OPTIMAL MULTIPARAMETRICAL CORRECTION OF DYNAMIC SYSTEMS: ANALYSIS AND OPTIMAL CHAOS SUPPRESSION

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Abstract

We review the results of multiparametrical analysis and optimal suppression of the chaotic dynamics. For the class of autonomous Lorenz-like chaotic systems it is shown how to study the condition and optimal ways to achieve superstability. In case of non-autonomous chaotic systems represented by dissipative nonlinear oscillators it allows comparing the efficiency of various forms of parametric perturbations which provide optimal elimination of nonregular dynamics.

Key words

Multiparametrical analysis, chaos suppression, optimal correction, superstability, Melnikov method

1 Introduction

For more than twenty years the attention of specialists in various fields of science has been focused on chaos control. The development of this field stimulated investigation of its various aspects such as stabilization, synchronization, chaotization, etc. And the areas of application of the results range from biomedicine to information security [Sanjuán and Grebogi, 2010; Zhang, Liu and Wang, 2009].

The advance in the field of chaos control led to the appearance of specific control goals unusual for the conventional control theory (the analysis of possible goals see in [Fradkov, 2007]). The investigation of these goals inspired the development of various methods of control of chaotic systems. And here arise two interrelated problems. On the one hand, the efficiency of chaotic behavior control schemes depends on how much we use the specific features of chaotic systems. For example, the efficiency of OGY [Ott, Grebogi and Yorke, 1990] method is provided by the use of recurrence property of chaotic trajectories. On the other hand, the systems with several available control parameters are rather common. This led to multiparametrical generalization of the methods [Barreto and Grebogi, 1995]. However our understanding of nonregular behavior appearance mechanisms is seriously hindered by parametric space multidimensionality of chaotic systems and high perturbation sensitivity of their parameters. It is often not clear which parameters and forms of perturbation are the best for control. That is why the advance in the field of chaos control requires development of reliable means of multiparametrical analysis that could be used for enhancing the efficiency of existing methods and for creating the new ones.

In this paper we present our summary of results recently gained in course of research of multiparametrical analysis and optimization of controllable chaotic systems. These results grew from our previous investigations [Talagaev and Tarakanov, 2008; Gorelik, Talagaev and Tarakanov, 2009] where we offered the concept of optimal correction applied to parameters of the systems which demonstrate chaotic dynamics. Now this correction became the pivot of multiparametrical research and opened new possibilities in chaotic systems analysis. The aim of correction is to provide the modification of chaotic system limit set (chaotic attractor) from the unstable to the stable one with regard to small perturbation demand. The peculiarity of this aim is that only the character of system stability, but not the quantitative characteristics of a target set are given beforehand. While looking for the way to provide the desired properties of the system we realize the small perturbation demand in minimization of the special optimality criterion. It is designed to consider the conditions of system chaotization for this particular class of chaotic systems. As a result it is possible compare the efficiency of various forms of parametric perturbations which provide optimal chaos suppression.

The paper is organized as follows. The solution of special optimization problem which allows considering the possibility of achieving the state of superstability by chaotic system is presented in part two. In practice the superstable behavior of chaotic systems is an important property and its achievement conditions by chaotic systems have never been analyzed before. Part 3 deals with multiparametrical studies of the picture of optimal chaotic dynamics suppression in dissipative nonlinear oscillators. First of all, on the basis of Melnikov criterion we formulate two original optimization problems. The solution of the first one is the optimal method of simultaneous parametric change needed for chaos suppression. The second problem allows finding an effective resonant correlation between frequencies of controlled and chaos-inducing periodic perturbations. By choosing this correlation we can minimize the amplitude of chaos suppressing perturbation. The results unexpectedly came into agreement with the solution of the third problem. It was achieved through the two-stage optimal correction scheme (based on the optimality conditions of maximum principle) which was developed earlier and already proved to be effective.

