# STABILIZATION OF A CLASS OF NONLINEAR MODEL WITH PERIODIC PARAMETERS IN THE TAKAGI-SUGENO FORM.

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**Abstract:** This paper deals with the stabilization of a class of discrete nonlinear models. The considered models are taken in the Takagi-Sugeno's form with periodic parameters. The main goal of this paper is to reduce the conservatism of the stabilization conditions using a special class of candidate Lyapunov functions.

## I. Introduction

Takagi-Sugeno models (Takagi and Sugeno 85) are strongly investigated for more than 10 years (Tanaka and Wang 2001, Sala et al. 2005 and references therein). This is surely because of the systematic ways both to derive them from a nonlinear model and to find stabilizing control laws including performances and/or robustness.
From most of the affine in the control nonlinear models a Takagi-Sugeno representation (Taniguchi et al. 2001) usable for its control can be derived. Takagi-Sugeno models are composed by a given number of linear models blended together by nonlinear functions. The main property of these functions is the convex sum property.
The main control law used is the PDC (Parallel Distributed Compensation). This control is based on the structure of the model, i.e. it is composed by linear state feedbacks blended together with the same nonlinear functions as in the model (Wang et al. 96).
Making the assumption that the state is measurable and the nonlinear functions use measurable variables, through the direct Lyapunov method, it is easy to derive stabilization conditions in a LMI (Linear Matrix inequality) problem formulation (Tanaka et al 1998). These problems can be effectively solved with some algorithms like the well-known interior point algorithm (Boyd et al. 94).
As usual for nonlinear models, the obtained conditions are only sufficient ones. The problem remaining is to evaluate the degree of conservatism introduced by the method. First of all, there exist an infinite number of Takagi-Sugeno (TS) models for a given nonlinear model. Then, the choice of the TS model can lead to unfeasible LMI problem even if theoretically there is a stabilizing control law. The second source of conservatism is the candidate Lyapunov function chosen. Generally, this function is the well-known quadratic one and the conservatism introduced in this case can be very important. At last, another source of conservatism is the way to tackle with multiple sum inequalities. For improvements in these different points the reader can refer to (Kruszewski et al. 2005), (Rhee and Won 2006), (Liu et Zhang 2004), (Sala et al. 2005), (Tuan et al. 2001).
The main point addressed in this paper is the choice of the candidate Lyapunov function. This is done in the discrete context and considering TS models with periodic parameters.
As in most papers dealing with periodic models, we can use the periodic candidate Lyapunov function (Bolzern and Colaneri 1988). This Lyapunov considers some Lyapunov functions cycling through time. Using this approach with quadratic Lyapunov functions can give quickly better results than using simple Lyapunov functions but the conservatism is still important.
Instead of using the classical quadratic function as a basis for the periodic one, we can consider more effective candidate Lyapunov functions like the nonquadratic ones proposed in (Guerra and Vermeiren 2004). The main idea is to define the Lyapunov function with the same structure as the model i.e. a set of quadratic function blended together by the same nonlinear functions as in the model.

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This extension seems to be straightforward and is not presented hereinafter.
A new way to deal with these models is investigated through this work. The key point of the work is the use of Lyapunov functions that are taking into account the states at different samples. With $k$ the number of samples considered in the function, the stabilization conditions are derived using the *k*-sample variation method presented in (Kruszewski and Guerra 2005). These first results are presented part 2.
The second result given uses a periodic Lyapunov function. Using the same key point as before, it can be shown that the results obtained are always better than using a periodic quadratic Lyapunov function.
The paper is organized as follows. The first part gives some generalities about TS models and the hypothesis done. This part also introduces the notation. The second part introduces the *k*-sample variation method. An example is given to show the effectiveness of the approach and its limits. The third part proposes the second approach and an example is given for sake of comparison with the first one and classical results. The last section gives some conclusions

## II. TS models and Materials

There exist a systematic way to go from an affine in the control nonlinear model to a Takagi-Sugeno model. This method is called sector nonlinearity approach (Tanaka and Wang 2001). The main property of this method is its ability to derive TS model equivalent to the considered nonlinear model in a compact set of the state space.

