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ALGORITHM FOR SOLVING POLYNOMIAL ALGEBRAIC RICCATI EQUATIONS AND ITS APPLICATION

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Abstract
The paper presents a numerical algorithm for solving algebraic Riccati equations with two-sided polynomials. In general, the solution is not a finite-order polynomial. Existence conditions for a finite-order solution are given. If these conditions do not hold, the non-finite-order solution is truncated, and the proposed algorithm returns a finite-order polynomial of an arbitrary high order. Illustrative examples are included.

Key words
Numerical algorithms, algebraic Riccati equation, spatially distributed systems, optimal control.

1 Introduction

We shall concentrate on linear, time-invariant and spatially invariant spatially distributed systems. They can be described by linear partial differential equations (PDEs) with constant coefficients. It was shown in [Bose, 1985] that such systems can be described by state-space equations with state-space matrices which are elements of a ring. Hence, the right mathematical concept to use here is *linear systems over rings*, see, e. g., [Kamen and Khargonekar, 1984; Bose, 1985].

Control of spatially distributed systems and its application has always been an active research topic, see, e. g., [Bamieh *et al.*, 2002; D'Andrea and Dullerud, 2003; Gorinevsky *et al.*, 2008; Cichy *et al.*, 2011; Cichy *et al.*, 2012]. This paper focuses on systems equipped with a dense and regular array of sensors and actuators. We consider that the dynamics of systems is localised, i. e., exciting the system at any location, response can only be observed in the neighbouring nodes. This assumption is not realistic, however, it simplifies the control design a lot.

It is well known that in the stabilisation and optimal control of linear systems *algebraic Riccati equations* (AREs) play a role. The controller optimal in the sense of the standard quadratic criterion can have the form of state feedback. Then, a feedback gain is dependent on the solution to the Riccati equation. It was shown in [Bose, 1985] and in [Bamieh *et al.*, 2002] that for spatially distributed systems the optimal control can be designed via solving AREs with polynomials.

While techniques to obtain a solution to the ARE with constant matrices are in common use, methods to solve the AREs with polynomial matrices have not been deeply explored. Authors of [Bamieh *et al.*, 2002] proposed a method based on an analytic continuation of the Fourier transform of the irrational positive solution to the ARE, however, it seems to be quite complicated and not efficient. Barabanov [Barabanov, 1997] developed an algorithm which uses the equivalence of ARE with polynomials and polynomial matrix factorisation. However, the weight matrix is assumed to be constant and the continuous-time systems are considered only.

In this paper we present an algorithm which is based on the discrete Fourier transform (DFT) theory and is intended for both continuous-time and disrete-time systems discrete in space. However, in contrast to [Barabanov, 1997], we consider only scalar AREs for the present. The weight is not restricted and can be a polynomial. The algorithm is an extension for the discrete-time case of the algorithm developed for continuous-time systems in [Augusta, 2012]. Tests and simulations of the algorithm were performed and are included.

The Sec. 2 deals with a description of spatially distributed systems. AREs with polynomials for both continuous-time and discrete-time systems are introduced in Sec. 3. In Sec. 4 we discuss when the ARE with polynomials is solvable in the set of polynomials. Numerical algorithms for solving AREs are derived in Sec. 5. A possible application is proposed in Sec. 6. Conclusions conclude the paper.

Let $\mathbb{Z}$ denote the set of integers, $\mathbb{R}$ the set of real numbers, $\mathbb{C}$ the set of complex numbers, $\mathbb{R}[w]$ the ring of polynomials in w over $\mathbb{R}$, $\mathbb{R}^{m\times l}[w]$ the set of $m \times l$ matrices with entries in $\mathbb{R}[w]$. Let A^* denote the Hermitian transpose (complex conjugate transpose) of A.

2 State-Space Equations

State-space equations for both continuous-time and discrete-time spatially distributed systems are briefly discussed in this section. The reader is referred to [Bose, 1985] for the detail information.

