OPTIMAL SYNTHESIS OF GYROMOMENT GUIDANCE AND CONTROL FOR SPACECRAFT AND FREE-FLYING ROBOTS

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Abstract
Problems by an optimal synthesis of the spacecraft attitude motion at its a spatial rotation maneuver are most actual for the spacecraft opto- or radio-electronic observing the Earth and for space free-flying robots. New statement of the optimization problem by the spacecraft rotation maneuver with the general kinematic boundary conditions is considered. Methods for exact numeric and approximate analytic solution of the stated problem, and also some results on synthesis of the spacecraft guidance and control laws of the gyromoment control cluster by multiply scheme on the base of six gyrodines, are presented. Methods for synthesis of the digital robust gyromoment spacecraft attitude control systems with obtaining the guaranteed quality's indexes, are also represented.

Key words
Motion control, nonlinear systems, applications

1 Introduction

Some author's results were before presented (Somov, 2006; Somov *et al.*, 2007) on analytic synthesis of the spacecraft (SC) and the free-flying space robotic module (SRM) gyromoment guidance laws at given time interval. Obtained moreover guidance laws are not unique, therefore there are appeared problems for synthesis both a strict optimal and approximate optimal the gyromoment guidance laws which are calculated by explicit analytic relations. In the paper new statement of the optimization problem is presented on the SC rotation maneuver with the general kinematic boundary conditions. Applied quality's index is most pactice important functional which have a clear physical sense. There are presented methods for exact numeric and approximate analytic solution of the stated problem, and also some results on synthesis of the SC guidance and robust digital control laws of the gyromoment cluster by multiply scheme on the base of six gyrodines.

2 Problem of a rotation maneuver optimization

The SC rotation maneuver in inertial reference frame (IRF) $\mathbf{I}_\oplus$ described by kinematic relations for its body reference frame (BRF)

$$\begin{gathered}\dot{\boldsymbol{\Lambda}}(t) = \tfrac{1}{2}\boldsymbol{\Lambda} \circ \boldsymbol{\omega}(t);\ \dot{\boldsymbol{\omega}}(t) = \boldsymbol{\varepsilon}(t);\\ \dot{\boldsymbol{\varepsilon}} = \mathbf{v} \equiv \boldsymbol{\varepsilon}^*(t) + \boldsymbol{\omega}(t) \times \boldsymbol{\varepsilon}(t)\end{gathered} \tag{1}$$

during given time interval $\mathrm{T_p} \equiv [t_0^p, t_f^p]$ where $t_f^p \equiv t_0^p + T_p$. The optimization problem consists in determination of time functions — quaternion $\boldsymbol{\Lambda}(t) = (\lambda_o(t), \boldsymbol{\lambda}(t)), \boldsymbol{\lambda} \equiv \{\lambda_i, i = 1 \div 3\}$ and vectors $\boldsymbol{\omega}(t) \equiv \{\omega_i(t)\},\ \boldsymbol{\varepsilon}(t) \equiv \{\varepsilon_i(t)\}$ for the boundary conditions on left ($t = t_0^p$) and right ($t = t_f^p$) trajectory ends

$$\boldsymbol{\Lambda}(t_0^p) = \boldsymbol{\Lambda}_0;\ \boldsymbol{\omega}(t_0^p) = \boldsymbol{\omega}_0;\ \boldsymbol{\varepsilon}(t_0^p) = \boldsymbol{\varepsilon}_0; \tag{2}$$

$$\boldsymbol{\Lambda}(t_f^p) = \boldsymbol{\Lambda}_f;\ \boldsymbol{\omega}(t_f^p) = \boldsymbol{\omega}_f;\ \boldsymbol{\varepsilon}(t_f^p) = \boldsymbol{\varepsilon}_f \tag{3}$$

with optimization of the integral quadratic index

$$\mathrm{I_o} = \frac{1}{2}\int_{t_0^p}^{t_f^p} \langle \mathbf{v}(\tau), \mathbf{v}(\tau)\rangle \, d\tau \Rightarrow \min. \tag{4}$$

Optimizing this functional is topologically equivalently to optimizing the most practice important functional

$$\mathrm{I_1} = \bar{\mathrm{v}} \equiv \frac{1}{T_p}\int_{t_0^p}^{t_f^p} |\mathbf{v}(\tau)| \, d\tau \Rightarrow \min, \tag{5}$$

which have the clear physical sense — the mean value $\bar{\mathrm{v}}$ of the "control" module $|\mathbf{v}(t)|$ — a module by derivative of the BRF acceleration vector during process of the SC (or the SRM) rotation maneuver with respect to inertial reference frame $\mathbf{I}_\oplus$.

3 Optimal one-axis motion

Of course this problem is elementary and have analytic solution by Pontryagin's maximum principle. In result, the SC optimal on index (4) motion with respect to any k axis is presented by the analytic function $\varphi_k(t)$ in a class of the five degree polynomials (splines) by normed time $\tau = (t - t_0^p)/T_p \subset [0, 1]$ with analytic relations

$$\begin{aligned}\ddot{\varphi}_k(\tau) &= \varepsilon_k(\tau) = \varepsilon_k^0 \\ &\quad + \tau(6a_3 + 12a_4\tau + 20a_5\tau^2)/T_p^2;\\ \dot{\varphi}_k(\tau) &= \omega_k(\tau) = \omega_k^0 + \varepsilon_k^0\, T_p\tau \\ &\quad + \tau^2(3a_3 + 4a_4\tau + 5a_5\tau^2)/T_p;\\ \varphi_k(\tau) &= \varphi_k^0 + \tau(\omega_k^0 T_p + \varepsilon_k^0 T_p^2\tau/2 \\ &\quad + \tau^2(a_3 + a_4\tau + a_5\tau^2)),\end{aligned} \tag{6}$$

where constant coefficients $a_s,\ s = 3 \div 5$ are defined by the vector-matrix relation

$$\begin{bmatrix} a_3 \\ a_4 \\ a_5 \end{bmatrix} = \mathbf{A} \begin{bmatrix} \varphi_k^f - \varphi_k^0 - \omega_k^0 T_p - \varepsilon_k^0 T_p^2/2 \\ T_p(\omega_k^f - \omega_k^0 - \varepsilon_k^0 T_p) \\ T_p^2\ (\varepsilon_k^f - \varepsilon_k^0) \end{bmatrix} \tag{7}$$

with matrix $\mathbf{A} = \begin{bmatrix} 10 & -4 & 0.5 \\ -15 & 7 & -1 \\ 6 & -3 & 0.5 \end{bmatrix}$ and boundary conditions $\varphi_k^s, \omega_k^s, \varepsilon_k^s,\ s = 0, f$ on angels, angular rates and accelerations for that elementary rotation.