Considering a nonlinear model given by the expression:

$$x(t+1) = f(z(t))x(t) + g(z(t))u(t) \tag{1}$$

with $f(\cdot)$ and $g(\cdot)$ are nonlinear functions, $x(t)$ the state, $u(t)$ the input and $z(t)$ a vector supposed measurable.

We can derive a TS model given by:

$$x(t+1) = \sum_{i=1}^{r} h_i(z(t))\left(A_i x(t) + B_i u(t)\right) \tag{2}$$

Looking to the structure of the Takagi-Sugeno models (2), we can notice that it is composed of linear models $(A_i, B_i)$ blended together with nonlinear functions $h_i(\cdot)$ sharing the convex sum property: $\sum_{1}^{r} h_i(\cdot) = 1$, $h_i(\cdot) \in [0 \quad 1]$. The integer $r$ present in every sum represents the number of linear models used in the TS model. This number grows exponentially according to the number of nonlinearities taken into account in the considered nonlinear model (1) (Tanaka and Wang 2001).
In the following the nonlinear model is supposed controllable and the linear models $(A_i, B_i)$ too.

In the literature, the main control law used is the PDC (Parallel Distributed Compensation). This control law is composed by several linear state feedback blended by the same nonlinear functions $h_i(\cdot)$ as in the model. Their expressions are given by:

$$u(t) = -\sum_{1}^{r} h_i(z(t)) L_i x(t) \tag{3}$$

Or equivalently introducing a free matrix $G$ $u(t) = -\sum_{1}^{r} h_i(z(t)) F_i G^{-1} x(t)$. This last expression is useful to simplify the notation.

In the following, the use of candidate Lyapunov function involving the state at different samples leads to complex expressions.
To simplify the expressions, the sums are eliminated of them, using the following notations. Considering well defined matrices $Y_i$ and $h_i(\cdot)$ some function sharing the convex sum property, a single sum at time $t$ will be noted: $Y_{z(t)} = \sum_{1}^{r} h_i(z(t)) Y_i$. With $Y_{ij}$ some matrices of the same dimension, $Y_{z(t+n),z(t+n)}$ represents a double sum considered at time $t+n$:

$$Y_{z(t+n),z(t+n)} \triangleq \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t+n)) h_j(z(t+n)) Y_{ij} \tag{4}$$

By extension an expression involving $n$ double sums at different times will be written as: $Y_{z(t),\ldots z(t+n-1),z(t),\ldots z(t+n-1)}$.

Using this notation, TS discrete model (2) can be written:

$$x(t+1) = A_{z(t)} x(t) + B_{z(t)} u(t) \tag{5}$$

As usual a star $(*)$ in a bloc defined matrix stands for the transpose term. $\begin{bmatrix} A & (*) \\ B & C \end{bmatrix} = \begin{bmatrix} A & B^T \\ B & C \end{bmatrix}$ with $A$, $B$ and $C$ some matrices of appropriate dimension.

All the conditions obtained using the Lyapunov theory gives stabilization conditions in the form of matrix inequalities involving multiple sums, for example: $\Upsilon_{z(t),\ldots z(t+n-1),z(t),\ldots z(t+n-1)} < 0$. To derive LMI conditions, i.e. removing the nonlinear functions $h_i(\cdot)$, several possibilities exist. Only one is investigated in this work, the other ways can be straightforwardly used. The one presented is an extension of the double sum case found in (Tuan et al. 2001).

**Lemma 1:** The expression $\Upsilon_{z(t),\ldots,z(t+k-1),z(t),\ldots,z(t+k-1)}$ with $\Upsilon_{i_0 i_1 \cdots i_{k-1}, j_0 j_1 \cdots j_{k-1}}$ a symmetric matrix with the following structure: $\Upsilon_{i_0 i_1 \cdots i_{k-1}, j_0 j_1 \cdots j_{k-1}} \triangleq \tilde{\Upsilon} + \tilde{\Upsilon}^{(0)}_{i_0, j_0} + \cdots + \tilde{\Upsilon}^{(k-1)}_{i_{k-1}, j_{k-1}}$ is negative definite if the following inequalities hold:

$$\Upsilon_{i_0 i_1 \cdots i_{k-1}, i_0 i_1 \cdots i_{k-1}} < 0 \tag{6}$$

$$\frac{2}{r-1}\Upsilon_{i_0 i_1\cdots i_{k-1},i_0 i_1\cdots i_{k-1}}+\Upsilon_{i_0 i_1\cdots i_{k-1},j_0 j_1\cdots j_{k-1}}+\Upsilon_{j_0 j_1\cdots j_{k-1},i_0 i_1\cdots i_{k-1}}<0$$
$$j_0,\cdots,j_{k-1}\neq i_0,i_1,\cdots,i_{k-1} \tag{7}$$

*Proof*: it is direct considering (Tuan et al. 2001) work.

Now, consider TS discrete models with periodic parameters. Their expressions are given by:

$$x(t+1)=A_{z(t)}^{(c)}x(t)+B_{z(t)}^{(c)}u(t),\ c=t \bmod p, \tag{8}$$

$p$ denotes the period and $c$ the period state.

The control used for this kind of model is directly derived from the PDC control law and is given by:

$$u(t)=-F_{z(t)}^{(c)}G^{(c)^{-1}}x(t),\ c=t \bmod p \tag{9}$$

The next theorem gives the results obtained using the following quadratic periodic Lyapunov function:

$$V_l(x)=x^T\left(G^{(l)}\right)^{-T}P^{(l)}\left(G^{(l)}\right)^{-1}x,\ l\in\{0,\ldots,p-1\} \tag{10}$$

Let define the following matrices:

$$\Upsilon_{ij}^{l}=\begin{bmatrix}-P^{(l \bmod p)} & (*)\\ \tilde{A}_{ij}^{(l)} & \Theta^{(l+1)}\end{bmatrix}<0 \tag{11}$$
$$\tilde{A}_{ij}^{(l)}\triangleq A_i^{(l \bmod p)}G^{(l \bmod p)}-B_i^{(l \bmod p)}F_j^{(l \bmod p)}$$
$$\Theta^{(l+1)}=-G^{((l+1)\bmod p)}-G^{((l+1)\bmod p)^T}+P^{((l+1)\bmod p)}$$

**Theorem 1:** Consider the matrices $\Upsilon_{ij}^{l}$ $l\in\{0,\ldots,p-1\}$, defined in (11). The model (8) together the control law (9) is globally asymptotically stable if there exits matrices $P^{(l)}$, $G^{(l)}$ and $F_i$ , such that for $l\in\{0,\ldots,p-1\}$ :

$$\Upsilon_{ii}^{l}<0 \text{ for } i\in\{1,\ldots,r\} \tag{12}$$
$$\frac{2}{r-1}\Upsilon_{ii}^{l}+\Upsilon_{ij}^{l}+\Upsilon_{ji}^{l}<0 \text{ for } i,j\in\{1,\ldots,r\},\ i\neq j \tag{13}$$

*Proof*: it is direct combining lemma 1 and results of (Guerra et Vermeiren 2004).

## III. First result

In this section, the stability of the closed loop ((8) with (9)) is studied with the following candidate Lyapunov function:

$$\mathcal{V}_k(X_k(t))=X_k(t)^T\mathbf{\Pi}X_k(t) \tag{14}$$

With $X_k(t)=\begin{bmatrix}x(t)^T & \cdots & x(t+k-1)^T\end{bmatrix}^T$ and

$$\mathbf{\Pi}=\begin{bmatrix}G^{(0)^{-T}}P^{(0)}G^{(0)^{-1}} & & 0\\ & \ddots & \\ 0 & & G^{(k-1)^{-T}}P^{(k-1)}G^{(k-1)^{-1}}\end{bmatrix}>0.$$

with $P^{(i)}$, $i\in\{0,\ldots,k-1\}$ some positive definite matrices and $G^{(i)}$, $i\in\{0,\ldots,k-1\}$ some full rank square matrices.

The closed loop is stable if the *k*-sample variation of (14) is negative along its trajectories (Kruszewski and Guerra 2005) i.e. $\mathcal{V}_k(X_k(t+k))-\mathcal{V}_k(X_k(t))<0$ ($k$ is an arbitrary chosen integer). In that case, the analysis is reduced to the time *t* such that $t \bmod k=0$ (Kruszewski 2006).

