A CONSPIRACY OF OSCILLATORS

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Abstract
We discuss nonlinear mechanical systems containing several oscillators whose frequencies are all much higher than frequencies associated with the remaining degrees of freedom. In this situation a near constant of the motion, an adiabatic invariant, exists which is the sum of all the oscillator actions. The phenomenon is illustrated, and calculations of the small change of the adiabatic invariant is outlined.

Key words
High-frequency oscillators, adiabatic invariants

1 Introduction
In a mechanical system with \( n \) degrees of freedom, where \( m \) of the degrees of freedom are associated with oscillatory modes whose characteristic time scale is much shorter than the characteristic time scale associated with the remaining \( n - m \) degrees of freedom, the sum of actions from all the \( m \) oscillatory modes forms an adiabatic invariant (see, e.g., [1]) of the system, its value evolving only exponentially slowly.

This is a nontrivial sum because the individual oscillator action are not adiabatically invariant, its invariance broken by resonance with the other high-frequency oscillators. We discuss two examples of this phenomenon. In section 2, a model 1-dimensional gas of mutually interacting diatomic molecules whose vibrational time scale is much shorter than the time scale associated with the collisions between the molecules; and in section 3 a three-dimensional gas of charged particles moving in a magnetic field so strong that the Larmor period of the charged particles is much shorter that the time scale for collisions along the field lines. In both systems there is a thus a separation between the oscillator time scale and the time scale associated with the remaining degrees of freedom. In both systems one finds that the many-body sum of actions is adiabatically invariant, exponentially well preserved by the interactions. The greater the time scale separation, the more the high-frequency oscillators together will conspire to effectively decouple themselves from the rest of the system, leaving the low frequency components to evolve on their own. Finally, section 4 will outline some analytical estimates of the decoupling.

2 Diatomic molecules
Consider a classical mechanical model of a one-dimensional gas consisting of \( n \) identical diatomic molecules. See figure 1.

Figure 1. Several diatomic molecules move in 1 dimension and interact with long-range forces between all the mass points.

Denoting by \( x = (x_1, \ldots, x_n) \) and \( p = (p_1, \ldots, p_n) \) the canonical variables of the centers of mass of the molecules, and by \( \xi = (\xi_1, \ldots, \xi_n) \), \( \pi = (\pi_1, \ldots, \pi_n) \) the (Cartesian) canonical variables describing the internal vibrations, the Hamiltonian has the overall form

\[ H(x, p, \xi, \pi) = H_{tr}(p) + H_{vib}(\xi, \pi) + V_{\text{int}}(x, \xi) \]  \hspace{1cm} (1)

with

\[ H_{tr} = \sum_{i=1}^{n} \left( \frac{p_i^2}{2m} + U(r_i) \right) , \quad H_{vib} = \sum_{i=1}^{n} \left( \frac{\pi_i^2}{2\mu} + \frac{\mu\omega^2\xi_i^2}{2} \right) \]

and

\[ V_{\text{int}} = \sum_{i=1}^{n-1} V(r_i, \xi_i, \xi_{i+1}) , \]

where \( m \) and \( \mu \) denote respectively the total mass and the reduced mass of the molecules, while \( \omega \) denotes...
the frequency of vibration for a single molecule, and 
\[ r_i = x_{i+1} - x_i - l, \] with \( l \) the proper length of the 
molecule; the separation between \( U \) and \( V \) is estab-
lished by requiring \( V(r, 0, 0) = 0 \). Both \( U \) and \( V \) are 
assumed to be smooth functions and to vanish for 
\( r \to \infty \). The Hamiltonian \( H_{\text{tr}} \) describes a gas of 
perfectly rigid molecules, with pairwise interaction de-
scribed by the potential \( U \); as is natural, \( U(r) \) will be 
assumed to diverge for \( r \to 0 \). The gas is assumed to 
be diluted, so that \( V_{\text{int}} \) is small compared to \( H_{\text{tr}} \) and 
\( H_{\text{vib}} \).