4 Approximate optimal spatial motion

Developed analytical approach to the problem is based on necessary and sufficient condition for solvability of Darboux problem. At general case the solution is presented as result of composition by three ($k = 1 \div 3$) simultaneously derived elementary rotations of embedded bases $\mathbf{E}_k$ about units $\mathbf{e}_k$ of Euler axes, which positions are defined from the boundary conditions (2) and (3) for initial spatial problem. For all 3 elementary rotations with respect to units $\mathbf{e}_k$ the boundary conditions are analytically assigned.

Into the IRF $\mathbf{I}_\oplus$ the quaternion $\mathbf{\Lambda}(t)$ is defined by the production

$$\mathbf{\Lambda}(t) = \mathbf{\Lambda}_0 \circ \mathbf{\Lambda}_1(t) \circ \mathbf{\Lambda}_2(t) \circ \mathbf{\Lambda}_3(t), \tag{8}$$

where $\mathbf{\Lambda}_k(t) = (C(\varphi_k(t)/2), S(\varphi_k(t)/2)\mathbf{e}_k)$, $C(\alpha) = \cos\alpha$, $S(\alpha) = \sin\alpha$, and functions $\varphi_k(t)$ present the elementary rotation angles in analytical form (6),(7).

Let the quaternion $\mathbf{\Lambda}^* \equiv (\lambda_0^*, \boldsymbol{\lambda}^*) = \tilde{\mathbf{\Lambda}}_0 \circ \mathbf{\Lambda}_f \neq \mathbf{1}$ have the Euler axis unit $\mathbf{e}_3 = \boldsymbol{\lambda}^*/S(\varphi^*/2)$ by 3-rd elementary rotation where angle $\varphi^* = 2\arccos(\lambda_0^*)$. For elementary rotations there are applied next the boundary quaternion values:

$$\begin{aligned}&\mathbf{\Lambda}_1(t_0^p) = \mathbf{\Lambda}_1(t_f^p) = \mathbf{\Lambda}_2(t_0^p) = \mathbf{\Lambda}_2(t_f^p) = \mathbf{1};\\ &\mathbf{\Lambda}_3(t_0^p) = \mathbf{1}; \mathbf{\Lambda}_3(t_f^p) = (C(\varphi_3^f/2), \mathbf{e}_3 S(\varphi_3^f/2)),\end{aligned} \tag{9}$$

where $\varphi_3^f = \varphi^*$ and $\mathbf{1}$ is a single quaternion. For

$$\mathbf{e}_0^\omega \equiv \frac{\boldsymbol{\omega}_0}{\omega_0};\ \mathbf{e}_f^\omega \equiv \frac{\boldsymbol{\omega}_f}{\omega_f};\ \mathbf{e}_0^\varepsilon \equiv \frac{\boldsymbol{\varepsilon}_0}{\varepsilon_0};\ \mathbf{e}_f^\varepsilon \equiv \frac{\boldsymbol{\varepsilon}_0}{\varepsilon_0};\ \ a \equiv |\mathbf{a}|$$

unit $\mathbf{e}_1$ of 1-st elementary rotation's on Euler's axis is selected by next simple algorithm:

$$\begin{aligned}&\mathbf{a} =: \mathbf{e}_0^\omega - \langle \mathbf{e}_0^\omega, \mathbf{e}_3\rangle \mathbf{e}_3, if\ a > 0, then\ \mathbf{e}_1 = \mathbf{a}/a,\\ &else\\ &\mathbf{a} =: \mathbf{e}_f^\omega - \langle \mathbf{e}_f^\omega, \mathbf{e}_3\rangle \mathbf{e}_3, if\ a > 0, then\ \mathbf{e}_1 = \mathbf{a}/a,\\ &else\\ &\mathbf{a} =: \mathbf{e}_0^\varepsilon - \langle \mathbf{e}_0^\varepsilon, \mathbf{e}_3\rangle \mathbf{e}_3, if\ a > 0, then\ \mathbf{e}_1 = \mathbf{a}/a,\\ &else\\ &\mathbf{a} =: \mathbf{e}_f^\varepsilon - \langle \mathbf{e}_f^\varepsilon, \mathbf{e}_3\rangle \mathbf{e}_3, if\ a > 0, then\ \mathbf{e}_1 = \mathbf{a}/a,\\ &\textit{else 3-rd rotation is only, e.g. } \mathbf{\Lambda}_1(t) = \mathbf{\Lambda}_2(t) \equiv \mathbf{1}.\end{aligned}$$

Unit $\mathbf{e}_2$ is defined as $\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1$. All vectors $\boldsymbol{\omega}_k(t) = \dot{\varphi}_k(t)\mathbf{e}_k$, $\boldsymbol{\varepsilon}_k(t) = \ddot{\varphi}_k(t)\mathbf{e}_k$ and $\dot{\boldsymbol{\varepsilon}}_k(t) = \dddot{\varphi}_k(t)\mathbf{e}_k$ have analytic form owing to function $\varphi_k(t)$ (6), (7) which is optimal on index (4) for each elementary rotation.

At initial notations $\boldsymbol{\omega}^{(1)}(t) = \boldsymbol{\omega}_1(t)$, $\boldsymbol{\varepsilon}^{(1)}(t) = \boldsymbol{\varepsilon}_1(t)$ and $\dot{\boldsymbol{\varepsilon}}^{(1)}(t) = \dot{\boldsymbol{\varepsilon}}_1(t)$ vectors $\boldsymbol{\omega}(t)$, $\boldsymbol{\varepsilon}(t)$ and $\dot{\boldsymbol{\varepsilon}}(t) \equiv \mathbf{v}(t)$ are analytically defined by next recurrent algorithm: for upper indexes $k = 2, 3$ there are consistently computed