In the **continuous-time** case a spatially distributed system can be described by the state-space equations

$$\begin{aligned} \frac{\mathrm{d}\chi(t,r)}{\mathrm{d}t} &= \sum_{l=-\infty}^{\infty} \alpha(r-l)\,\chi(t,l)+ \\ &\quad + \sum_{l=-\infty}^{\infty} \beta(r-l)\,\upsilon(t,l) \\ \psi(t,r) &= \sum_{l=-\infty}^{\infty} \gamma(r-l)\,\chi(t,l)+ \\ &\quad + \sum_{l=-\infty}^{\infty} \delta(r-l)\,\upsilon(t,l), \end{aligned} \tag{1}$$

where $\chi(t,r) \in \mathbb{R}^n$ is the state at time t and spatial point $r \in \mathbb{Z}$, $\upsilon(t,r) \in \mathbb{R}^m$ is the input at time t and spatial point r, and $\psi(t,r) \in \mathbb{R}^p$ is the output at time t and spatial point r, $\alpha, \beta, \gamma, \delta$ are matrices of dimension $m \times m, n \times m, p \times n, p \times m$, respectively, with entries being absolutely summable functions from $\mathbb{Z}$ into $\mathbb{R}$, i. e., $\sum_{j=-\infty}^{\infty} |\alpha(j)| < \infty$ and similarly for β, γ, δ.

Let the $\mathcal{Z}$-transform be defined by $\hat{f}(w) = \sum_{l=-\infty}^{\infty} f(l)\,w^{-l}$. Applying the $\mathcal{Z}$-transform entry-by-entry to the matrices $\alpha, \beta, \gamma, \delta$ of (1) we obtain a family of finite-dimensional systems parametrised by a complex variable w

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} x(t,w) &= A(w)\,x(t,w) + B(w)\,u(t,w) \\ y(t,w) &= C(w)\,x(t,w) + D(w)\,u(t,w), \end{aligned} \tag{2}$$

where $x \in \mathbb{C}^n$, $u \in \mathbb{C}^m$ and $y \in \mathbb{C}^p$ denote continuous-time state variable, input and output, respectively, $A(w) \in \mathbb{R}^{n\times n}[w], B(w) \in \mathbb{R}^{n\times m}[w], C(w) \in \mathbb{R}^{p\times n}[w], D(w) \in \mathbb{R}^{n\times m}[w]$.

Similarly, in the **discrete-time** case a spatially distributed system can be described by

$$\begin{aligned} \chi(k+1,r) &= \sum_{l=-\infty}^{\infty} \alpha(r-l)\,\chi(k,l)+ \\ &\quad + \sum_{l=-\infty}^{\infty} \beta(r-l)\,\upsilon(k,l) \\ \psi(k,r) &= \sum_{l=-\infty}^{\infty} \gamma(r-l)\,\chi(k,l)+ \\ &\quad + \sum_{l=-\infty}^{\infty} \delta(r-l)\,\upsilon(k,l). \end{aligned} \tag{3}$$

Applying the $\mathcal{Z}$-transform entry-by-entry to the matrices $\alpha, \beta, \gamma, \delta$ of (3) we obtain a family of finite-dimensional systems parametrised by w

$$\begin{aligned} x(k+1,w) &= F(w)\,x(k,w) + G(w)\,u(k,w) \\ y(k,w) &= H(w)\,x(k,w) + J(w)\,u(k,w), \end{aligned} \tag{4}$$

where $x \in \mathbb{C}^n$, $u \in \mathbb{C}^m$ and $y \in \mathbb{C}^p$ denote discrete-time state variable, input and output, respectively, $F(w) \in \mathbb{R}^{n\times n}[w], G(w) \in \mathbb{R}^{n\times m}[w], H(w) \in \mathbb{R}^{p\times n}[w]$ and $J(w) \in \mathbb{R}^{n\times m}[w]$.