A similar model was considered in 1903 by Jeans [6], 
in the investigation of the time scale of the approach to 
equilibrium between translational and vibrational de-
grees of freedom in diatomic gases. The basic idea 
of Jeans (following Boltzmann) was that energy ex-
changes among the translational degrees of freedom, as 
well as among the vibrational degrees of freedom, are 
easy, so that, the two sub–systems described by \( H_{\text{tr}} \) and 
by \( H_{\text{vib}} \) separately reach their thermodynamical equi-
librium, with temperatures \( T_{\text{tr}} \) and \( T_{\text{vib}} \) not necessarily 
equal; whereas collisional energy exchanges between 
\( H_{\text{tr}} \) and \( H_{\text{vib}} \), in case of large \( \omega \), are exponentially 
small, so that an effective “freezing out” of the vibra-
tional energy for quite large times (“billions of years”, 
in the words by Jeans!) is expected to occur, if \( \omega \) is suf-
ciently large. ¹ The conclusion by Jeans was that, in 
principle, the experimentally observed ‘freezing’ phe-
nomena of various kind could be explained classically, 
without Planck’s quantization. Jeans conjectured, and 
supported heuristically, an exponential law for the col-
sisional transfer of energy \( E_0 \), between translational and 
vibrational modes in a single binary collision to be of 
the form

\[ E_0 \approx E_0 e^{-\tau_0 \omega}, \]

where \( E_0 \) is a typical microscopic energy, while the 
(crucial) constant \( \tau_0 \) represents some time scale asso-
ciated to the translational motion. Here, unfortunately, 
Jeans was not really precise, though he qualitatively 
identifies \( \tau_0 \) with the typical duration of the collision 
process. The conclusion is that in typical conditions \( E_0 \) 
is negligibly small, and long–time freezing out occurs.

Today we model molecular collisions by quantum me-
chanics and the ‘freezing out’ of vibrational degrees of 
freedom is seen as a manifestation of the quantized 
energy levels and the inability of collisional energies 
to excite the molecules away from their vibrational 
ground state.

3 Charged particles in strong magnetic field

Consider a particle of mass \( m = 1 \) and charge \( e = 1 \) 
moving in \( \mathbb{R}^3 \). A homogeneous magnetic field \( \mathbf{B} = B \mathbf{e}_z \) acts on the particle. Denote by \( \mathbf{A} = \mathbf{A}(r) \) the 
magnetic vector potential at the point \( r \), so that \( \mathbf{B} = \text{curl} \mathbf{A} \). It is natural to take \( \mathbf{A} = \frac{1}{2} \mathbf{B} \times \mathbf{r} \), but the vector potential can be taken as any representative 
from an equivalence class of potentials that differ by the 
gradiant of some scalar field, see [8]. The Hamiltonian 
for the system is:

\[ H = \frac{(p - \mathbf{A}(r))^2}{2}, \]

Note that we can write down the Hamiltonian function 
in this vector form without specifying any particular 
set of phase space coordinates. The motion of the sin-
gle particle is integrable, and, for the above choice of 
direction of the magnetic field, consists of uniform cir-
cular motion (around the ‘guiding center’) with angular 
frequency \( \Omega_c \equiv B \) (the cyclotron frequency) in the \( xy \)-
plane (perpendicular to the \( z \)-direction), and free iner-
tial drift in the \( z \)-direction.

We shall employ here a particular set of coordinates 
for the perpendicular directions. These (canonical) co-
ordinates \((\Theta, p_\Theta, \psi, p_\psi)\) are defined such that the an-
gle \( \Theta \) and its associated momentum \( p_\Theta \) specify the po-
lar coordinates for the guiding center, and the angle \( \psi \) 
and its associated momentum \( p_\psi \) specify the polar co-
ordinates for the particle position relative to the guiding 
center and the direction given by \( \Theta \). See figure 2.

In terms of these polar guiding center coordinates, the 
single particle Hamiltonian is simply

\[ H = \Omega_c p_\psi + \frac{p_\psi^2}{2}, \]

which manifestly is integrable. If we perturb \( H \) to a 
slow spatial or temporal variation of, say, the mag-
cnet field, the quantity \( p_\psi \) will be adiabatically invari-
ant [1], and the motion of the single particle will such 
that the value of this phase space function is approxi-
mately conserved.