$$\begin{aligned}&\boldsymbol{\omega}_q^k(t) =: \tilde{\mathbf{\Lambda}}_k(t) \circ \boldsymbol{\omega}^{(k-1)}(t) \circ \mathbf{\Lambda}_k(t);\\ &\boldsymbol{\omega}^{(k)}(t) = \boldsymbol{\omega}_k(t) + \boldsymbol{\omega}_q^k(t);\\ &\boldsymbol{\varepsilon}_q^k(t) =: \tilde{\mathbf{\Lambda}}_k(t) \circ \boldsymbol{\varepsilon}^{(k-1)}(t) \circ \mathbf{\Lambda}_k(t)\\ &\boldsymbol{\varepsilon}^{(k)}(t) = \boldsymbol{\varepsilon}_k(t) + \boldsymbol{\varepsilon}_q^k(t) + \boldsymbol{\omega}_q^k(t) \times \boldsymbol{\omega}_k(t);\\ &\dot{\boldsymbol{\varepsilon}}_q^k(t) =: \tilde{\mathbf{\Lambda}}_k(t) \circ \dot{\boldsymbol{\varepsilon}}^{(k-1)}(t) \circ \mathbf{\Lambda}_k(t);\\ &\dot{\boldsymbol{\varepsilon}}^{(k)}(t) = \dot{\boldsymbol{\varepsilon}}_k(t) + \dot{\boldsymbol{\varepsilon}}_q^k(t) + (2\boldsymbol{\varepsilon}_q^k(t)\\ &+\boldsymbol{\omega}_q^k(t) \times \boldsymbol{\omega}_k(t)) \times \boldsymbol{\omega}_k(t) + \boldsymbol{\omega}_q^k(t) \times \boldsymbol{\varepsilon}_k(t),\end{aligned} \tag{10}$$

in result one can obtain $\boldsymbol{\omega}(t) = \boldsymbol{\omega}^{(3)}(t)$, $\boldsymbol{\varepsilon}(t) = \boldsymbol{\varepsilon}^{(3)}(t)$, $\boldsymbol{\varepsilon}^*(t) = \dot{\boldsymbol{\varepsilon}}^{(3)}(t)$ and $\dot{\boldsymbol{\varepsilon}}(t) = \boldsymbol{\varepsilon}^*(t) + \boldsymbol{\omega}(t) \times \boldsymbol{\varepsilon}(t)$. Functions $\varphi_k(t), k = 1 \div 3$ must have boundary conditions

$$\begin{aligned}&\dot{\varphi}_k^0 = \langle \boldsymbol{\omega}_0, \mathbf{e}_k\rangle,\ \ k = 1, 2, 3;\ \ \ \dot{\varphi}_3^f = \langle \boldsymbol{\omega}_f, \mathbf{e}_3\rangle;\\ &\dot{\varphi}_k^f = \langle \mathbf{\Lambda}^* \circ (\boldsymbol{\omega}_f - \mathbf{e}_3\dot{\varphi}_3^f) \circ \tilde{\mathbf{\Lambda}}^*, \mathbf{e}_k\rangle,\ \ k = 1, 2;\\ &\ddot{\varphi}_1^0 = \langle \boldsymbol{\varepsilon}_0, \mathbf{e}_1\rangle - \dot{\varphi}_2^0\dot{\varphi}_3^0;\ \ddot{\varphi}_2^0 = \langle \boldsymbol{\varepsilon}_0, \mathbf{e}_2\rangle + \dot{\varphi}_1^0\dot{\varphi}_3^0;\\ &\ddot{\varphi}_3^0 = \langle \boldsymbol{\varepsilon}_0, \mathbf{e}_3\rangle - \dot{\varphi}_1^0\dot{\varphi}_2^0;\ \tilde{\varepsilon}_k^f \equiv \langle \mathbf{\Lambda}^* \circ \boldsymbol{\varepsilon}_f \circ \tilde{\mathbf{\Lambda}}^*, \mathbf{e}_k\rangle;\\ &\ddot{\varphi}_1^f = \tilde{\varepsilon}_1^f - \dot{\varphi}_2^f\dot{\varphi}_3^f; \ddot{\varphi}_2^f = \tilde{\varepsilon}_2^f + \dot{\varphi}_1^f\dot{\varphi}_3^f; \ddot{\varphi}_3^f = \tilde{\varepsilon}_3^f - \dot{\varphi}_1^f\dot{\varphi}_2^f,\end{aligned}$$

which are carried out by (9) and the algorithm (10).

Suggested approach have large advantages with respect to optimal rotations by standard Euler-Krylov angels, see numerical results in (Somov, 2007).

5 Optimal spatial motion

For nonlinear problem (1) – (4) Hamilton function is

$$\begin{aligned}H &= -\tfrac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle + \tfrac{1}{2}\langle \mathbf{\Psi}, \mathbf{\Lambda} \circ \boldsymbol{\omega}\rangle + \langle \boldsymbol{\mu}, \boldsymbol{\varepsilon}\rangle + \langle \boldsymbol{\nu}, \mathbf{v}\rangle\\ &= -\tfrac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle + \tfrac{1}{2}\langle \mathrm{vect}(\tilde{\mathbf{\Lambda}} \circ \mathbf{\Psi}, \boldsymbol{\omega}\rangle + \langle \boldsymbol{\mu}, \boldsymbol{\varepsilon}\rangle + \langle \boldsymbol{\nu}, \mathbf{v}\rangle\end{aligned}$$

with associated quaternion $\mathbf{\Psi}(t) = \mathbf{C}_\varphi \circ \mathbf{\Lambda}(t)$, where $\mathbf{C}_\varphi = (\mathrm{c}_{\varphi 0}, \mathbf{c}_\varphi)$ is the normed quaternion (Branetz and Shmyglevsky, 1973) with a vector part $\mathbf{c}_\varphi = \{\mathrm{c}_{\varphi k}\}$. The associated differential system have the form

$$\dot{\mathbf{\Psi}} = \tfrac{1}{2}\mathbf{\Psi} \circ \boldsymbol{\omega}; \;\; \dot{\boldsymbol{\mu}} = -\tfrac{1}{2}\tilde{\mathbf{\Lambda}} \circ \mathbf{c}_\varphi \circ \mathbf{\Lambda}; \;\; \dot{\boldsymbol{\nu}} = -\boldsymbol{\mu}. \quad (11)$$

The optimality condition $\partial H/\partial \mathbf{v} = -\mathbf{v} + \boldsymbol{\nu} = \mathbf{0}$ give the optimal " control "

$$\mathbf{v}(t) = \mathbf{c}_\varepsilon - \mathbf{c}_\omega(t - t_0^p) + \frac{1}{2}\int_{t_0^p}^{t}(\int_{t_0^p}^{\tau}\tilde{\mathbf{\Lambda}}(s)\circ\mathbf{c}_\varphi\circ\mathbf{\Lambda}(s)ds)\, d\tau,$$

where vectors $\mathbf{c}_\varphi, \mathbf{c}_\omega = \{\mathrm{c}_{\omega k}\}$ and $\mathbf{c}_\varepsilon = \{\mathrm{c}_{\varepsilon k}\}$ must be numerically defined using known analytical structure of solution for direct system (1) and taking into account the boundary conditions (2) and (3).

