean value, gives a set of equations for the steady state solutions I_*, n_* and $\phi = \omega_* t$ without feedback (the solitary laser mode). In [17] the authors showed, that this laser mode always exists, for arbitrary K, τ and φ and ψ, and that it is the only laser mode if

$$
K < K_c = \frac{1}{\tau\sqrt{1+\alpha^2}}.
$$

We will only consider this regime and investigate investigate how the control term influences the oscillations, caused by the noise, around this mode. Linearizing equations (10) around the steady state $X(t) = X_* + \delta X(t)$, with $X(t) = (I,\, \phi,\, n)$ gives

$$
\frac{d}{dt}X(t) = U\,X(t) - V\,[X(t) - X(t-\tau)] + F(t), \tag{11}
$$

with

$$
\begin{aligned}
U &= \begin{bmatrix} n_* - \Gamma_I & 0 & I_* + \beta \\ 0 & 0 & \frac{1}{2}\alpha \\ -\frac{1}{T}(1+n_*) & 0 & -\frac{1}{T}(1+I_*) \end{bmatrix}, \\
V &= \mathrm{diag}(K,\, K,\, 0)
\end{aligned}
$$

and

$$
F = (F_I,\, F_\phi,\, 0).
$$

Fourier transformation of (11) gives

$$
\widehat{X}(\omega) = \underbrace{[i\omega - U + V\,(1 - e^{-i\omega\tau})]^{-1}}_{\equiv M}\,\widehat{F}(\omega).
$$

The transformed covariance matrix of the noise is

$$
\langle \widehat{F}(\omega)\,\widehat{F}(\omega')^\dagger \rangle = \frac{1}{2\pi}\mathrm{diag}(2R_{sp}I_*,\, \frac{R_{sp}}{2I_*},\, 0)\,\delta(\omega-\omega'),
$$

with the adjoint $\dagger$. The matrix valued power spectrum $S(\omega)$ can then be defined through

$$
S(\omega)\,\delta(\omega-\omega') = \langle \widehat{X}(\omega)\,\widehat{X}(\omega)^\dagger \rangle
$$

and is thus given by

$$
\begin{aligned}
S(\omega) &= \mathrm{diag}\,(S_{\delta I}(\omega),\, S_{\delta\phi}(\omega),\, S_{\delta n}(\omega)) \\
&= \frac{1}{2\pi}\,M\,\mathrm{diag}(2R_{sp}I_*,\, \frac{R_{sp}}{2I_*},\, 0)\,M^\dagger.
\end{aligned}
$$

The frequency power spectrum is related to the phase power spectrum by [19]

$$
S_{\delta\dot\phi}(\omega) = \omega^2\,S_{\delta\phi}(\omega).
$$

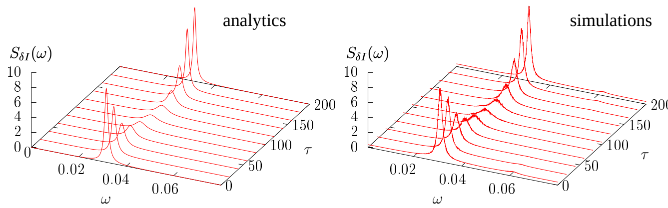

Fig. 3. Analytical and numerical results for the power spectrum of the intensity for different values of the delay time τ.
Parameters: $p = 1,\ T = 1000,\ \alpha = 2,\ \beta = 10^{-5},\ n_0 = 10,\ K = 0.002$

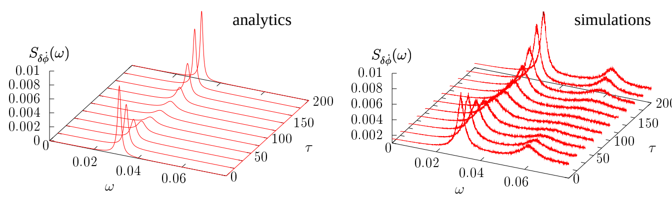

Fig. 4. Analytical and numerical results for the power spectrum of the frequency for different values of the delay time τ.
Parameters: $p = 1,\ T = 1000,\ \alpha = 2,\ \beta = 10^{-5},\ n_0 = 10,\ K = 0.002$

Figures 3 and 4 display the intensity and the frequency power spectra for different values of the delay time τ. All spectra have a main peak at the relaxation oscillation frequency $\Omega_{RO} \approx 0.03$. The higher harmonics can also be seen in the spectra obtained from simulations. The main peak decreases with increasing τ and reaches a minimum at

$$
\tau_{opt} \approx \frac{T_{RO}}{2} = \frac{2\pi}{2\Omega_{RO}} \approx 100.
$$

With further increasing τ the height increases again until it reaches approximately its original height at $\tau \approx T_{RO}$. A small peak in the power spectra indicates that the relaxation oscillation are strongly damped. This means that the fluctuations around the steady state values I_* and n_* are small.

As a measure for the steadiness of the intensity we will use the variance of the intensity

$$
\Delta I \equiv \left\langle \left(\langle I \rangle - I\right)^2 \right\rangle.
$$

This measure corresponds to the quantity $\langle r^2 \rangle$ we considered in the first section. Figure 5 displays the variance as a function of the delay time. The variance is minimal at $\tau \approx T_{RO}/2$, thus for this value of τ, the intensity is most steady and relaxation oscillations excited by the noise have a small amplitude.

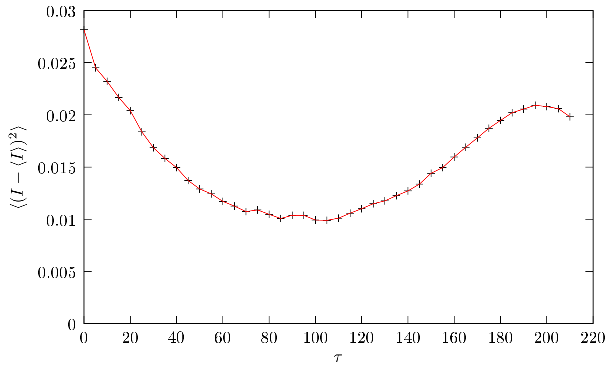

Fig. 5. Variance of the intensity vs. the delay time. Parameters: $p = 1$, $T = 1000$, $\alpha = 2$, $\beta = 10^{-5}$, $n_0 = 10$, $K = 0.002$

IV. Conclusion

In this paper we showed that time-delayed feedback can suppress noise induced oscillations.

In the first part we investigated a generic model consisting of a stable focus with noise and control. We found an analytic expression for the mean square radius of the oscillations. This quantity is modulated with a period of $T_0 = \frac{2\pi}{\omega_0}$ over τ. For $\tau = T_0/2$ the oscillations have the smallest amplitude.

In the second part we considered a laser coupled to a Fabry-Perot resonator. In the laser spontaneous emission noise excites relaxation oscillations. By tuning the cavity round trip time to half the relaxation oscillation period $\tau_{opt} \approx T_{RO}/2$ the oscillations can be suppressed considerably. This is demonstrated in the power spectra of the intensity and the frequency, where the relaxation oscillation peak has a minimal height at τ_{opt}. The variance of the intensity ΔI shows a minimum at τ_{opt}, thus the intensity is most steady at this τ value.

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