For two interacting particles moving in the homoge-
nous magnetic field, one can separate the (now 6 de-

[¹] This idea was stressed by Jeans in his book on gas theory, [7], 
but only up to the second edition (1916), see chapter XVI. In the next 
edition edition (1920) the chapter was removed.
degree of freedom) problem
\[ H = \frac{(p_1 - A(r_1))^2}{2} + \frac{(p_2 - A(r_2))^2}{2} + \frac{1}{|r_1 - r_2|}, \]
by introducing, as in the celestial mechanics two-body problem, relative and center-of-mass frames \( r \equiv r_1 - r_2 \), and \( R \equiv \frac{1}{2}(r_1 + r_2) \). In terms of these vectors,
\[ H = H^{CM} + H^{rel} \equiv \frac{(P - A)^2}{2} + \frac{(p - a)^2}{2} + \frac{1}{|r|}, \]
where \( A = A(r_1) + A(r_2) \) and \( a = \frac{1}{2}(A(r_1) - A(r_2)) \). The center-of-mass dynamics decouples completely, reducing the problem to one of single particle motion. Thus, the nontrivial dynamics takes place only in the relative coordinates. The relative Hamiltonian \( H^{rel} \) describes (as in celestial mechanics) the collision of a single particle with another particle held fixed at the origin of the coordinate frame. Using the polar guiding-center coordinates \( (\Theta, \rho, \psi, \psi_0) \) defined above, \( H^{rel} \) takes the coordinate form
\[ H^{rel} = \Omega_c p_\psi + \frac{\rho^2}{2m} + \frac{1}{\sqrt{\rho^2 + z^2}}, \]
where
\[ \rho^2 = \frac{4p_\Theta}{\Omega_c} + \frac{8\Omega_c}{\Omega_c} \sqrt{\rho_0 (p_\Theta - p_\psi)} \cos \psi \]
specifies the \( xy \)-radial position of the guiding center of the moving particle.
We now introduce the criterion for strong magnetization. We say the magnetic field is strong, when the initial values of the \( z \)-velocity, and the value of \( \Omega_c \) respectively, are such that the cyclotron period \( 2\pi/\Omega_c \) is much shorter than the duration of one collision (this duration is taken to be the distance of closests approach divided by the \( z \)-velocity). The criterion simply states that the moving particle completes many cyclotron orbits during its collision with the charge fixed at the origin, or equivalently, that the interaction with the field of the fixed particle is a slow perturbation. The total perpendicular kinetic energy
\[ E_{\perp} = \Omega_c p_\psi = E_{\perp}^{in} + E_{\perp}^{rel} = E_{\perp} (\text{particle 1}) + E_{\perp} (\text{particle 2}), \]
will thus change only by an exponentially small amount in a binary collision.
Note that \( E_{\perp} (\text{particle 1}) \) and \( E_{\perp} (\text{particle 2}) \) each will undergo substantial change in the course of a collision, since each individual particle moves in a rapidly fluctuating electric field from the other particle. But their \( \text{sum} \) will only change by a small amount, if the magnetic field is strong. As in the diatomic molecules, the near constancy of the \( \text{sum} \) is not the result of a near constancy of each of its parts.