Standard Newton iteration method was applied for numerical obtaining the "control" $\mathbf{v}(t)$ which is a strict optimal on index (4) for the nonlinear optimization problem (1) – (4). Moreover analytical solution of the "start" problem (initial point) was applied in the form of approximate optimal motion (8) and (6), (7) with the constant vectors $\mathbf{c}_\varphi$, $\mathbf{c}_\omega$ and $\mathbf{c}_\varepsilon$. Values of these constant vectors are numerically corrected by an iteration procedure using a combine numerical integration of direct (1) and associated (11) differential systems which are linearizated at neighbourhood of numerical solution on previous iteration. At such initial point the Newton's iteration process have a rapid convergence: usually there is needed only 2 – 3 iterations for obtaining a numerical solution with fine accuracy. Difference between approximate optimal spatial motion (analytic solution of "start" problem) and strict optimal spatial motion is very light — up to 5 % by functional I_1 (5) for the SC (or the SRM) practical rotational maneuvers.

6 Gyro Moment Cluster

At precession theory of control moment gyros a simplest modeling the gyro moment cluster (GMC), based on several gyrodines (GDs), is carried out as follows. Let each GD have angular momentum (AM) vector $\mathbf{H}_p \equiv h_g \mathbf{h}_p$, $p = 1 \div m$ with the same module h_g. Each GD is connected the right trihedron of its axes: unit $\mathbf{h}_p(\beta_p)$ of the AM vector which position is determined by angle β_p in own device basis, the unit $\mathbf{g}_p$ of the GD suspension axis fixed in the BRF Oxyz and unit $\mathbf{p}_p(\beta_p) = \mathbf{h}_p(\beta_p) \times \mathbf{g}_p$. Normed to h_g vector of the GMC summary AM is $\mathbf{h}(\boldsymbol{\beta}) \equiv \sum \mathbf{h}_p$, where vector-column $\boldsymbol{\beta} = \{\beta_p\}$. The unit $\mathbf{m}_p^g(\beta_p) = \partial\mathbf{h}_p(\beta_p)/\partial\beta_p = \mathbf{g}_p \times \mathbf{h}_p(\beta_p)$ of the GD $\#p$ gyroscopic torque is always contrary with respect to unit $\mathbf{p}_p(\beta_p)$.

Collinear pair of the stop-less GDs was named *Scissored Pair Ensemble* (*SPE*) in known original work *J.W. Crenshaw* (1973), and redundant multiply scheme based on three collinear GD pairs ($m = 6$) was named as *3-SPE*, see (Somov *et al.*, 2003) and fig. 1. At the

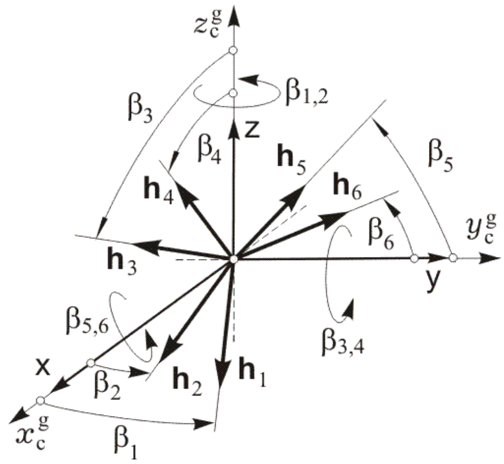

Figure 1. The *3-SPE* scheme of the GMC

park state of the GMC a vector of its normed summary AM $\mathbf{h}(\boldsymbol{\beta}) = \mathbf{0}$. Into orthogonal canonical basis Oxyz, see fig. 1, the GD's AM units have next projections:

$$x_1 = C_1;\; x_2 = C_2;\; y_1 = S_1;\; y_2 = S_2;$$
$$x_3 = S_3;\; x_4 = S_4;\; z_3 = C_3;\; z_4 = C_4;$$
$$y_5 = C_5;\; y_6 = C_6;\; z_5 = S_5;\; z_6 = S_6,$$

where $S_p \equiv \sin\beta_p$ and $C_p \equiv \cos\beta_p$. Then vector-column $\mathbf{h}(\boldsymbol{\beta}) = \{\mathsf{x}, \mathsf{y}, \mathsf{z}\}$ of normed GMC's summary AM vector and matrix $\mathbf{A}_\mathrm{h}(\boldsymbol{\beta}) = \partial\mathbf{h}/\partial\boldsymbol{\beta}$ have the form

$$\mathbf{h}(\boldsymbol{\beta}) = \{\sum x_p, \sum y_p, \sum z_p\};$$

$$\mathbf{A}_\mathrm{h}(\boldsymbol{\beta}) = \begin{bmatrix} -y_1 & -y_2 & z_3 & z_4 & 0 & 0 \\ x_1 & x_2 & 0 & 0 & -z_5 & -z_6 \\ 0 & 0 & -x_3 & -x_4 & y_5 & y_6 \end{bmatrix}.$$

For *3-SPE* scheme a singular state is appeared when the matrix Gramme $\mathbf{G}(\boldsymbol{\beta}) = \mathbf{A}_\mathrm{h}(\boldsymbol{\beta})\mathbf{A}_\mathrm{h}^\mathrm{t}(\boldsymbol{\beta})$ loses its full rang, e.g. when $\mathsf{G} \equiv \det \mathbf{G}(\boldsymbol{\beta}) = 0$.

7 Synthesis of the gyromoment guidance laws

The problem consists in elaboration of algorithms for angular removing each gyrodine into the GMC for exact angular guidance of the SC (or SRM) body, for example at optimal spatial rotational maneuver.

For the GMC control torque vector

$$\mathbf{M}^\mathrm{g}(\boldsymbol{\beta}, \mathbf{u}^\mathrm{g}) = -\dot{\boldsymbol{\mathcal{H}}} = -h_g \mathbf{A}_\mathrm{h}(\boldsymbol{\beta})\,\mathbf{u}^\mathrm{g};\;\; \dot{\boldsymbol{\beta}} = \mathbf{u}^\mathrm{g} \quad (12)$$

and the SC model as a free rigid body the simplified controlled object is considered:

$$\dot{\mathbf{\Lambda}} = \mathbf{\Lambda} \circ \boldsymbol{\omega}/2;\;\; \mathbf{J}\dot{\boldsymbol{\omega}} + [\boldsymbol{\omega}\times]\mathbf{G}^o = \mathbf{M}^\mathrm{g};\;\; \dot{\boldsymbol{\beta}} = \mathbf{u}^\mathrm{g}. \quad (13)$$

Here all vectors and tensor $\mathbf{J}$ of the SC body inertia are presented in the BRF, and $\mathbf{G}^o = \mathbf{J}\boldsymbol{\omega} + \boldsymbol{\mathcal{H}}(\boldsymbol{\beta})$ with the GMC summary AM's vector $\boldsymbol{\mathcal{H}}(\boldsymbol{\beta}) = h_g \mathbf{h}(\boldsymbol{\beta})$.

