OPTIMAL HARVESTING PROBLEM IN A SIMPLE AGE STRUCTURE POPULATION

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Abstract

In this paper, we investigate a discrete model of population dynamics for the species with simple age structure. The cases when survival rates represent the functions of both age groups numbers are considered. We classify the types of dynamical modes and investigate the scenario of transition from regular to chaotic behavior, and vice versa. The optimization problem is stated and investigated. The problem was to find optimum catch quotas and a steady value of population number providing a theoretically maximum possible sustainable yield. It is shown that a single age class harvesting is the optimal one, and a choice of the age class is determined by the values of population parameters and prices ratio. It is found that stationary harvesting strategy with constant quota stabilizes the population dynamics for definite values of parameters. However, there is a range of the parameters at which there appear two-year fluctuations of number in the population. It leads to the necessity of transfer from harvesting based on constant catch quotas to threshold harvesting. It is shown that the threshold strategy always stabilizes the systems dynamics.

Key words

Population models, dynamic modes, harvesting strategy, constant catch quotas, threshold harvesting.

1 Introduction

The optimal harvesting problem is one of the important problems in mathematical biology. It has been considered in many articles [Beddington, Taylor, 1973; Clark, 1990; Jensen, 1996; Kokko, Lindstrom, 1998; Idels, Wang, 2008]. For many specific populations the optimum strategies of exploitation have been developed [Skaletskaya, et. all, 1979; Aanes, et. all, 2002]. To develop the optimal harvesting strategies, a wide range of mathematical models was built. In spite of the extensive study of this problem, still there is a possibility of finding its solution and more precise results for specific mathematical models, depending on the operation objectives.

In this paper, we consider the exploited population which, by the end of each reproduction season, consists of two age groups: juveniles (immature individuals) and adults (participants of the reproduction process). We assume that the time between two reproduction seasons is enough for the juveniles to become adults. Increase in the population number is regulated by density-dependent limitation of a younger class survival.

We study the harvesting strategy not leading to instabilities or extinctions, which results in maximum sustainable yields. The optimal control problem is solved by means of determining the optimal catch quota at differentiated harvesting in the population age groups. It is important that the development of harvesting strategies should be closely connected with the study of population structure, in particular, age structure. First, the population increment is a complex process involving the juveniles survival and growth, etc. and each of these characteristics may be affected differently by a change in the population density and harvest. Secondly, in most cases the field men are interested only in a part of the exploited population (eg, mature trees, big fish of marketable size, adult seals and their cubs).

2 Mathematical model and its research

In our mathematical model, n is a reproductive season number, x is a number of juveniles, y is a number of adults. The dynamic equations for our model are as follows

$$\left.\begin{aligned} x_{n+1} &= a y_n \\ y_{n+1} &= s(x_n, y_n) x_n + v y_n \end{aligned}\right\} \qquad (1)$$

where a is the birth rate, and v is the survival rate of adults and $s(x, y)$ is the survival rate of juveniles as a function of the age groups sizes

We consider two versions of the survival rate of juveniles a) $s(x, y) = \exp(-\alpha x - \beta y)$; b) $s(x, y) = 1 - \alpha x - \beta y$, where α and β are parameters which characterizes the ratio of intensity of the limitation of juvenile survival rate due to the number of adults and the self-limitation. To simplify the analysis of this system, we introduce a new parameter $q = \beta/\alpha$. The equilibrium points of special cases for the system (1) are found. The conditions of existence and stability for every one are determined.

We illustrate modeling results for the case when the survival rate of juveniles is a linear function ($s(x, y) = 1 - x - qy$).

Depending on the mode of stability loss by the nonzero solution for the system (1), according to which the population dynamics scenario will be developing, we can identify the following intervals for the parameter q values (fig.1).

1) In case of $0 \leq q \leq 1$, loss of stability may happen only at the moment of the pair of complex- conjugate roots of characteristic equation of system (1) passing through a unit circle ($\mid \lambda \mid = 1$, $\lambda = e^{i\varphi}$) (fig. 1a).

As a result quasi-periodic oscillations of the number of individuals appear in the system; they become chaotic if the systems parameters are further changed (fig. 2 a).

2) Subject to $1 < q \leq 5$ a new boundary of stability adds ($\lambda = -1$). As a result, a nonzero equilibrium becomes unstable, and the cycle of length two appears in the system.

