Optimal Terminal Control Problem for Discrete-Time Dynamical System

Andrey F. Shorikov

***Abstract*— In this report we consider the dynamical system that consist of one controlled object. The motion of this object is described by linear discrete-time recurrent vector equation. It is assumed that the set constraining control action is known and is convex, closed and bounded polyhedron (with a finite number of vertices) in the corresponding Euclidean vector space. Under these assumptions, we formulate and solve the optimal terminal control problem with a convex functional for such linear discrete-time dynamical system. In order to solve this problem we suggest a recurrent numerical algorithm which reduce the initial multistep problem to solving a sequences of direct and inverse one-step linear and convex programming problems. The results obtained in this report can be used for computer simulation of an actual physical processes and for designing controlling and navigation systems.**

I. Introduction

In this report we consider the dynamical system that consist of one controlled object. The motion of this object is described by linear discrete-time recurrent vector equation. It is assumed that the set constraining control action is known and is convex, closed and bounded polyhedron (with a finite number of vertices) in the corresponding Euclidean vector space. Under these assumptions, we formulate and solve the optimal terminal control problem with a convex functional for such linear discrete-time dynamical system (see [1], [2]). In order to solve of optimal terminal control problem we suggest a recurrent numerical algorithm which reduce the initial multistep problem to solving a sequence of direct and inverse one-step linear and convex programming problems. The results obtained in this report are based on [2]–[4] and can be used for computer simulation of an actual technical and economics processes and for designing of optimal digital controlling and navigation systems for technological and transportation systems. Mathematical models of such systems had considered, for example, in [1]–[5].

II. Formulation of the Problem

On a given integer-valued time interval $\overline{0, \mathrm{T}} = \{0, 1, \cdots, \mathrm{T}\}$ $(\mathrm{T} > 0)$ we consider the controlled dynamical object, whose motion is described by the linear discrete-time recurrent vector equation

$$x(t+1) = A(t)x(t) + B(t)u(t),\ x(0) = x_0. \qquad (1)$$

Here $t \in \overline{0, \mathrm{T}-1}$; $x \in \mathbf{R}^n$ is the phase vector of the system ($n \in \mathbf{R}^n$, where $\mathbf{N}$ is the set of all natural numbers; for $k \in \mathbf{N}$, $\mathbf{R}^k$ is the k-dimensional Euclidean space of column vectors); initial vector $x(0) = x_0$ is fixed; $u(t) \in \mathbf{R}^p$ is the vector of the control action (control), constrained by the given set

$$u(t) \in \mathrm{U}_1 \subset \mathbf{R}^p \ (p \in \mathbf{N} :\ p \leq n); \qquad (2)$$

$A(t)$ and $B(t)$ are real matrices of orders $(n \times n)$ and $(n \times p)$ respectively, and such, that for all $t \in \overline{0, \mathrm{T}-1}$ the matrix $A(t)$ is assumed to be invertible, and therefore for it exist corresponding inverse matrix $A^{-1}(t)$, and rank of the matrix $B(t)$ is equal p (dimension of the vector $u(t)$); the set U_1 is convex, closed, and bounded polyhedron (with a finite number of vertices) in the space $\mathbf{R}^p$.

This control process performance is estimated by the convex functional which determine on the terminal phase states of the object and is continuously differentiable.

For a strict mathematical formulation of optimal terminal control problem for the discrete-time dynamical system (1), (2), we introduce some definitions.

For a fixed number $k \in \mathbf{N}$ and $\overline{\tau, \vartheta} \subseteq \overline{0, \mathrm{T}}$ $(\tau \leq \vartheta)$, we denote by $\mathbf{S}_k(\overline{\tau, \vartheta})$ the metric space of functions $\varphi : \overline{\tau, \vartheta} \longrightarrow \mathbf{R}^k$ of the integer argument, where the metric ρ_k is defined by

$$\rho_k(\varphi_1(\cdot), \varphi_2(\cdot)) = \max_{t \in \overline{\tau, \vartheta}} \parallel \varphi_1(t) - \varphi_2(t) \parallel_k$$

$$((\varphi_1(\cdot), \varphi_2(\cdot)) \in \mathbf{S}_k(\overline{\tau, \vartheta}) \times \mathbf{S}_k(\overline{\tau, \vartheta})),$$

and by $\mathrm{comp}(\mathbf{S}_k(\overline{\tau, \vartheta}))$ we denote the set of all subsets of the space $\mathbf{S}_k(\overline{\tau, \vartheta})$ that are nonempty and compact in the sense of this metric, and $\parallel \cdot \parallel_k$ is the Euclidean norm in $\mathbf{R}^k$.