At given the SC body angular programmed motion $\mathbf{\Lambda}^p(t), \boldsymbol{\omega}^p(t), \varepsilon^p(t) = \dot{\boldsymbol{\omega}}^p(t)$ with respect to the IRF $\mathbf{I}_\oplus$ during time interval $t \in \mathrm{T_p}$ and for forming the vector of corresponding continuous control torque $\mathbf{M}^\mathrm{g}(\boldsymbol{\beta}, \mathbf{u}^\mathrm{g})$

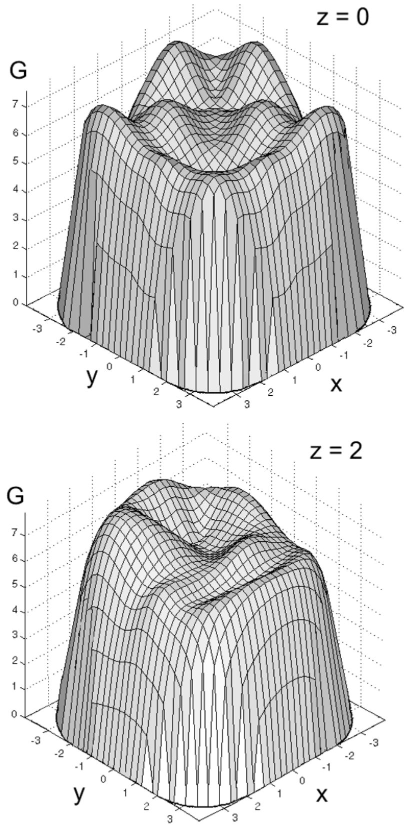

Figure 2. Gramme determinant G for the GMC canonical scheme *3-SPE* at the vector explicit tuning law (16) with $\rho = 0.65$.

(12), the vector-columns $\dot{\boldsymbol{\beta}}$ and $\ddot{\boldsymbol{\beta}}$ must be component-wise module restricted:

$$|\dot{\beta}_p(t)| \le \bar{\mathrm{u}}_{\mathrm{g}} < \bar{\mathrm{u}}_{\mathrm{g}}^{\mathrm{m}}, \; |\ddot{\beta}_p(t)| \le \bar{\mathrm{v}}_{\mathrm{g}}, \; \forall t \in \mathrm{T_p}, \quad (14)$$

where values $\bar{\mathrm{u}}_{\mathrm{g}}, \bar{\mathrm{u}}_{\mathrm{g}}^{\mathrm{m}}$ and $\bar{\mathrm{v}}_{\mathrm{g}}$ are constant, and $p = 1 \div 6$. Onboard algorithms have been developed for gyromoment SC guidance by its SRM on the explicit time functions $\boldsymbol{\Lambda}^p(t)$, $\boldsymbol{\omega}^p(t)$, $\boldsymbol{\varepsilon}^p(t)$ for the boundary conditions (2), (3) and also for given condition

$$\dot{\boldsymbol{\varepsilon}}(t_f^p) = \dot{\boldsymbol{\varepsilon}}_f \equiv \boldsymbol{\varepsilon}_f^* + \boldsymbol{\omega}_f \times \boldsymbol{\varepsilon}_f, \quad (15)$$

which presents requirements to a *smooth conjugation* of guidance by a SRM with guidance at next the SC's spatial course motion (SCM), taking into account the restrictions (14) to vectors $\dot{\boldsymbol{\beta}}(t)$ and $\ddot{\boldsymbol{\beta}}(t)$ Developed approach to the problem is based on the SC approximate optimal motion with given boundary conditions (2), (3) and (15). Here functions $\varphi_k(t)$ are selected in a class of splines by five and six degree, moreover a module of a angular rate $\dot{\varphi}_3(t)$ in a position transfer $(k=3)$ may be limited when functions $\dot{\varphi}_1(t) = \dot{\varphi}_2(t) \equiv 0$ and $\dot{\varphi}_3(t) = \omega_* = \mathrm{const}$. The technique is based on the generalized integral's properties for the AM $\mathbf{G}^o(t)$ of the mechanical system "SC+GMC" and allows to evaluate vectors $\boldsymbol{\beta}(t)$, $\dot{\boldsymbol{\beta}}(t)$, $\ddot{\boldsymbol{\beta}}(t)$ in analytical form for a preassigned SC motion $\boldsymbol{\Lambda}(t)$, $\boldsymbol{\omega}(t)$, $\boldsymbol{\varepsilon}(t)$, $\dot{\boldsymbol{\varepsilon}}(t)\, \forall t \in \mathrm{T_p}$.

At introducing the denotations

$$x_{12} = x_1 + x_2; \;\; x_{34} = x_3 + x_4; \;\; y_{12} = y_1 + y_2;$$
$$y_{56} = y_5 + y_6; \;\; z_{34} = z_3 + z_4; \;\; z_{56} = z_5 + z_6;$$
$$\tilde{x}_{12} = x_{12}/\sqrt{4 - y_{12}^2}\,; \;\; \tilde{x}_{34} = x_{34}/\sqrt{4 - z_{34}^2};$$
$$\tilde{y}_{12} = y_{12}/\sqrt{4 - x_{12}^2}\,; \;\; \tilde{y}_{56} = y_{56}/\sqrt{4 - z_{56}^2};$$
$$\tilde{z}_{34} = z_{34}/\sqrt{4 - x_{34}^2}\,; \;\; \tilde{z}_{56} = z_{56}/\sqrt{4 - y_{56}^2}$$

components of the GMC explicit vector tuning law

$$\mathbf{f}_\rho(\boldsymbol{\beta}) \equiv \{f_{\rho 1}(\boldsymbol{\beta}), f_{\rho 2}(\boldsymbol{\beta}), f_{\rho 3}(\boldsymbol{\beta})\} = \mathbf{0} \quad (16)$$

are applied in the form

$$f_{\rho 1}(\boldsymbol{\beta}) \equiv \tilde{x}_{12} - \tilde{x}_{34} + \rho\,(\tilde{x}_{12}\,\tilde{x}_{34} - 1);$$
$$f_{\rho 2}(\boldsymbol{\beta}) \equiv \tilde{y}_{56} - \tilde{y}_{12} + \rho\,(\tilde{y}_{56}\,\tilde{y}_{12} - 1);$$
$$f_{\rho 3}(\boldsymbol{\beta}) \equiv \tilde{z}_{34} - \tilde{z}_{56} + \rho\,(\tilde{z}_{34}\,\tilde{z}_{56} - 1).$$

The analytical proof have been elaborated that vector tuning law (16) ensures absent of singular states by this GMC scheme for all values of the GMC AM vector $\mathbf{h}(t)$ inside all its variation domain, see fig. 2.