With the parameter a further growth a regular cycle destroys and passes to the equilibrium point, which becomes unstable through the boundary $\lambda = e^{i\varphi}$, and quasi-periodic oscillations appear in the system (fig. 2 b).

This change in modes of the dynamics (from stabilization of the system to its destabilization, and vice versa) is realized within the range of v values, satisfying the inequality: $0 < v \leq v_1$, where $v_1 = (5q + 3 - 4\sqrt{q(q + 3)})/9$. If $v > v_1$, loss of stability under this model happens when the eigenvalues are conjugates and $\mid \lambda \mid$ transitions through 1 (fig. 1b).

3) At $5 < q < 9.8$, there are three possible scenarios of stability loss: emergence of 2-cycles, when $0 < v \leq v_2$, where $v_2 = (q + 3 - \sqrt{q(q - 2) + 49})/2$, the change of dynamics modes (equilibrium, 2-cycles, equilibrium, quasi-periodic oscillations), when $v_2 < v < v_1$, and the invariant curve emergence and destruction at $v_1 < v < 1$.

4) If $9.8 \leq q < 12$, then there are two possible scenarios of stability loss (fig. 1 d): the emergence of 2-cycle when $0 < v \leq v_2$ (fig. 2 c), and the emergence of the invariant curve, when $v_2 < v < 1$.

5) With further growth of the parameter $q \geq 12$ loss of the equilibrium stability may happen with the transition of one of the eigenvalues through -1 and is accompanied by a cascade of period-doubling bifurcations (fig. 1e).

We have obtained similar research results for the case when the survival rate of juveniles is an exponential function ($s(x, y) = \exp(-x - qy)$).

3 Formulation and solution of the problem of optimal control over the population dynamics

We assume that both age groups (juveniles and adults) are of commercial value. The population exploitation is characterized by the extraction of some individuals from every age group at fixed timing.

When the population is harvested the model (1) is to be:

$$\left.\begin{aligned} x_{n+1} &= (ay_n)(1 - u_1) \\ y_{n+1} &= (s(x_n, y_n)x_n + vy_n)(1 - u_2) \end{aligned}\right\} \quad (2)$$

where u_1 and u_2 are the catch quotas of immature and mature individuals respectively.

We consider the strategy with a stationary character of exploitation not leading to the population destruction.

The catch quotas u_1 and u_2 at stationary harvesting ensure the yield, its total income (I) defined by

$$I = c_1u_1(a\overline{y}) + c_2u_2(s(\overline{x}, \overline{y})\overline{x} + v\overline{y}) \quad (3)$$

or $I = c_1R_1 + c_2R_2$,

where $\overline{x}, \overline{y}$ are stationary numbers of the system (2), c_1 and c_2 - an average price of a single individual from both younger and older age classes, respectively, R_i ($i = 1, 2$) is the number of caught individuals belonging to the respective group.

The optimization problem is to determine the optimal catch quotas (u_1, u_2) and equilibrium values of the population size to provide a sustainable yield and maximum sales return.

In fact, we need to maximize the functional (3).

As a result of the optimization problem solution we have found out that the functional (3) maximum is achieved only at the borders $u_1 = 0$ or $u_2 = 0$.

In this connection, the optimization problem was considered for the cases: 1) $u_1 = 0$, ie, the control action is completely determined by the catch quota for mature individuals, and 2) $u_2 = 0$, ie only juveniles are exploited.

The equilibrium values of population size and optimal catch quota are found for each special case. The conditions of existence and stability for every one are determined.

To find the function absolute maximum (3) we have compared the income values of stationary hunting for adults ($\overline{I}_2$) and juveniles ($\overline{I}_1$).