2 Superstability and optimal multiparametrical suppression of chaotic dynamics in the Lorenz-like systems

A rather wide class of chaotic Lorenz-like systems is described as follows:

$$\begin{aligned} &\dot{x}(t) = Ax(t) + f(x(t)), \\ &x \in R^3,\ x(0) = x_0,\ t > 0, \end{aligned} \tag{1}$$

where $A = (a_{ij}) \in R^{3\times 3}$ – the matrix of system parameters, $f(x)$ – nonlinear part ($f : R^3 \to R^3$) that characterizes nonlinear perturbation.

While suppressing chaotic dynamics, usually, we set a goal and try to find out under what conditions can the system (1) achieve stable behavior. However there exists a narrow class of systems which demonstrate a special type of regular behavior. They are superstable systems (for essential information on superstability see [Polyak, Shcherbakov, 2002]). A superstable system is always stable, but the opposite statement is not necessary true. The analysis of the superstability conditions of the system is of practical importance as the transition from unstable regime to the state of superstability goes smoothly without an undesirable jump rise of solution norm which is possible at the initial stage of the transient process. It is essential that, unlike stability, the superstability is preserved at nonlinear perturbations. It grounds the possibility of analysis of superstability achievement conditions with regard to system (1).

System (1) can have several states of equilibrium $x_e^{(k)}$ ($\dot{x}_e^{(k)} = 0$), where $k = \overline{1,s}$ – the number of states of equilibrium. Then Jacobian matrix for k-state of equilibrium of the system (1) will be written as $J_k = A + G_k$, where $G_k = (g_{ij}^{(k)}) \in R^{3\times 3}$ – is the matrix with the elements $g_{ij}^{(k)} = (\partial f_i(x)/\partial x_j)|_{x=x_e^{(k)}}$, $i, j = 1, 2, 3$. The generalization of *superstability condition* for the systems with multiple states of equilibrium (1) leads to the condition

$$\sigma^{(k)} = \min_i \sigma_i^{(k)} > 0, \tag{2}$$

where

$$\sigma_i^{(k)} = -(a_{ii} + g_{ii}^{(k)}) - \sum_{j\neq i} |a_{ij} + g_{ij}^{(k)}|,\ i, j = 1, 2, 3.$$

The quantity $\sigma^{(k)}$ is called the *superstability degree*. For every k-state of equilibrium the conditions $\sigma_i^k > 0$ are linear restrictions on J_k matrix coefficients that show that negative diagonal dominance should necessarily be present.

2.1 Problem statement

Let initial values of parameters of the system (1) lie in the area of parametric space that corresponds to chaotic dynamics. Then according to the conditions of dynamics chaotization the Jacobian matrix J_k of the system (1) will be unstable for every k and condition (2) will not be performed.

Consider the system stabilization problem in the following form: for the given unstable matrix J_k should be found the closest superstable matrix $\widetilde{J}_k$. Let's define the closeness as the least value $||J_k - \widetilde{J}_k||$, where $||\cdot||$ – some matrix norm (from here on $||M|| = \left(\sum_{i,j=1}^n m_{i,j}^2\right)^{1/2}$). Then the task will be reduced to finding the *optimal corrective* matrix $H_{k^*}^*$ that provides superstabilty of the matrix $\widetilde{J}_{k^*} = J_{k^*} + H_{k^*}^*$ with condition

$$||H_{k^*}^*|| = \min_k \min_{H_k \in C} ||H_k||, \tag{3}$$

where C – the set of corrective matrixes $H_k = (h_{ij}^k) \in R^{3\times 3}$, which allow performing conditions (2) for the coefficients of the corrected matrix $\widetilde{J}_k$.

The problem formulated above can be regarded as a generalization of the problem of static output stabilization of state space MIMO systems. The peculiarity of the problem is that we must provide not just stability, but superstability. At the same time optimization criterion (3) is built with regard to the possibility of the system having several states of equilibrium. Hence the solution of the problem is the pair: the $H_{k^*}^*$ matrix and state of equilibrium $x_e^{(k^*)}$ correspondent to it. Together they define the corrected matrix $\widetilde{J}_{k^*}$ which is the superstable matrix closest to J_k.