### 3.1 Many Particles
For \( N >> 1 \) particles there is little to be gained in separating out the center-of-mass motion, since this will rid the problem of only 3 degrees of freedom. Nevertheless the key result from the 2-particle problem is still valid, as the following argument demonstrates. The Hamiltonian, can, in the polar guiding center coordinates be written as
\[ H = \sum_{j=1}^{N} \left[ \Omega_c p_{\psi_j} + \frac{p_{\psi_j}^2}{2} + \sum_{i>j} \frac{1}{|r_i - r_j|} \right] \]
where \( r_i - r_j \) are assumed to be expressed as a function of \( \Theta_j, \psi_j, p_{\Theta_j}, p_{\psi_j}, \psi_1, \psi_j \), etc. To uncover the many-particle invariant we introduce a new set of angle variables, \( \{\varphi_j\} \), given by
\[ \varphi_1 = \psi_1, \]
\[ \varphi_j = \psi_j - \psi_1, \quad j = 2, \ldots, N. \]
The corresponding momenta are:
\[ p_{\varphi_1} = \sum_{j=1}^{N} p_{\psi_j}, \]
\[ p_{\varphi_j} = p_{\psi_j}, \quad j = 2, \ldots, N. \]
so that, in terms of the \( (\varphi, p_\varphi, \Theta, p_\Theta, z, p_z) \) variables, the Hamilton function takes the form
\[ H = \Omega_c p_{\varphi_1} + \sum_{j=1}^{N} \left[ \frac{p_{\varphi_j}^2}{2} + \sum_{i>j} \frac{1}{|r_i - r_j|} \right]. \]
Under the condition of strong magnetization, the variable \( \varphi_1 \) is the only fast angle in the problem. Consequently, \( p_{\varphi_1} \) is an adiabatic invariant. For \( \Omega_c \) constant, \( \Omega_c p_{\varphi_1} = \sum E_{\perp} \) is therefore nearly conserved. Thus, generalizing the 2-particle result, the \( \text{sum} \) of all the individual particle perpendicular actions, for a homogeneous, strong magnetic field, is an adiabatic invariant of the full \( N \)-particle dynamics.

### 4 How good is the conspiracy?
Since Landau and Teller’s pioneering work in the 1930’s, the ‘breaking’ or shift in an adiabatically invariant quantity when subjected to a given perturbation has been to calculate along unperturbed orbits, i.e., evaluate the oscillatory integral of the time derivative of the action.
For the case of binary collision of two charged particles in a strong magnetic field (equivalent to the collision of one charged particle with a charge fixed at the origin), one can explicitly compute the exponentially small change in $E_{\perp}^{rel} = \Omega_c p \psi$ through a collision where initially and finally the moving particle is at infinity.

The change in perpendicular kinetic energy is calculated as

$$\Delta E_{\perp}^{rel} = \int_{-\infty}^{+\infty} \frac{\cos(\Omega_c t)}{(\rho^2 + z(t)^2)^{3/2}} dt$$

where $\rho$ is the perpendicular distance from the guiding centre to the origin, and $z(t)$ is a solution to the differential equation

$$\dddot{z}(t) + \frac{1}{[\rho^2 + z(t)^2]} = \dddot{z}(-\infty)$$

corresponding to the 'infinite field' motion of a particle sliding on the field line.

By extending the integral to complex values of the time parameter and deforming the contour away from the real line as far as permitted by the branch cuts of the function in the denominator, one finds that the contour integral has the form

$$\Delta E_{\perp}^{rel} = h(\rho) \exp(-g(\rho) \Omega_c) \cos \delta,$$

where $\delta$ is (a random) initial phase, the function $h(\rho)$ is neither exponentially small or large, and the function $g(\rho)$ is given by

$$g(\rho) = \int_{1}^{\rho} \frac{x^{3/2}}{\sqrt{(x-1)(\rho^2 - x^2)}} dx.$$

To estimate the evolution of the many-particle invariant, one can assume that the evolution proceeds as a sequence of binary collisions, and then use statistical mechanics to find evolution of the average perpendicular kinetic energy of an ensemble of particles. This process is discussed in [3].

## 5 Conclusion

The deep mechanical phenomenon of adiabatic invariance manifests itself not only as a single-particle phenomenon, but may also take the form of a non-trivial collective behavior, a conspiracy of high-frequency oscillators. The result also generalizes to the case where there are a number of well-separated time scales; the oscillators associated with each time scale will exchange energy among themselves and tend to decouple from oscillators associated with other time scales.

### References


Figure 3. The function $g(\rho)$ which determines the exponential smallness of the change of the adiabatic invariant in a single collision.

The function $g(\rho)$ is monotone with $g(0) = \pi/2$. Details can be found in [5].