For example, in case of $s(x, y) = 1 - x - qy$ the maximum (3) is

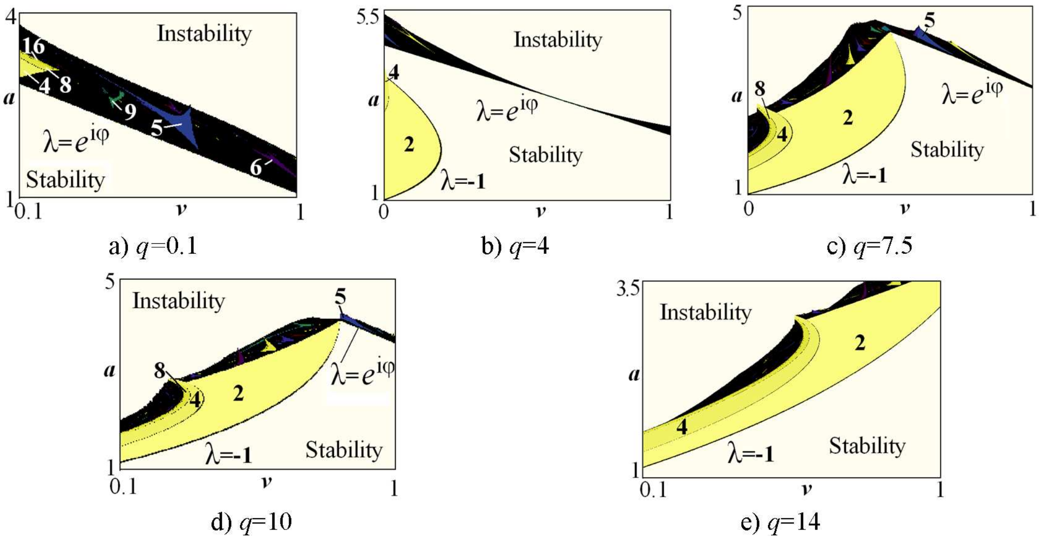

Figure 1. The dynamic modes maps of the system (1) on the (v, a) parameters plane at $s(x, y) = 1 - x - qy$ and at various values of the parameter q. The periods of oscillations are numerated.

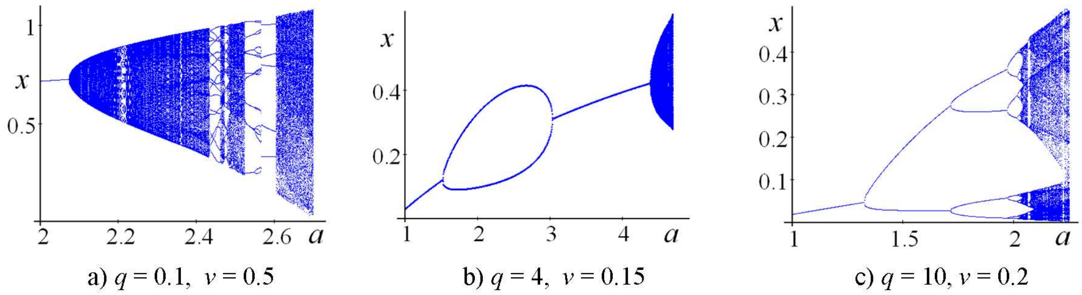

Figure 2. Scenarios of changes in the limit distributions of the number of juveniles (x) in the attractors of system (1) as dependent on the values of the parameter a at $s(x, y) = 1 - x - qy$. The behavior of the limit distribution of the number of adults (y) is similar.

$\overline{I} = \max\{\overline{I}_2, \overline{I}_1\} = \frac{(a+v-1)^2}{4a(a+q)} \max\{c_2, \Delta c_1\}$,
where $\Delta = \frac{(4a(a+q)(H-a(a+q))((1-v)(1+q-v)-H)}{(a+v-1)^2(a(1-v)+H)((a+q)(1+q-v)+H)}$,
$H = \sqrt{a(1-v)(1+q-v)(a+q)}$.

Consequently, if $c_2 > \Delta c_1$, then, there is the maximum at the boundary $u_1 = 0$, and the adult individuals are exploited. If $c_2 < \Delta c_1$, then $u_2 = 0$, and only immature individuals are harvested (fig. 3).

It is shown that there is a domain of population parameters, characterized by the loss of equilibrium stability at its transition into, and the emergence of 2-cycles, even in case of harvesting strategy based on constant catch quotas. Harvesting promotes a decrease in the amplitude of oscillations, not leading to their complete disappearance (fig. 4).

In this connection, we consider the strategy of threshold harvesting [Skaletskaya, et. all, 1979; Svirezhev, Elizarov, 1972]. The optimal control takes the form

$$\left.\begin{array}{ll} V_i = z_i - z_M & at\ z_i \geq z_M \\ V_i = 0 & at\ z_i < z_M \end{array}\right\} \quad (4)$$

where i is the year of exploitation, z is the size of exploited age group, z_M is the number at which the population growth is maximum possible, V is the yield.

This harvesting strategy always stabilizes the systems dynamics (fig. 4).