2.2 The corrective matrix structure and superstability achievement conditions

It may seem that in problem (3) the correction rule of the matrix J_k should be presented as transformation of its coefficients $a_{ij} + g_{ij}^{(k)} \Rightarrow \widetilde{d}_{ij}^k = (a_{ij} + g_{ij}^{(k)}) + h_{ij}^k$, where h_{ij}^k – corrective amendments which define matrixes H_k and $\widetilde{J}_k = J_k + H_k$. However, as superstability conditions (2) are strict there arises a question. Is there always exist the corrective matrix $H_k$for the system (1), which allows fulfilling superstability conditions?

The work [Talagaev and Tarakanov, 2011] describes the cases when the correction of the matrix J_k coefficients allows finding the best way to change the parameters and modify a chaotic system into the superstable one. The structural analysis of the matrix $J_k = A+G_k$ substantionally restricts the correction rule: 1) the matrix A is often sparse (that is some a_{ij} are equal to zero) and its zero coefficients should not be influenced by; 2) the matrix G_k can not be corrected at all. These restriction are caused by our attempts to apply multiparametrical correction to the system while preserving its properties. If these restrictions are not fulfilled the system undergo structural changes (new terms appear in the equation). With regard to these restrictions the correction rule will be written in the form

$$\widetilde{J}_k = (A + H_k) + G_k,\ k = \overline{1, s}, \tag{4}$$

where H_k – corrective matrix structurally equivalent to A.

The rule (4) defines only the allowed method of correction, but does not provide us with any information about superstability accessibility. Let's write superstability conditions of the corrected system:

$$\begin{aligned} &-d_{ij}^{(k)} - g_{ii}^{(k)} > \sum_{j\neq i} |d_{ij}^{(k)} + g_{ij}^{(k)}|, \\ &d_{ij}^{(k)} = a_{ij} + h_{ij}^{(k)},\ i, j = 1, 2, 3. \end{aligned} \tag{5}$$

Then superstability accessibility means that there exists matrix H_k (according to (4)) with coefficients that satisfy these equations. It is easy to name the situations when superstability is initially impossible. It happens when one of diagonal elements of the matrix J_k is equal to zero. Then the existing condition (5) results in an unavoidable contradiction.

If superstability is accessible (i.e. set C in (3) is not empty) the problem (3) can be solved in two steps. At step I we should find the matrixes $H_k^*,\ k = \overline{1, s}$, that is to solve s quadratic programming problems $||H_k||^2 \to \min$ with constraints (5). At step II we find $x_e^{(k^*)}$ with matrix $H_{k^*}^*$ correspondent to it. To do this it is necessary to solve the problem $||H_k^*|| \to \min\limits_k$, $k = \overline{1, s}$. Once these steps are performed we get the optimal method of parametrical correction which provides superstable dynamic behavior of the system.

Thus superstability is a unique property peculiar to few chaotic systems. It can serve as a foundation for an additional classification (*superstabilizable* systems). For example the optimal multiparametrical correction problem (3) can easily be solved for the subclass of chaotic systems investigated in [Celikovsky and Chen, 2002]

$$\begin{aligned} \dot{x} &= Ax + f(x) \\ &= \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} x + x_1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} x, \end{aligned}$$

where $x = (x_1, x_2, x_3)^T$. If $a_{12}a_{21} > 0$, $a_{12}a_{21} < 0$ and $a_{12}a_{21} = 0$ we get Lorenz, Chen and L$\ddot{u}$ systems, correspondingly. The existence of pairs $H_{k^*}^*$, $x_e^{(k^*)}$ for these systems means that the character of their dynamics can be transformed from chaotic to the stable one through parametric correction. So the systems are superstabilizable.