**Proof:** Assume that $\mathcal{V}_k(X_k(t+k))-\mathcal{V}_k(X_k(t))<0$ for all $t$ such that $t \bmod k=0$. So we have $\lim_{t\to+\infty}\mathcal{V}_k(X_k(t))=0$ for each $t$ such that $t \bmod k=0$. $\mathcal{V}_k(X_k(t))$ is a candidate Lyapunov function. This implies that $\lim_{\substack{t\to+\infty\\ t \bmod k=0}}\|X_k(t)\|=0$ for each $t$ such that $t \bmod k=0$ or $\lim_{\substack{t\to+\infty\\ t \bmod k=0}}\|x(t+i)\|=0$, $i\in\{0,\ldots,k-1\}$. This proves that the model is stable.

The first version of the Finsler's lemma is needed to derive the stabilization conditions (De Oliveira and Skelton 2001).

**Lemma 2:** Consider a vector $x\in\mathbb{R}^n$ and two matrices $Q=Q^T\in\mathbb{R}^{n\times n}$ and $R\in\mathbb{R}^{m\times n}$ such that $rank(R)<n$. The two following expressions are equivalent:

1. $x^TQx<0 \quad \forall x\in\{x\in\mathbb{R}^n \mid x\neq 0, Rx=0\}$ (15)
2. $\exists M\in\mathbb{R}^{n\times m}$ such that $Q+MR+R^TM^T<0$ (16)

Consider the following TS discrete model:

$$x(t+1)=A_{z(t)}^{(c)}x(t)+B_{z(t)}^{(c)}u(t),\ c=t \bmod p, \tag{17}$$

the control law:

$$u(t)=-F_{z(t)}^{(c)}G^{(c)^{-1}}x(t),\ c=t \bmod p \tag{18}$$

The goal is to find through a LMI formulation, the parameters of the control law ($F_i^{(l)}$ and $G^{(l)}$ $i\in\{1,\ldots,r\}$, $l\in\{0,\ldots,p-1\}$) and the parameters of the Lyapunov function ($P^{(l)}$, $l\in\{0,\ldots,p-1\}$) to ensure the closed loop stability. The closed loop can be written:

$$x(t+1)=\left(A_{z(t)}^{(c)}-B_{z(t)}^{(c)}F_{z(t)}^{(c)}G^{(c)^{-1}}\right)x(t),\ c=t \bmod p \tag{19}$$

(19) is stable if $\mathcal{V}_k(X_k(t+k))-\mathcal{V}_k(X_k(t))<0$. We choose $k=p$.

This latter inequality can be written as:

$$X_k(t+k)^T\mathbf{\Pi}X_k(t+k)-X_k(t)^T\mathbf{\Pi}X_k(t)<0 \tag{20}$$

Or $X_{2k}(t)^T\begin{bmatrix}-\mathbf{\Pi} & 0\\ 0 & \mathbf{\Pi}\end{bmatrix}X_{2k}(t)<0$ (21)

with $X_{2k}(t)=\begin{bmatrix}x(t)^T & \cdots & x(t+2k-1)^T\end{bmatrix}^T$.

In order to apply Finsler's lemma, the following constraint: $RX_{2k}(t)=0$ is defined. Considering the expression of the closed loop (19):

$$x(t+1)=\left(A_{z(t)}^{(c)}-B_{z(t)}^{(c)}F_{z(t)}^{(c)}G^{(c)^{-1}}\right)x(t)$$

$$\Leftrightarrow\left(A_{z(t)}^{(c)}-B_{z(t)}^{(c)}F_{z(t)}^{(c)}G^{(c)^{-1}}\right)x(t)-x(t+1)=0$$

$$\Leftrightarrow\begin{bmatrix}\overline{A}_{z(t)z(t)}^{(c)}G^{(c)^{-1}} & -I\end{bmatrix}\begin{bmatrix}x(t)\\ x(t+1)\end{bmatrix}=0$$

with $\overline{A}_{z(t)z(t)}^{(c)}=A_{z(t)}^{(c)}G^{(c)}-B_{z(t)}^{(c)}F_{z(t)}^{(c)}$