For the representation

$$x_{12} = (\mathsf{x} + \Delta_x)/2; \;\; x_{34} = (\mathsf{x} - \Delta_x)/2;$$
$$y_{56} = (\mathsf{y} + \Delta_y)/2; \;\; y_{12} = (\mathsf{y} - \Delta_y)/2;$$
$$z_{34} = (\mathsf{z} + \Delta_z)/2; \;\; z_{56} = (\mathsf{z} - \Delta_z)/2$$

and denotation $\boldsymbol{\Delta} = \{\Delta_x, \Delta_y, \Delta_z\}$ one can obtain the nonlinear vector equation $\boldsymbol{\Delta}(t) = \boldsymbol{\Phi}(\mathbf{h}(t), \boldsymbol{\Delta}(t))$. At a known vector $\mathbf{h}(t)$ this equation have single solution $\boldsymbol{\Delta}(t)$, which is readily computed by method of a simple iteration. Further the units $\mathbf{h}_p(\beta_p(t))$ and vector-columns $\boldsymbol{\beta}(t)$, $\dot{\boldsymbol{\beta}}(t)$, $\ddot{\boldsymbol{\beta}}(t)$ are calculated by the explicit analytical relations $\forall t \in \mathrm{T_p}$. Fig. 3 and fig. 4 present dynamic characteristics of the SC's SRM and the GMC by *3-SPE* scheme during time $t \in \mathrm{T_p} = [0, T_p]$ with $T_p = 45$ sec, for a possible limitation $|\boldsymbol{\omega}(t)| \le \omega_* = 2°/s$ and next boundary conditions:

$$\boldsymbol{\Lambda}_0 = (0.06255029449, -0.35479160599, -0.67663869314, -0.64216077108);$$
$$\boldsymbol{\Lambda}_f = (0.04168181290, -0.35479620846, -0.89901121936, -0.25330042320);$$
$$\boldsymbol{\omega}_0 = \{0.060345, 0.355995, 0.071572\}°/s;$$
$$\boldsymbol{\omega}_f = \{-0.084455, -0.333483, 0.060107\}°/s;$$
$$\boldsymbol{\varepsilon}_0 = 10^{-2} \cdot \{0.2960, -0.0643, 0.0303\}°/s^2;$$
$$\boldsymbol{\varepsilon}_f = 10^{-2} \cdot \{-0.2784; 0.1417; -0.0074\}°/s^2;$$
$$\dot{\boldsymbol{\varepsilon}}_f = 10^{-5} \cdot \{0.05, 0.38, 0.01\}°/s^3.$$

8 Synthesis of the gyromoment control law

For a fixed position of the SC flexible structures with some simplifying assumptions and $t \in \mathrm{T}_{t_0} = [t_0, +\infty)$ a SC angular motion model is as follows:

$$\dot{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda} \circ \boldsymbol{\omega}/2; \;\; \mathbf{A}^o\,\{\dot{\boldsymbol{\omega}}, \ddot{\mathbf{q}}\} = \{\mathbf{F}^\omega, \mathbf{F}^q\}, \quad (17)$$

$$\mathbf{F}^\omega = \mathbf{M}^{\mathbf{g}}(\boldsymbol{\beta}, \mathbf{u}^{\mathrm{g}}) - \boldsymbol{\omega} \times \mathbf{G} + \mathbf{M}_d^o + \mathbf{Q}^o; \quad \dot{\boldsymbol{\beta}} = \mathbf{u}^{\mathrm{g}};$$
$$\mathbf{F}^q = \{-((\delta/\pi)\Omega_j^q \dot{q}_j + (\Omega_j^q)^2 q_j) + \mathrm{Q}_j^q(\boldsymbol{\omega}, \dot{q}_j, q_j)\}.$$