3 Multiparametrical picture of optimal chaotic dynamics suppression in dissipative nonlinear oscillators

The following statements became the starting point of development of nonfeedback methods (see [Rega, Lenci and Thompson, 2010] and references) of chaos control: 1) only one parameter of the system is under controllable perturbation; 2) the structure of perturbation is known beforehand (time periodic function). In this case Melnikov theory appears to be a handy analytical tool. It allows finding critical values of perturbation parameters that lead to the appearance (disappearance) of chaotic behavior. Having applied the information about system's dynamic behaviour gained from Melnikov criterion, we could evaluate the efficiency of controlling perturbation in nonautonomous chaos suppression [Chacon, 1995]. Today further development of the method is closely related to minimization of the amplitude of the applied perturbation. The new results in the field were gained due to more profound studies of inherent features of perturbation (phase control of chaos [Qu, Hu, Yang and Qin, 1995; Zambrano, Allaria, Brugioni, Leyva, Meucci, Sanjuán, and Arecchi, 2006]).

The integral development of multiparametrical concept requires correspondent generalizations for nonfeedback methods of chaos suppression as well. At the same time we suppose that the improvement of chaos suppression methods should not be limited just to the examination of the case of several parameters of the system being simultaneously perturbed. It is more important to find optimal multiparametric methods of chaos suppression at various forms of parameter perturbations.

In this part we present the results of a multiparametrical analysis of the process of optimal chaotic dynamics suppression in dissipative nonlinear oscillators. As a reference model of this class of chaotic systems we chose a double-well Duffing oscillator

$$\begin{aligned} &\ddot{x} - \alpha_0(1 + h_\alpha(t))x + \beta_0(1 + h_\beta(t))\, x^3 \\ &\quad = -\gamma_0\, \dot{x} + f_0 \cos(\omega_0\, t), \end{aligned} \tag{6}$$

where $\{\alpha_0, \beta_0, \gamma_0, f_0, \omega_0\} = \{1, 4, 0.154, 0.095, 1.1\}$ – parametric configuration leading to the typical chaotic behavior of the oscillator, $h_\alpha(t)$ and $h_\beta(t)$ – corrective functions.

3.1 Optimization based on Melnikov criterion

To develop the concept of multiparametrical analysis of the chaotic system we solve two optimization problems via Melnikov criterion. Their specific feature is that the solution is sought right on the "regular dynamics – chaos" boundary (RC boundary) thus providing optimal chaos suppression. Such approach allows exposing subtle links between key parameters of the system. It deepens our understanding of chaos suppression phenomenon in various types of oscillators.

Optimization problem I. We begin multiparametrical analysis of optimal suppression of oscillator's chaotic dynamics (6) with studying the situation of correction of initial values of parameters α_0 and β_0.

Let $\dot{h}_\alpha(t) = 0$, $\dot{h}_\beta(t) = 0$, $\dot{h}_\beta(t) = 0$. We must find corrective amendments h_α and h_β that satisfy the condition

$$\text{OP I: } h_\alpha^2 + h_\beta^2 \rightarrow \min_{h_\alpha, h_\beta \in K},$$

where $K = \{h_\alpha, h_\beta \mid M_1(\theta) = 0 \quad \forall\theta\}$ is a set of admissible values of amendments lying on the *RC* boundary, $M_1(\theta)$ – Melnikov function for (6). While solving the problem we found the connection $h_\beta = h_\beta(h_\alpha)$ between the values of corrective amendments h_α and h_β which are situated on the RC boundary. Actually each point of the curve $h_\beta = h_\beta(h_\alpha)$ can be a solution, but there is only one which is the best. To find the optimal pair $(h_{\alpha\,\min}, h_{\beta\,\min})$ we should define the minimum of the function $Q(h_\alpha) = h_\alpha^2 + (h_\beta(h_\alpha))^2$ while moving along the RC boundary. The results are shown in Fig.1. The use of optimal corrective amendments $h_{\alpha\,\min}$ and $h_{\beta\,\min}$ provides optimal special transition of parameters from chaotic dynamics to the stable one.