Note that since we have assumed the localised dynamics of systems, A, B, C, D, F, G, H, J are finite-order polynomial matrices.

3 AREs for Spatially Distributed Systems

In the continuous-time case the ARE is the quadratic equation with two-sided polynomial matrices

$$\begin{aligned} &X(w)\,A(w) + A^*(w)\,X(w)- \\ &\quad - X(w)\,B(w)\,B^*(w)\,X(w) + Q(w) = 0, \end{aligned} \tag{5}$$

where $A(w)$, $B(w)$ are matrices of (2), $Q(w)$ is a positive definite for all $|w| = 1$ Hermitian matrix of size $n \times n$ and $X(w)$ is an unknown positive definite for all $|w| = 1$ Hermitian matrix of size $n \times n$. In the discrete-time case the ARE has the form

$$\begin{aligned} &F^*(w)\,X(w)\,F(w) - X(w)- \\ &\quad - F^*(w)\,X(w)\,G(w)[G^*(w)\,X(w)\,G(w)+ \\ &\qquad + I]^{-1}G^*(w)\,X(w)\,F(w) + Q(w) = 0, \end{aligned} \tag{6}$$

where $F(w)$, $G(w)$ are matrices of (4), $Q(w)$ is a positive definite for all $|w| = 1$ Hermitian matrix of size $n \times n$ and $X(w)$ is an unknown positive definite for all $|w| = 1$ Hermitian matrix of size $n \times n$. The following theorems adopted from [Bose, 1985] formulate the necessity to solve ARE.

Theorem 1. *The system (2) is stabilisable if and only if (5) has a unique positive definite Hermitian solution $X(w)$ for all w. In this case, $K_\mathrm{c} = B^* X$ is a stabilising feedback.*

Theorem 2. *The system (4) is stabilisable if and only if (6) has a unique positive definite Hermitian solution $X(w)$ for all w. In this case, $K_\mathrm{d} = (B^* X\,B)^{-1} B^* X\,A$ is a stabilising feedback.*

The stabilising feedbacks are optimal. In the continuous-time case, the control $u = -K_\mathrm{c}\, x$ minimises the quadratic criterion

$$\int_0^\infty \left(x^*\,Q(w)\,x + u^*\,u\right)\mathrm{d}t. \tag{7}$$

In this paper, we shall concentrate on the scalar versions of (5) and (6). For simplicity, we suppose spatial symmetry of the system. Hence, A is considered in the form $A(w) = \sum_{i=0}^{m_A} A_i(w^i + w^{-i})$,where $A_i \in \mathbb{R}$, $m_A \in \mathbb{Z}$, $m_A \geq 0$, $A_{m_A} \neq 0$, and similarly for $B(w), F(w), G(w), Q(w)$ and $X(w)$. Let

$Q(w) > 0, X(w) > 0$ for all $|w| = 1$. Then, (5) and (6) read

$$-B^2(w)\,X^2(w) + 2\,A(w)\,X(w) + Q(w) = 0, \quad (8)$$

$$-\frac{F^2(w)\,G^2(w)}{G^2(w)\,X(w)+1}\,X^2(w) + \\ +[F^2(w)-1]\,X(w) + Q(w) = 0, \quad (9)$$

respectively.

4 Solvability of Scalar AREs with Polynomials

Consider (8) has two solutions $M, N \in \mathbb{R}[w]$. Then

$$-B^2\,X^2 + 2\,A\,X + Q = -B^2(X-M)(X-N) = \\ = -B^2\,X^2 + (B^2\,M + B^2\,N)X - B^2\,M\,N.$$

Comparing coefficients of the left-hand and the right--hand sides, we get

$$B^2(w)\,M(w) + B^2(w)\,N(w) = 2\,A(w) \quad (10)$$
$$-B^2(w)\,M(w)\,N(w) = Q(w). \quad (11)$$