The extension of this expression to the vector $X_{2k}(t)$ is:

$$RX_{2k}(t)=0 \tag{22}$$

with:

$$R=\begin{bmatrix}\overline{A}_{z(t)z(t)}^{(t\bmod p)}G^{(t\bmod p)^{-1}} & -I & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \overline{A}_{z(t+2k-1)z(t+2k-1)}^{((t+k-1)\bmod p)}G^{((t+k-1)\bmod p)^{-1}} & -I\end{bmatrix}$$

Now using the fact that the analysis can be reduced to the time $t$ such that $t \bmod k=0$ ( $k=p$ ). $R$ can be written:

$$R=\begin{bmatrix}\overline{A}_{z(t)z(t)}^{(0)}G^{(0)^{-1}} & -I & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & \overline{A}_{z(t+2k-1)z(t+2k-1)}^{(k-1)}G^{(k-1)^{-1}} & -I\end{bmatrix} \tag{23}$$

The following conditions of stability are obtained applying Finsler's lemma:

$$\exists M \text{ such that } Q+MR+R^TM^T<0 \tag{24}$$

where $R$ is given in (23) and:

$$Q=\begin{bmatrix}-\mathbf{\Pi} & 0\\ 0 & \mathbf{\Pi}\end{bmatrix} \tag{25}$$

Note that the condition (24) is nonlinear according to the decision variables. To avoid this usual problem (see for example in an LPV context (Ebihara et al. 2005)) a reduced form for the $M$ matrix must be chosen. In this case, we may loose the necessity of the condition. Thus, $M$ is set as:

$$M=\begin{bmatrix}0_{1\times(2k-1)}\\ diag\left(\begin{bmatrix}G^{(1)^{-T}} & \cdots & G^{(k-1)^{-T}} & G^{(0)^{-T}} & \cdots & G^{(k-1)^{-T}}\end{bmatrix}\right)\end{bmatrix}$$

then pre and post multiplying by the following full rank matrix: $diag\left(\begin{bmatrix}G^{(0)} & \cdots & G^{(k-1)} & G^{(0)} & \cdots & G^{(k-1)}\end{bmatrix}\right)$.

We get the LMI condition $\Upsilon_{z(t),\dots z(t+k-1),z(t),\dots z(t+k-1)}<0$ with:

$$\Upsilon_{i_0i_1\cdots i_{k-1},\,j_0j_1\cdots j_{k-1}}=$$

$$\begin{bmatrix}-P^{(0)} & (*) & 0 & & & \cdots & & 0\\ \overline{A}_{i_0j_0}^{(0)} & \Phi^{(1)} & \ddots & & & & & \\ 0 & \ddots & \ddots & (*) & \ddots & & & \\ & & \overline{A}_{i_{k-2}j_{k-2}}^{(k-2)} & \Phi^{(k-1)} & (*) & & & \vdots\\ \vdots & & \ddots & \overline{A}_{i_{k-1}j_{k-1}}^{(k-1)} & \Theta^{(0)} & (*) & & \\ & & & & \overline{A}_{i_kj_k}^{(0)} & \Theta^{(1)} & \ddots & 0\\ & & & & & \ddots & \ddots & (*)\\ 0 & & \cdots & & 0 & & \overline{A}_{i_{2k-2}j_{2k-2}}^{(k-2)} & \Theta^{(k-1)}\end{bmatrix} \tag{26}$$

$\overline{A}_{ij}^{(l)}=A_i^{(l)}G^{(l)}-B_i^{(l)}F_j^{(l)}$, $\Phi^{(l)}=-G^{(l)}-G^{(l)^T}-P^{(l)}$ and $\Theta^{(l)}=-G^{(l)}-G^{(l)^T}+P^{(l)}$.