Here vector $\mathbf{M}_d^o$ presents the external disturbance

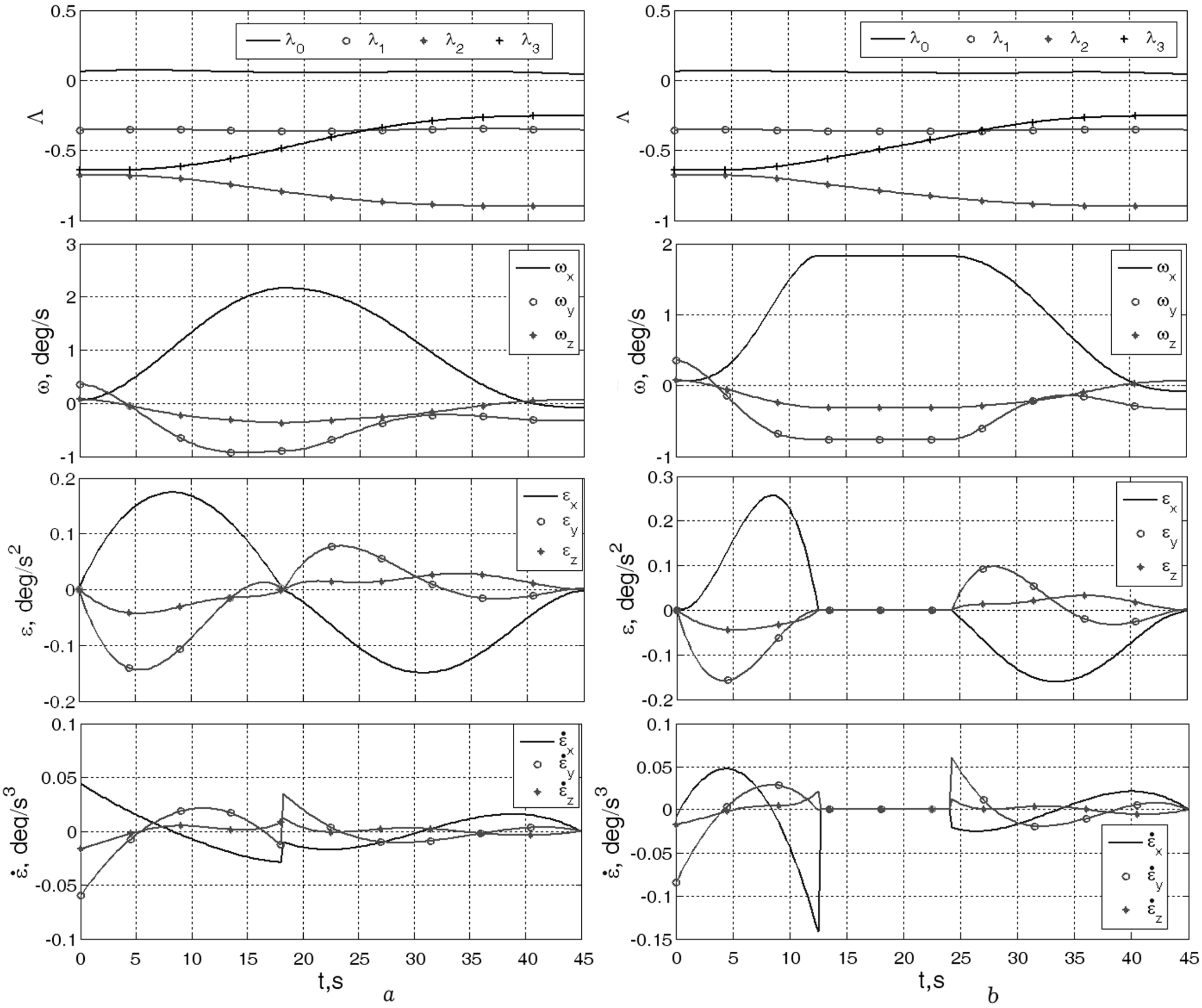

Figure 3. The SC's SRM: a — without a limit on module of the SC angular rate vector; b — with such limit.

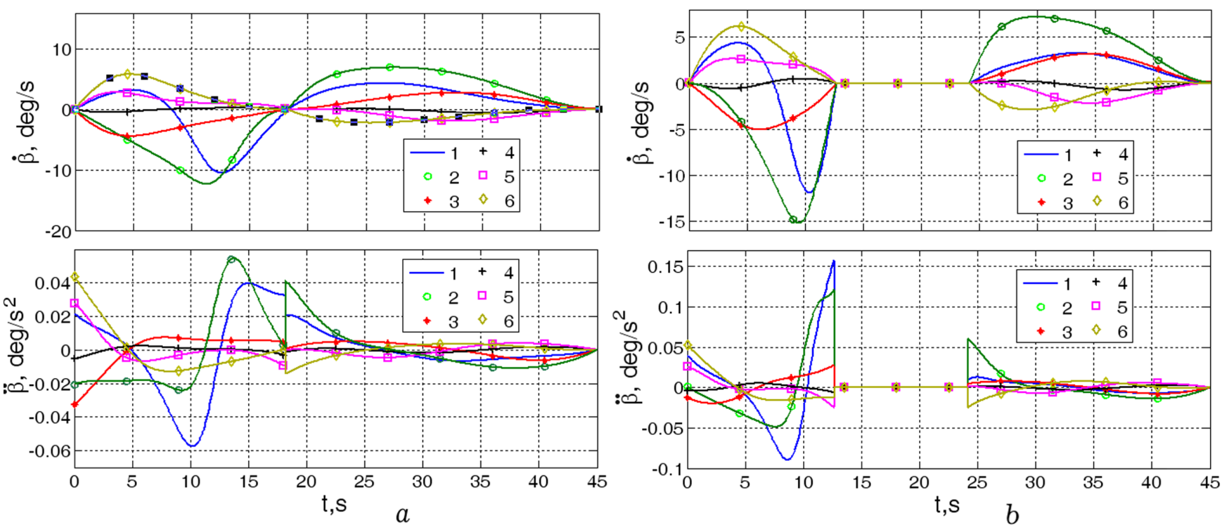

Figure 4. The GMC *3-SPE* scheme coordinates at the same SC's SRM: a — without a limit; b — with such limit.

torques, $\mathbf{Q}^o$ and $\mathrm{Q}_j^q(\boldsymbol{\omega}, \dot{q}_j, q_j)$ are nonlinear functions, and

$$\mathbf{A}^o = \begin{bmatrix} \mathbf{J} & \mathbf{D}_q \\ \mathbf{D}_q^{\mathrm{t}} & \mathbf{I} \end{bmatrix}; \quad \begin{array}{l} \mathbf{G} = \mathbf{G}^o + \mathbf{D}_q\,\mathbf{q}; \\ \mathbf{q} = \{q_j, j = 1 \div n_q\}. \end{array}$$

Applied onboard measuring subsystem is based on inertial gyro unit corrected by the fine fixed-head star trackers. Applied contemporary filtering $\&$ alignment calibration algorithms and a discrete astatic observer give finally a fine discrete estimating the SC angular motion by the quaternion and angular rate vector, when a measurement period $T_q = t_{s+1} - t_s \le T_u$ is multiply with respect to a control period T_u. The GMC torque vector $\mathbf{M}^{\mathbf{g}}$ is presented in the form (12), where $\mathrm{u}_p^{\mathrm{g}}(t) = a^{\mathrm{g}}\,\mathrm{Zh}[\mathrm{Sat}(\mathrm{Qntr}(u_{pk}^g, b_u), \bar{\mathrm{u}}_{\mathrm{g}}^{\mathrm{m}}), T_u]$ with a constant a^{g} and a control period $T_u = t_{k+1} - t_k$, $k \in \mathbb{N}_0 \equiv [0, 1, 2, ...)$; discrete functions u_{pk}^g are outputs of nonlinear control law, and functions $\mathrm{Sat}(x, a)$, $\mathrm{Qntr}(x, a)$ and $\mathrm{Zh}[x, T]$ are general-usage ones.

Applied approach to synthesis of nonlinear control system with a partial measurement of its state is based on method of vector Lyapunov functions (VLF) in cooperation with the exact feedback linearization (EFL) technique.