Optimization problem II. Let $h_\alpha(t) \equiv 0$, $h_\beta(t) = b\cos(\Omega t)$, where b and Ω – amplitude and frequency of a controlled parametrical perturbation, correspondingly. A well-known result is a minimal amplitude evaluation $b_{\min}$ (analytical expression for $b_{\min}$ see in [Chacon, 1995]). In this case an obligatory condition for chaos suppression is the performance of resonant correlation of frequencies $\Omega_z = z\omega_0$, z is an integer. However evaluation for $b_{\min}$ is "local" as it is true only for fixed values $\alpha_0, \beta_0, \gamma_0, f_0, \omega_0$ and z. For instance, the result of calculation for the frequency $\omega_0 = 1.1$ at $z = 1$ will be $a_{min} = 0.095$, but at $z = 2$ we get $a_{min} = 0.081$. To clarify the peculiarities of chaotic dynamics suppression by the function

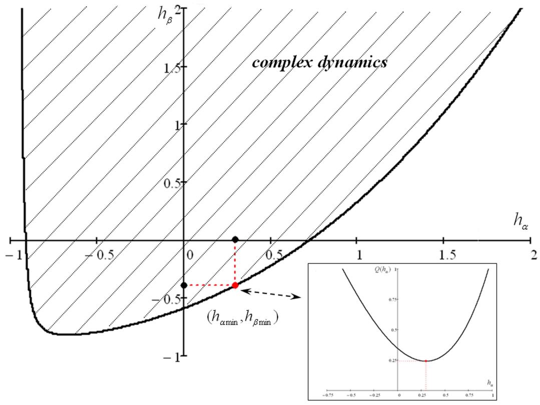

Figure 1. The RC boundary of the oscillator and the position of the pair $(h_{\alpha min}, h_{\beta min})$, which gives optimal combination of corrective amendments that provide chaos suppression.

$u(t) = b_{\min}\cos(\Omega_z\, t)$ we solved an optimization problem

$$\text{OP II : } b_{\min}(\omega_0, z) \rightarrow \min_z.$$

It was found that, actually, minimal value $b^*_{\min}$ can be achieved only at some z^* and only for a rather specified interval of frequencies $(\underline{\omega}, \bar{\omega})$ (see. Fig. 2). At the same time there exists the *optimal* boundary of oscillator between regions of regular and complex dynamics (Fig. 2), generated from the *effective resonant correlation* $\Omega_z = z^*\omega_0$. As a result we can specify the effective resonance $z^* = 2$ for the frequency $\omega_0 = 1.1$ of the outer chaos-inducing perturbation and minimize the amplitude of chaos-suppressing parametric perturbation.

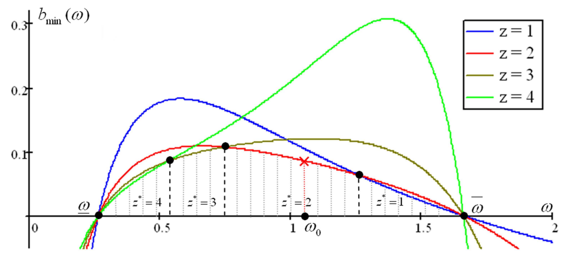

Figure 2. Optimal "chaos-regular dynamics" boundary, formed by the effective resonance correlation in the OP II.

3.2 Dynamic correction with unknown perturbation structure

Optimization problem III. The last step of the investigation is solving the optimal chaos suppression problem for the oscillator (1) when we do not know the structure of parametric perturbations, just the fact that they are restricted by the set U: $|h_\alpha(t)| \le a$, $|h_\beta(t)| \le b$. The application of Melnikov criterion becomes a problem as the parametric perturbation structure is unknown. Thus, in this case the local instability of the system's trajectories is used as a chaos criterion. In this case there appears a two-criteria chaos suppression optimization problem

OP III:

$$\max\left(\max_{0\le t\le T}|h_\alpha^0(t)|;\ \max_{0\le t\le T}|h_\beta^0(t)|\right)\to\min_{\substack{a,b\in K_\Lambda\\ h^0\in U^0}},$$

$$h^0=(h_\alpha^0,h_\beta^0)\in U^0=Arg\min_{h\in U}\int_0^T(h_\alpha^2+h_\beta^2)\,dt.$$

To solve this problem we use a *two-stage correction scheme* [Gorelik, Talagaev and Tarakanov, 2009], which combines the methods of optimal control theory with computer simulations of system's dynamic behavior. While applying this scheme, first, on the basis of maximum principle we should analytically determine the structure of the stabilizing function of parametric perturbation h^0. The optimality of the structure h^0 complies with the demand of minimal energy expenses of chaos suppression process. At stage two we apply the procedure of numerical testing of chaos suppression quality, which allows minimizing the restriction values a and b.