The expression (10) is the linear Diophantine equation with polynomials with unknowns M and N. It has a solution if and only if $B^2(w)$ divides $A(w)$. See, e. g., [Kučera, 1979] for the details. The solution is then parametrised by

$$M = -V(w) + \frac{2\,A(w)}{B^2(w)}, \qquad N = V(w), \quad (12)$$

where $V \in \mathbb{R}[w]$ can be arbitrary. Substitution (11) into (8) gets

$$-\,B^2(w)\,X^2(w) + 2\,A(w)\,X(w) + \\ +\,B^2(w)\,V^2(w) - 2\,A(w)\,V(w) = 0, \quad (13)$$

where we put

$$Q(w) = B^2(w)\,V^2(w) - 2\,A(w)\,V(w) \quad (14)$$

and $V(w)$ must be such a two-sided polynomial in w that (14) is a self-adjoint polynomial positive definite for all $|w| = 1$. AREs of the form (13) have two solutions in the set of polynomials. An example follows.

Example 1. *Consider $A(w) = w + 5 + w^{-1}$, $B(w) = 1$, $Q(w) = 4\,w + 24 + 4\,w^{-1}$, then (8) is of the form (13) and reads $-X^2 + 2(w + 5 + w^{-1})X + 4\,w + 24 + 4\,w^{-1} = 0$. The factorisation (14) exists and the above ARE has solutions $X_1 = -2$, $X_2 = 2\,w + 12 + 2\,w^{-1}$. The solution X_2 is a self-adjoint polynomial positive definite for all $|w| = 1$.*

Remark 1. *The weight $Q(w)$ is generally a polynomial in w, however, the aim is often to design a controller with $Q(w)$ being the real constant. Consider $A \in \mathbb{R}[w]$, but $B, Q \in \mathbb{R}$. To Q be a real constant, it follows from (11) that both solutions of the ARE must be real constants too, what it is not possible for the general A. Hence, (8) with $Q \in \mathbb{R}$ is unsolvable in $\mathbb{R}[w]$.*

In the discrete-time case the situation is similar. The ARE (9) has solutions $M, N \in \mathbb{R}[w]$ if and only if $F^2\,G^2 + G^2$ divides $F^2\,G^2 + G^2\,Q - 1$ and Q can be factorised to

$$Q = \frac{(F^2G^2 + G^2)V^2 - (F^2G^2 - 1)V - F^2}{G^2V + 1}, \quad (15)$$

where $V(w)$ must be such a two-sided polynomial in w that (15) is a self-adjoint polynomial positive definite for all $|w| = 1$. AREs (9) of the form

$$-\frac{F^2(w)G^2(w)}{G^2(w)X(w)+1}X^2(w) + (F^2(w)-1)X(w) + \\ \frac{(F^2G^2 + G^2)V^2 - (F^2G^2 - 1)V - F^2}{G^2V + 1} = 0, \quad (16)$$

have two solutions in the set of polynomials.

5 Algorithm

After analysis the solutions we propose numerical algorithms for solving AREs with polynomials. From the previous section it is obvious that problems leading to the ARE having a solution in $\mathbb{R}[w]$ occur very rarely. In practise, problems have a non-finite-order solution that must be approximated by a polynomial. Thus, if the solution is used in the optimal control design, the resulting controller is practically always suboptimal.

Algorithms introduced in this section can find all solutions from $\mathbb{R}[w]$ if they exist or truncate non-finite-order solutions and return them in the form of a polynomial of an arbitrary high finite order. Algorithms are based on the DFT theory. Recently, the same principle was used for computing the polynomial spectral factorisation [Hromčík *et al.*, 2001] and the polynomial plus/minus factorisation [Hromčík and Šebek, 2006].