Using the constraint (2), we define the set $\mathbf{U}(\overline{\tau, \vartheta}) \in \mathrm{comp}(\mathbf{S}_p(\overline{\tau, \vartheta - 1}))$ of all admissible program controls $u(\cdot) = \{u(t)\}_{t \in \overline{\tau, \vartheta-1}}$ on the time interval $\overline{\tau, \vartheta}$ $(\tau < \vartheta)$ by

$$\mathbf{U}(\overline{\tau, \vartheta}) = \{u(\cdot) :\ u(\cdot) \in \mathbf{S}_p(\overline{\tau, \vartheta - 1}),$$

$$\forall\ t \in \overline{\tau, \vartheta - 1},\ u(t) \in \mathrm{U}_1\}.$$

Then for estimating the quality of the control process in discrete-time dynamical system (1), (2) on the time interval $\overline{0, \mathrm{T}}$ we define the functional $\gamma_{\overline{0,\mathrm{T}}} :\ \mathbf{R}^n \times \mathbf{U}(\overline{0, \mathrm{T}}) \longrightarrow \mathbf{R}^1$ in such a way that for any realization of the collection $(x_0, u(\cdot)) \in \mathbf{R}^n \times \mathbf{U}(\overline{0, \mathrm{T}})$, its value is defined by the following terminal functional

$$\gamma_{\overline{0,\mathrm{T}}}(x_0, u(\cdot)) = \mathrm{F}(x(\mathrm{T})), \qquad (3)$$

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A. F. Shorikov is with Department of Differential Equations, Institute of Mathematics and Mechanics of UB RAS, Urals State University of Economics, 8 Marta Str., 62, 620219 Ekaterinburg, Russia; shorikov@usue.ru

where the vector $x(\mathrm{T}) = \tilde{x}(\mathrm{T}; \overline{0,\mathrm{T}}, x_0, u(\cdot))$, and $\mathrm{F}(x)$ is the convex and continuously differentiable functional for all vectors $x \in \mathbf{R}^n$. Here and below, by $\tilde{x}(\cdot; \overline{0,\mathrm{T}}, x_0, u(\cdot)) \in \mathbf{S}_n(\overline{0,\mathrm{T}})$ we denote the trajectory of the object from the initial phase state $x_0 \in \mathbf{R}^n$ on the time interval $\overline{0,\mathrm{T}}$, generated by the program control $u(\cdot) \in \mathbf{U}(\overline{0,\mathrm{T}})$, namely

$$\tilde{x}(\cdot; \overline{0,\mathrm{T}}, x_0, u(\cdot)) = \{x(\cdot) : \ x(\cdot) \in \mathbf{S}_n(\overline{0,\mathrm{T}}), \ \forall\, t \in \overline{0,\mathrm{T}-1},$$

$$x(t+1) = A(t)x(t) + B(t)u(t), \ x(0) = x_0\},$$

and will be denote by $\tilde{x}(t; \overline{0,\mathrm{T}}, x_0, u(\cdot)) \in \mathbf{R}^n$ the section of this trajectory at the time moment $t \in \overline{0,\mathrm{T}}$.

Now we formulate the following multistep problem of optimal terminal control for discrete-time dynamical system (1)–(3).