In stage 1, the GMC control torque vector $\mathbf{M}^{\mathbf{g}}$ (12) and the SC simplest model (13) are considered. The error quaternion is $\mathbf{E} = (e_0, \mathbf{e}) = \tilde{\boldsymbol{\Lambda}}^p(t) \circ \boldsymbol{\Lambda}$, Euler parameters' vector is $\boldsymbol{\mathcal{E}} = \{e_0, \mathbf{e}\}$, and the attitude error's matrix is $\mathbf{C}_e \equiv \mathbf{C}(\boldsymbol{\mathcal{E}}) = \mathbf{I}_3 - 2[\mathbf{e}\times]\mathbf{Q}_e$, where $\mathbf{Q}_e \equiv \mathbf{Q}(\boldsymbol{\mathcal{E}}) = \mathbf{I}_3 e_0 + [\mathbf{e}\times]$ with $\det(\mathbf{Q}_e) = e_0$. If error $\delta\boldsymbol{\omega} \equiv \tilde{\boldsymbol{\omega}}$ in the rate vector $\boldsymbol{\omega}$ is defined as $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} - \mathbf{C}_e \boldsymbol{\omega}^p(t)$, and the GMC's required control torque vector $\mathbf{M}^{\mathbf{g}}$ is formed as

$$\mathbf{M}^{\mathbf{g}} = \boldsymbol{\omega} \times \mathbf{G}^o + \mathbf{J}(\mathbf{C}_e \dot{\boldsymbol{\omega}}^p(t) - [\boldsymbol{\omega}\times]\mathbf{C}_e \boldsymbol{\omega}^p(t) + \tilde{\mathbf{m}}),$$

then the simplest nonlinear model for the SC's attitude error is as follows:

$$\dot{e}_0 = -\langle \mathbf{e}, \tilde{\boldsymbol{\omega}} \rangle / 2; \;\; \dot{\mathbf{e}} = \mathbf{Q}_e \tilde{\boldsymbol{\omega}} / 2; \;\; \dot{\tilde{\boldsymbol{\omega}}} = \tilde{\mathbf{m}}.$$

By the relations $\mathbf{Q}_e^{-1}\mathbf{Q}_e^{\mathrm{t}} = \mathbf{C}_e$; $\mathbf{Q}_e^{-1} = \mathbf{Q}_e^{\mathrm{t}} + \mathbf{e}\,\mathbf{e}^{\mathrm{t}}/e_0$; $\mathbf{Q}_e^{-1}\mathbf{e} = \mathbf{e}/e_0$; $\mathbf{I}_3 - e_0\mathbf{Q}_e^{-1} = \mathbf{Q}_e^{\mathrm{t}}[\mathbf{e}\times]$, which are used for $e_0 \neq 0$, a non-local nonlinear coordinate transformation is applied at analytical synthesis by the EFL. That results to the nonlinear control law

$$\tilde{\mathbf{m}}(\boldsymbol{\mathcal{E}}, \tilde{\boldsymbol{\omega}}) = -\mathbf{A}_0\,\mathbf{e}\,\mathrm{Sgn}(e_0) - \mathbf{A}_1\,\tilde{\boldsymbol{\omega}},$$

where $\mathbf{A}_0 = ((2a_0^* - \tilde{\omega}^2/2)/e_0)\mathbf{I}_3$; $\mathbf{A}_1 = a_1^*\mathbf{I}_3 - \mathbf{R}_{e\omega}$, $\mathrm{Sgn}(e_0) = (1, \text{ if } e_0 \ge 0) \vee (-1, \text{ if } e_0 < 0)$, matrix $\mathbf{R}_{e\omega} = \langle \mathbf{e}, \tilde{\boldsymbol{\omega}} \rangle \mathbf{Q}_e^{\mathrm{t}}[\mathbf{e}\times]/(2e_0)$, and constants a_0^*, a_1^* are calculated on spectrum $\mathrm{S}_{ci}^* = -\alpha_c \pm j\omega_c$. Simultaneously the VFL $\boldsymbol{\upsilon}(\boldsymbol{\mathcal{E}}, \tilde{\boldsymbol{\omega}})$ is analytically constructed.

In stage 2, the problems of synthesising nonlinear control law are solved for model (17) of flexible spacecraft. Furthermore, the selection of parameters in the structure of the GMC nonlinear robust control law is fulfilled by a multistage numerical analysis and parametric optimization of the comparison system for the VLF. Thereto, the VLF have the structure derived above for the error coordinates $\boldsymbol{\mathcal{E}}, \tilde{\boldsymbol{\omega}}$ and the structure of other VLF components in the form of sublinear norms for vector variables $\mathbf{q}(t)$, $\dot{\mathbf{q}}(t))$ using the vector $\boldsymbol{\beta}(t)$.

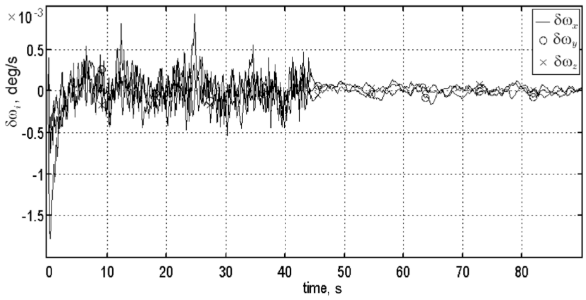

Figure 5. The rate errors for consequence of the SC rotational maneuver and a course motion

9 Computer Simulation

Fig. 5 presents some results on computer simulation of digital gyromomemt control system for Russian remote sensing SC by the *Resource-DK* type. Here the rate errors are represented at consequence of the SC spatial rotational maneuver for time $t \in [0, 45)$ sec and the SC course motion for time $t \in [45, 90]$ sec.

10 Conclusion

Contemporary approaches and some new results were presented on optimization of attitude guidance and nonlinear robust gyromoment control applied for the agile remote sensing spacecraft. These results were also successfully applied for a SRM at transportation of a flexible mechanical payload (Somov, 2006)

Acknowledgements

The work was supported by the RAS' Presidium (Pr. 22), the Division on EMMCP of the RAS (Pr. 15, 17), and the RFBR (Grants 05-08-18175, 07-08-97611, 08-08-99101 and 08-08-00512).

References

Branetz, B.N. and I.P. Shmyglevsky (1973). *Application of Quaternions for Problems of Rigid Body Attitude*. Nauka. Moscow.

Somov, Ye.I. (2006). Analytic synthesis of a programme gyromoment conrol by free-flying space robot. *Conrol Problems* (6), 72–78.

Somov, Ye.I. (2007). Optimization of a rotation maneuver and synthesis of gyromoment guidance laws for spacecraft and free-flying robots. *Izvestiya of Samara Scientific Center RAS* **9**(3), 824–834.

Somov, Ye.I., S.A. Butyrin, A.V. Sorokin and V.N. Platonov (2003). Steering the spacecraft control moment gyroscope clusters. In: *Proceedings of 10th St.-Petersburg Internetional conference on integrated navigation systems*. St.-Petersburg. pp. 403–419.

Somov, Ye.I., S.A. Butyrin, S.Ye Somov and G.P. Anshakov (2007). Guidance and robust gyromoment attitude control of agile flexible remote sensing spacecraft. In: *Preprints of 17th IFAC Symposium on Automatic control in Aerospace, Paper We-04-1/4*. ONERA. Toulouse. pp. 1–6.