Let us recall the following well-known definitions. For a vector of complex numbers $p = (p_0, p_1, \ldots, p_n)$, its direct DFT is the vector $P = (P_0, P_1, \ldots, P_n)$, where $P_k = \sum_{i=0}^{n-1} p_i\, e^{-j\frac{2\pi}{n} i\, k}$. If $P = (P_0, P_1, \ldots, P_n)$ is given, its inverse DFT is $p = (p_0, p_1, \ldots, p_n)$, where $p_i = \frac{1}{n} \sum_{k=0}^{n-1} P_k\, e^{j\frac{2\pi}{n} i\, k}$. The DFT is in common use in many engineering areas. For a numerical computation, the efficient fast Fourier transform (FFT) algorithms are available. The most common one is the Cooley--Tukey algorithm, see [Cooley and Tukey, 1965]. FFT algorithms play an important role and they are available as built-in functions in many computing packages.

In the continuous-time case, the algorithm deals with (8) and its solution in the form

$$X_{1,2} = \frac{A \pm \sqrt{A^2 + Q\,B^2}}{B^2}, \qquad (17)$$

where A, B, Q are generally two-sided polynomials. In the discrete-time case, the algorithm deals with (9) and its solution in the form

$$X_{1,2} = \frac{\pm\sqrt{\sigma} - G^2\,Q - F^2\,G^2 + 1}{2(F^2\,G^2 + G^2)}, \qquad (18)$$

where $\sigma = F^4\,G^4 + 4\,F^4\,G^2 + 2\,F^2\,G^4\,Q + 4\,F^2\,G^2\,Q + 2\,F^2\,G^2 + G^4\,Q^2 + 2\,G^2\,Q + 1$, and F, G, Q are generally two-sided polynomials. Algorithms can be described as follows.

Algorithm 1 (Continuous time). *Inputs:* $A(w)$, $B(w)$, $Q(w)$. *Outputs:* $X_{1,2}(w)$ *given by (17).*

1. *Put* $n = 2^N$, $N \in \mathbb{Z}$.
2. *Define coefficient matrices (vectors) of dimension* $1 \times n$ *with* $n \geq 1 + 2\,\max(m_A, m_B, m_Q)$,

$$p_A = \left(A_0\ A_1\ \cdots\ A_{m_A}\ 0 \cdots 0\ A_{m_A}\ \cdots\ A_1 \right),$$

and similarly for p_B *and* p_Q.
3. *Sample* $A(w), B(w), Q(w)$ *in* n *points on the unit circle. Simply, perform the FFT on* p_A, p_B, p_Q *with* n *interpolation points. Your get* 1*-by-*n *matrices* P_A, P_B, P_Q.
4. *Compute* $P_{X_1} = \frac{P_A + \sqrt{P_A^2 + P_Q \cdot P_B^2}}{P_B^2}$, $P_{X_2} = \frac{P_A - \sqrt{P_A^2 + P_Q \cdot P_B^2}}{P_B^2}$, *where all operations (addition, subtraction, multiplication, division, square and square root) are performed element by element.*
5. *Perform inverse FFT on* P_{X_1} *and* P_{X_2}. *You get*

$$p_{X_1} = \left(X_{10}\ X_{11}\ \cdots\ X_{1m_{X_1}}\ *\ \cdots \right.$$
$$\left. \cdots\ *\ X_{1m_{X_1}}\ \cdots\ X_{11} \right)$$

$$p_{X_2} = \left(X_{20}\ X_{21}\ \cdots\ X_{2m_{X_2}}\ *\ \cdots \right.$$
$$\left. \cdots\ *\ X_{2m_{X_2}}\ \cdots\ X_{21} \right),$$

where the star $(*)$ *denotes any number, see the below remarks. The desirable solutions (17) is then given by*

$$X_1(w) = \sum_{i=0}^{m_{X_1}} X_{1i}\,(w^i + w^{-i}),$$
$$X_2(w) = \sum_{i=0}^{m_{X_2}} X_{2i}\,(w^i + w^{-i}).$$

Algorithm 2 (Discrete time). *Inputs:* $F(w)$, $G(w)$, $Q(w)$. *Outputs:* $X_{1,2}(w)$ *given by (18).*