**Theorem 2:** The closed loop composed by the periodic TS model (17) and the control law (18) is globally asymptotically stable if there exist some matrices $F_i^{(l)}$, $G^{(l)}$ and $P^{(l)}>0$ $i\in\{1,\dots,r\}$, $l\in\{0,\dots,p-1\}$ such that the conditions (6) and (7) hold for $\Upsilon_{i_0i_1\cdots i_{k-1},\,j_0j_1\cdots j_{k-1}}$ defined in (26).

**Example:** Considering the model (17) with the matrices given hereinafter there is no solution using the theorem 1 conditions (classical result) but there is one with the theorem 2 conditions. This proves the interest of the approach.

$A_1^1=\begin{bmatrix}1 & 1\\ 0,5 & 1\end{bmatrix}$, $A_2^1=\begin{bmatrix}1 & 10\\ 0,5 & 1\end{bmatrix}$, $A_1^2=\begin{bmatrix}1,5 & 1\\ 0 & 0,5\end{bmatrix}$, $A_2^2=\begin{bmatrix}1,5 & 10\\ 0 & 0,5\end{bmatrix}$, $B_1^1=B_2^1=\begin{bmatrix}1\\ 0\end{bmatrix}$ et $B_1^2=B_2^2=\begin{bmatrix}0\\ 0\end{bmatrix}$

Note that it exists opposite examples where only theorem 1 conditions can find a solution to the stabilization problem of a periodic TS model. Next section gives another result ensuring the enhancement of the classical theory.

## IV. Second result

The second idea is an improvement of the periodic Lyapunov functions. It uses another basis function which introduces the state at different times. The function used is:

$$\mathrm{V}_{k,l}\left(X_k(t)\right)=X_k(t)^T\,\mathbf{\Pi}^{(l)}X_k(t) \tag{27}$$

$$\mathbf{\Pi}^{(l)}=\begin{bmatrix}G^{(l\bmod p)^{-T}}P_{(0)}^{(l)}G^{(l\bmod p)^{-1}}\\ \vdots\\ G^{((k-1+l)\bmod p)^{-T}}P_{(k-1)}^{(l)}G^{((k-1+l)\bmod p)^{-1}}\end{bmatrix}>0,$$

$l\in\{0,\dots,p-1\}$.

The closed loop (19) is stable if the following inequality holds along its trajectories:

$V_0\left(x(t)\right) > V_1\left(x(t+1)\right) > \ldots > V_{p-1}\left(x(t+t-1)\right) > V_0\left(x(t+p)\right)$
Using the same proof scheme we get the following theorem.

**Theorem 3:** The closed loop composed by the model (17) and the control law (18) is globally asymptotically stable for a given $k$ if there exist some matrices $F_i^{(l)}$, $G^{(l)}$ and $P_j^{(l)} > 0$ $i \in \{1,\ldots,r\}$, $l \in \{0,\ldots,p-1\}$, $j \in \{0,\ldots,k-1\}$ such that the conditions (6) and (7) hold for $\Upsilon^l_{i_0 i_1 \cdots i_{k-1}, j_0 j_1 \cdots j_{k-1}}$ $l \in \{0,\ldots,p-1\}$, defined by:

$$\Upsilon^l_{i_0 i_1 \cdots i_{k-1}, j_0 j_1 \cdots j_{k-1}} = \begin{bmatrix} -P_{(0)}^{(l \bmod p)} & (*) & 0 & \cdots & 0 \\ \tilde{A}^{(l)}_{i_0,j_0} & \Omega_0^{(l)} & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & (*) & 0 \\ \vdots & \ddots & \tilde{A}^{(l+k-2)}_{i_{k-2},j_{k-2}} & \Omega^{(l+k-2)}_{k-2} & (*) \\ 0 & \cdots & 0 & \tilde{A}^{(l+k-1)}_{i_{k-1},j_{k-1}} & \bar{\Omega} \end{bmatrix} \tag{28}$$

with $\tilde{A}_{ij}^{(l)} = A_i^{(l \bmod p)} G^{(l \bmod p)} - B_i^{(l \bmod p)} F_j^{(l \bmod p)}$,
$\Omega_i^{(l)} = -G^{((l+1) \bmod p)} - G^{((l+1) \bmod p)^T} + P_{(i)}^{((l+1) \bmod p)} - P_{(i+1)}^{(l \bmod p)}$
and $\bar{\Omega} = -G^{((l+k) \bmod p)} - G^{((l+k) \bmod p)^T} + P_{(k-1)}^{((l+1) \bmod p)}$.