The final solution is the process of optimal transition from chaotic dynamics to the regular one (Fig. 3). Having compared it to the solutions of the problems I and II we found two unexpected correspondences between the applied methods (Melnikov analysis and Maximum Principle). At the moment of stabilization optimal perturbations $h_\alpha^0(t)$ and $h_\beta^0(t)$ that influence on the system become saturated along the upper ($h_\alpha^0(t)=+a_{\min}$) and lower ($h_\beta^0(t)=-b_{\min}$) restriction boundaries. It corresponds exactly to the signs

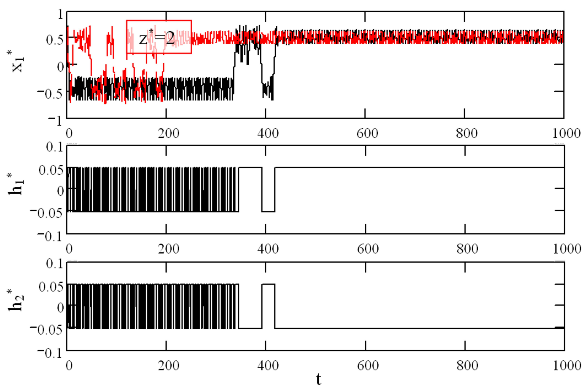

Figure 3. The solution of the OP III: optimal dynamics of the corrected system (6) with $\alpha_{min}=\beta_{min}=0.05$. For comparison the solution of OP II (shown in red) which gives effective resonant correlation is added.

of amendments h_α and h_β in OP I, where $h_\alpha>0$, $h_\beta<0$. Localization of unique limit cycle that corresponds to the stabilized dynamics of the system (1) corresponds precisely to the solution of OP II, that is satisfies effective resonant correlation.

4 Conclusion

Two results presented in the paper illustrate the effectiveness of implementation of optimal parametric correction to the analysis of peculiarities of chaotic dynamics suppression.

The first result was gained while studing two opposite states of autonomous dynamic systems - chaos and superstability. The conditions of achievement of the stable behavior by the system are often become the object of investigation in chaotic system control. Still the question whether it is possible for the system to achieve superstability remains unanswered. The conditions of achievement of the superstable dynamics were studied for a class of Lorenz-like chaotic systems. Based on superstability criteria the optimal multiparametrical (matrix) correction problem was formulated and an effective method of solution was presented. The offered method allows finding superstabilizable chaotic systems and transforming the chaotic dynamics to the superstable one using optimal parametric correction.

The second group of results deals with the problem of studying the multiparametrical picture of optimal chaos suppression in dissipative nonlinear oscillators. We show that Melnikov criterion can be used for stating and solving important optimization problems which deepen our understanding of chaos suppression process. While searching the optimal method of turning the system into the nonchaotic one we: 1) reveal the connection between corrective amendments to the parameters and found their minimal values; 2) expose the effective resonant correlation between the frequencies of controllable parametric and external chaos-inducing perturbations.

The comparison of these results with the results of the two-stage correction scheme resulted in an amazing correspondence. The corrective functions which are found with the help of maximum principle optimality conditions and provide the process of optimal chaos suppression, behave as Melnikov theory predicts. Such correspondence between two independent methods (Melnikov method and maximum principle) is by no means accidental. It shows that two offered methods of solution for the optimal chaos suppression problem supplement each other in multiparametrical analysis of the system.

In tote the named methods of multiparametrical analysis are applied to a wide range of chaotic systems. They help compare the efficiency of different forms of parametrical perturbations and choose the ones which provide optimal suppression of chaotic dynamics.

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