1. *Put* $n = 2^N$, $N \in \mathbb{Z}$.
2. *Define coefficient matrices (vectors) of dimension* $1 \times n$ *with* $n \geq 1 + 2\,\max(m_F, m_G, m_Q)$,

$$p_F = \left(F_0\ F_1\ \cdots\ F_{m_F}\ 0 \cdots 0\ F_{m_F}\ \cdots\ F_1 \right),$$

and similarly for p_F *and* p_Q.
3. *Perform the FFT on* p_F, p_G, p_Q *with* n *interpolation points. Your get* 1*-by-*n *matrices* P_F, P_G, P_Q.
4. *Compute* $\sigma = P_F^4\,P_G^4 + 4\,P_F^4\,P_G^2 + 2\,P_F^2\,P_G^4\,P_Q + 4\,P_F^2\,P_G^2\,P_Q + 2\,P_F^2\,P_G^2 + P_G^4\,P_Q^2 + 2\,P_G^2\,P_Q + 1$ *and* $P_{X_1} = \frac{\sqrt{\sigma} - P_G^2\,P_Q - P_F^2\,P_G^2 + 1}{2(P_F^2\,P_G^2 + P_G^2)}$, $P_{X_1} = \frac{-\sqrt{\sigma} - P_G^2\,P_Q - P_F^2\,P_G^2 + 1}{2(P_F^2\,P_G^2 + P_G^2)}$, *where all operations are performed element by element.*
5. *Perform inverse FFT on* P_{X_1} *and* P_{X_2}. *You get*

$$p_{X_1} = \left(X_{10}\ X_{11}\ \cdots\ X_{1m_{X_1}}\ *\ \cdots \right.$$
$$\left. \cdots\ *\ X_{1m_{X_1}}\ \cdots\ X_{11} \right)$$

$$p_{X_2} = \left(X_{20}\ X_{21}\ \cdots\ X_{2m_{X_2}}\ *\ \cdots \right.$$
$$\left. \cdots\ *\ X_{2m_{X_2}}\ \cdots\ X_{21} \right),$$

where the star $(*)$ *denotes any number, see the below remarks. The desirable solutions (18) is then given by*

$$X_1(w) = \sum_{i=0}^{m_{X_1}} X_{1i}\,(w^i + w^{-i}),$$
$$X_2(w) = \sum_{i=0}^{m_{X_2}} X_{2i}\,(w^i + w^{-i}).$$

Remarks and illustrative examples follow.

Remark 2. *Since we assumed* $Q(w) \geq 0$ *for all* $|w| = 1$, *all square roots occurring in algorithms exits.*

Remark 3. *The solution is a polynomial if and only if conditions given in Sec. 4 are satisfied. Then, entries of* p_{X_1} *and* p_{X_2} *marked by star* $(*)$ *are equal to zero. If a condition of Sec. 4 does on hold, the solution is not a finite-order polynomial and the algorithm truncates the solution. Its order* m_{X_1} *and* m_{X_2} *must be chosen by user. It can be shown that if the truncated polynomial is of sufficiently high order, the stability is preserved. In this case, entries of* p_{X_1} *and* p_{X_2} *marked by star* $(*)$ *are non-zero numbers.*

Example 2. *Consider the ARE of Example 1. We have* $A = w + 5 + w^{-1}$, $B = 1$, $Q = 4\,w + 24 + 4\,w^{-1}$. *We use Algorithm 1.*

1. *Let* N *be, for example, equal to 7, hence* $n = 128$.

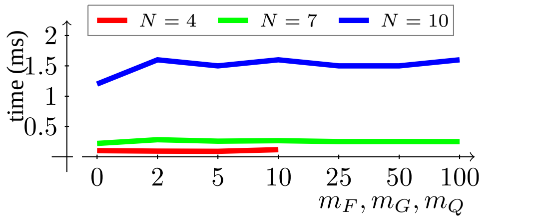

Figure 1. The time required by the Algorithm 2 to solve an ARE with polynomials of degree m_F, m_G, m_Q.