## V. Discussion and conclusion

Using the same proof as in (Kruszewski and Guerra 2005) it can be proven that there exists a partial order relation between the conditions involving different $k$ values. The bigger $k$ is, the less conservative are the conditions (and more complex). This result is given in the following lemma:

**Lemma 3**: If there exit some matrices $F_i^{(l)}$, $G^{(l)}$ and $P_j^{(l)} > 0$ $i \in \{1,\ldots,r\}$, $l \in \{0,\ldots,p-1\}$, $j \in \{0,\ldots,k-1\}$ such that the conditions of the theorem 3 are fulfilled for a given $k = m$, $m \in \mathbb{N}^*$ then they are also fulfilled for all $k = m \times l$, $l \in \mathbb{N}^*$.

**Remark:** In this theorem, the parameter $k$ allows choosing the ratio complexity/conservatism of the conditions. The number of LMI conditions obtained depends exponentially according to $k$. For high values of $k$ (more than 10) the obtained LMI problem can be inconsistent for actual LMI solvers (it depends also on the size of the state, the number of linear models …).

**Examples:**
To illustrate the different approaches, we consider the three following cases. The first and the second ones show the fact that the sets of results of each method only overlap, i.e. no inclusion can be derived. Effectively first example shows a big improvement of the second approach results in comparison with the first one and the second example shows exactly the opposite. The last example illustrates the interest of increasing the parameter $k$.

Consider the TS model (17) and the matrices with $a \in \mathbb{R}$:
$A_1^1 = \begin{bmatrix} 1{,}5 & 10 \\ 0 & 0.5 \end{bmatrix}$, $A_2^1 = \begin{bmatrix} 0.5 & 10 \\ 0 & 0 \end{bmatrix}$, $A_1^2 = \begin{bmatrix} 1+a & 1 \\ 0 & 0{,}5 \end{bmatrix}$,
$A_2^2 = \begin{bmatrix} 1 & 10 \\ 0 & 0.5 \end{bmatrix}$, $B_1^1 = B_2^1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ et $B_1^2 = B_2^2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $a \in \mathbb{R}$
the first approach gives solutions for $a \in [-1.66 \quad -0.3]$ and the second approach (with $k = 2$) for $a \in [-250 \quad 250]$. Note that the solution set of the second is larger than the solution set of the first one (more than 300 times).

Now consider the following matrices with $a \in \mathbb{R}$:
$A_1^1 = \begin{bmatrix} 1 & 1 \\ a & 1 \end{bmatrix}$, $A_2^1 = \begin{bmatrix} 1 & 10 \\ a & 1 \end{bmatrix}$, $A_1^2 = \begin{bmatrix} 1{,}5 & 1 \\ 0 & 0{,}5 \end{bmatrix}$,
$A_2^2 = \begin{bmatrix} 1{,}5 & 10 \\ 0 & 0{,}5 \end{bmatrix}$, $B_1^1 = B_2^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $B_1^2 = B_2^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
According to $a \in \mathbb{R}$, the first approach gives solutions for $a \in [-255 \quad -254]$ and the second approach (with $k = 2$) for $a \in [-0.22 \quad 0.22]$. Note that the solution set of the first is larger than the solution set of the second one (more than 500 times).

The last example uses the following parameters:
$A_1^1 = \begin{bmatrix} 2 & 1 \\ 0 & 0.5 \end{bmatrix}$, $A_2^1 = \begin{bmatrix} 1.5 & 10 \\ 0.2 & 0.5 \end{bmatrix}$, $A_1^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$,
$A_2^2 = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$, $B_1^1 = B_2^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ et $B_1^2 = B_2^2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
No solution is obtained using the classical periodic Lyapunov function (approach 2 with $k = 1$) but the approach 2 with $k = 2$ gives one.

Regarding these examples, it is not possible to conclude on a “best” approach. Both of them must be tested and the results compared. At last, the open question to be solved is: what is a (the?) good value of the parameter $k$?

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