4 Conclusion

We have considered a nonlinear model of the population, which has a complex structure of solutions depending on the parameters values (equilibrium, the cycles of varying length, quasi-periodic oscillations, and the transition to cycles and back to equilibrium). One of the mechanisms for population number stabilization is optimal control over harvesting. It is shown that optimal harvesting means exploitation of only one age class and its choice is determined by the values of population parameters and prices ratio. The highest sustainable yield is achieved with threshold harvesting. First the population is reduced to the size at which its increment maximum, and then it is maintained at this rate by harvesting up to the end of the exploitation process.

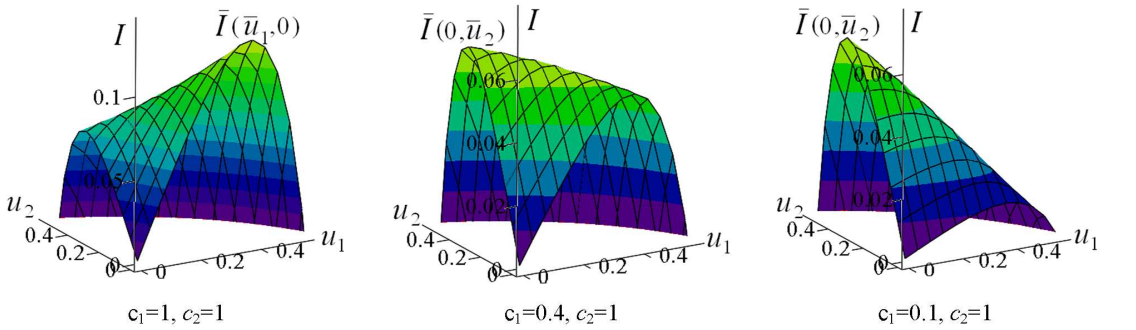

Figure 3. Surface of the function of income, dependent on price values for $s(x,y) = 1 - x - qy$, $a = 1.8$, $v = 0.2$, $q = 0.5$.

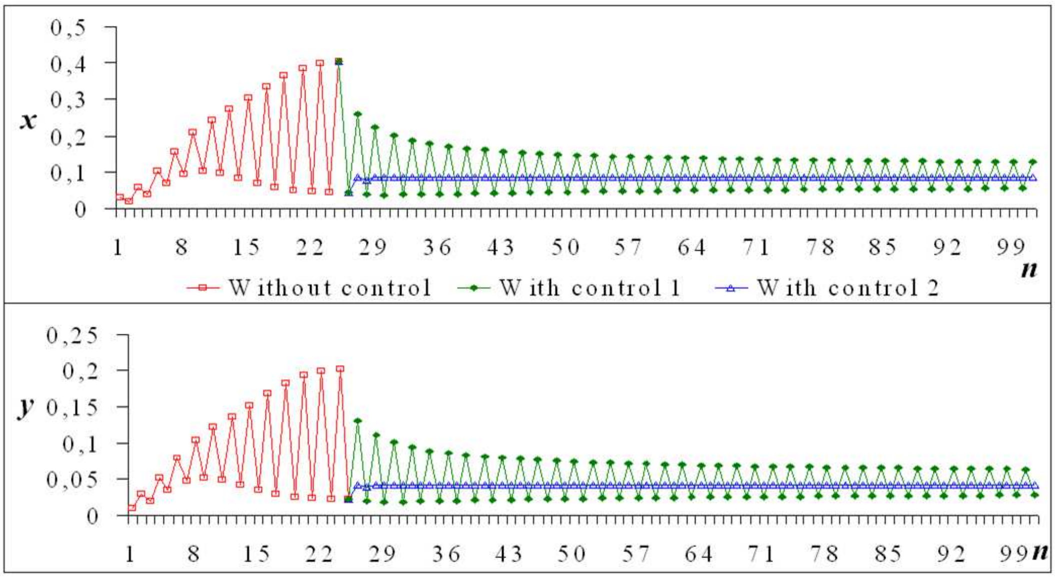

Figure 4. Dynamics of the juveniles number (x) and the adult number (y) under harvesting based on constant catch quotas of mature individuals (control 1) and under threshold harvesting (control 2) and without control. The harvest begins after 25-th generation. Population parameters are: $a = 2, q = 4.5, v = 0.1, x = 0.03, y = 0.01, s(x,y) = 1 - x - qy$.

Acknowledgements

This work is in part supported by the Far Eastern Branch of Russian Academy of Sciences (nos. 09-I-OBN-12, 11-III-V-01M-002); Russian Foundation for the Fundamental Research (no. 09-04-00146-a).

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