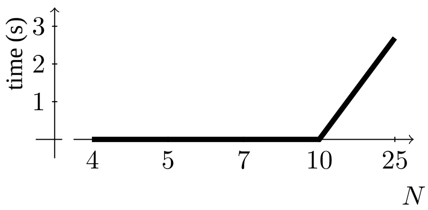

Figure 2. The time required by the Algorithm 2 to solve an ARE with 2^N interpolation points.

2. *Define* $p_A = \begin{pmatrix} 5 & 1 & 0 & \cdots & 0 & 1 \end{pmatrix}, p_B = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}, p_Q = \begin{pmatrix} 24 & 4 & 0 & \cdots & 0 & 4 \end{pmatrix}$ *of dimension* 1×128.
3. *The FFT on* p_A, p_B, p_Q *with* 128 *interpolation points gives* 1*-by-*128 *matrices* P_A, P_B, P_Q.
4. *Compute* P_{X_1} *and* P_{X_2}.
5. *Perform inverse FFT on* P_{X_1} *and* P_{X_2}. *You get* $p_{X_1} = \begin{pmatrix} 12 & 2 & 0 & \cdots & 0 & 2 \end{pmatrix}, p_{X_2} = \begin{pmatrix} -2 & 0 & \cdots & 0 \end{pmatrix}$. *The solution (17) is* $X_1(w) = 2\,w + 12 + 2\,w^{-1}$, $X_2(w) = -2$.

The following example illustrates the case, when the solution is not of the finite order.

Example 3. *Let* $A = w + 5 + w^{-1}, B = 1, Q = 1$, Q *cannot be factorised to (14). The Algorithm 1 returns*

$$p_{X_1} = \big(10.1075\ \ 1.9781\ \ 0.0044\ \ {-0.0009}\ \ 0.0002\ \ \cdots\ \ 0.0002\ \ {-0.0009}\ \ 0.0044\ \ 1.9781\big)$$

$$p_{X_2} = \big({-0.1075}\ \ 0.0219\ \ {-0.0044}\ \ 0.0009\ \ {-0.0002}\ \ \cdots\ \ {-0.0002}\ \ 0.0009\ \ {-0.0044}\ \ 0.0219\big).$$

The time required to solve an ARE does not depend on the degree of polynomials, but goes up with a number of interpolation points, as you can see in Figs. 1 and 2.

6 An Application: A Heat Conduction in a Rod

The above described concept is used to design a distributed LQ controller of a heat conduction in a metal rod. The rod is equipped with an array of temperature sensors and heaters and is sketched in Fig. 3, adopted from [Augusta and Hurák, 2013]. The system can be described by the state-space equation $\dot{x} = 0.33\,(w - 2 + w^{-1})\,x + u$.

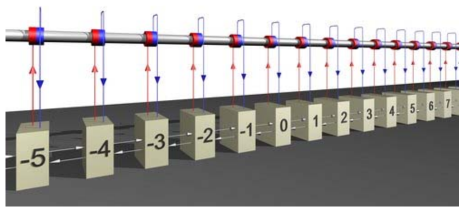

Figure 3. Distributed control of a spatially distributed system: a rod with an array of heaters and sensors and a distributed controller.

The controller optimal in the sense of minimizing (7) is designed. The Algorithm 1 finds the positive definite solution to (8), truncated to the finite-order form, $X =$

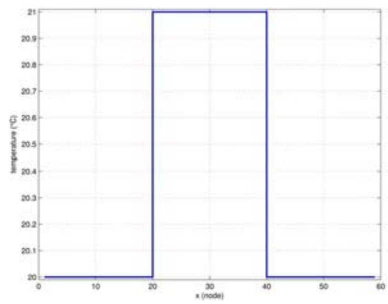

Figure 4. Initial conditions.

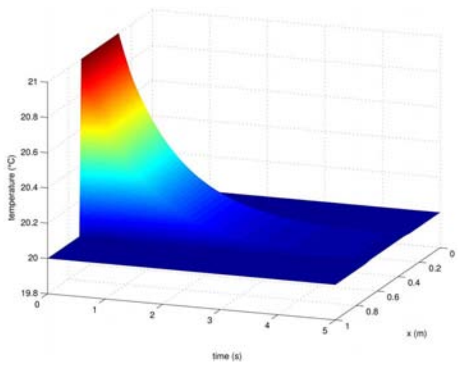

Figure 5. Output of the plant, $Q = 1$.

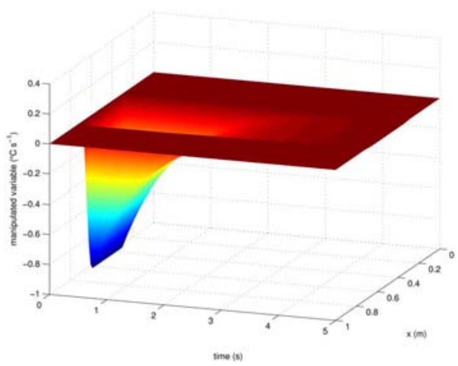

Figure 6. Manipulated variable, $Q = 1$.

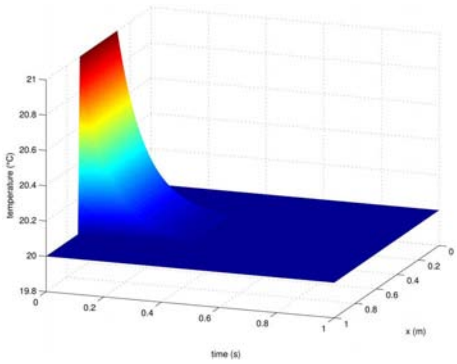

Figure 7. Output of the plant, $Q = 100$.

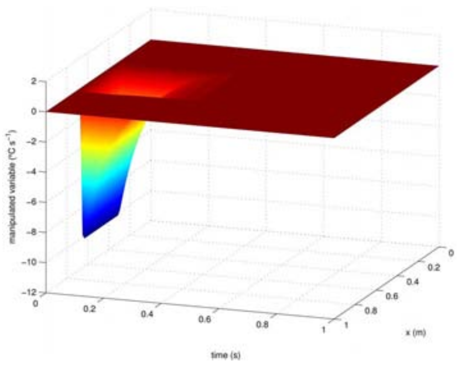

Figure 8. Manipulated variable, $Q = 100$.

$0.6 + 0.16\,(w + w^{-1}) + 0.03\,(w^2 + w^{-2})$ for $Q = 1$ and $X = 9.3 + 0.3\,(w + w^{-1})$ for $Q = 100$.

In simulations we consider a rod of length 1 m with 59 nodes, zero boundary conditions and initial condition of Fig. 4. Output of the plant and manipulated variable are plotted in Figs. 5–8. The closed-loop systems are stable. For $Q = 100$ we get responses with shorter settling time than for $Q = 1$, what is in accordance with higher power of manipulated variable.

7 Conclusion

In the paper algebraic Riccati equations with polynomials that arise in some applications of control theory are considered. Existence conditions for the solution in the set of polynomials are given. The presented numerical algorithms find all solutions from the set of polynomials if they exist or truncate non-finite-order solutions and return them in the form of polynomial of an arbitrary high finite order. It is also possible to combine this algorithm with optimisation routine to produce an approximation. The presented algorithms are available for scalar algebraic Riccati equations only. An extension for the matrix case is not straightforward and requires further research.

Possible applications of algorithms can be find in, e. g., optimal control of spatially distributed systems. A simple example of an application is considered, and design of optimal control of a heat conduction in a rod is shown using the presented algorithm.

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