Problem 1. For the initial phase state x_0 of the controlled discrete-time dynamical system (1)–(3), defined on the time interval $\overline{0,\mathrm{T}}$, it is required to find the set $\mathbf{U}^{(e)}_{\gamma_{\overline{0,\mathrm{T}}}}(x_0)$ of all optimal controls $u^{(e)}(\cdot) \in \mathbf{U}(\overline{0,\mathrm{T}})$, such that the terminal vectors $x^{(e)}(\mathrm{T}) = \tilde{x}(\mathrm{T}; \overline{0,\mathrm{T}}, x_0, u^{(e)}(\cdot))$ should supply a minimum by the convex terminal functional $\gamma_{\overline{0,\mathrm{T}}}$, i.e. to calculate the following set,

$$\mathbf{U}^{(e)}_{\gamma_{\overline{0,\mathrm{T}}}}(x_0) = \{u^{(e)}(\cdot) : \ u^{(e)}(\cdot) \in \mathbf{U}(\overline{0,\mathrm{T}}),$$

$$\min_{u(\cdot) \in \mathbf{U}(\overline{0,\mathrm{T}})} \gamma_{\overline{0,\mathrm{T}}}(x_0, u(\cdot)) = \gamma_{\overline{0,\mathrm{T}}}(x_0, u^{(e)}(\cdot)) =$$

$$= \mathrm{F}(x^{(e)}(\mathrm{T})) = \mathrm{F}(\tilde{x}(\mathrm{T}; \overline{0,\mathrm{T}}, x_0, u^{(e)}(\cdot))) =$$

$$= \min_{u(\cdot) \in \mathbf{U}(\overline{0,\mathrm{T}})} \mathrm{F}(\tilde{x}(\mathrm{T}; \overline{0,\mathrm{T}}, x_0, u(\cdot))) = \mathrm{F}^{(e)}(x_0)\}, \qquad (4)$$

and to calculate the number $\mathrm{F}^{(e)}(x_0)$, which is the optimal value of the result for this problem, as realization of a finite sequence of one-step operations only.

We mention that the solution of the formulated problem exists, since it reduces to solving the problem of minimization of the convex functional F defined by the relation (3) on a closed and boundary set from $\mathbf{R}^n$, and to realization of a finite sequence of one-step operations only.

III. General scheme of the solution the Problem 1

In this part of the report, we will develop analysis of the multistep Problem 1 of optimal terminal control for the discrete-time dynamical system (1)–(3) and description of a general scheme of its solving.

Then for a fixed time interval $\overline{\tau,\vartheta} \subseteq \overline{0,\mathrm{T}}$ $(\tau < \vartheta)$ and any fixed collections $(\tau, \ X(\tau)) \in \{\tau\} \times \mathbf{2}^{\mathbf{R}^n}$ and $(\vartheta, \ X(\vartheta)) \in \{\vartheta\} \times \mathbf{2}^{\mathbf{R}^n}$ (where here and below, for any set X we denote by a symbol $\mathbf{2}^X$ the set of all subset of the set X) we define by virtue (1), (2) the following sets

$$\mathbf{X}^{(+)}(\tau, X(\tau), \vartheta) = \{x(\vartheta) : \ x(\vartheta) \in \mathbf{R}^n, \ \forall\ t \in \overline{\tau,\vartheta-1},$$

$$x(t+1) = A(t)x(t) + B(t)u(t),$$

$$x(\tau) \in X(\tau), \ u(t) \in \mathrm{U}_1\}; \qquad (5)$$

$$\mathbf{X}^{(-)}(\vartheta, X(\vartheta), \tau) = \{x(\tau) : \ x(\tau) \in \mathbf{R}^n,$$

$$\forall\ t \in \{\vartheta, \vartheta-1, \cdots, \tau+2, \tau+1\},$$

$$x(t-1) = A^{-1}(t-1)[x(t) - B(t-1)u(t-1)],$$

$$x(\vartheta) \in X(\vartheta), \ u(t-1) \in \mathrm{U}_1\}, \qquad (6)$$

where the set $\mathbf{X}^{(+)}(\tau, X(\tau), \vartheta)$ is the direct attainability domain [2] of the possible states of the phase vector the object at the time moment ϑ, which correspond the collection $(\tau, X(\tau))$, and the set $\mathbf{X}^{(-)}(\vartheta, X(\vartheta), \tau)$ is the inverse attainability domain [2] of the possible states of the phase vector the object at the time moment τ, which correspond the collection $(\vartheta, X(\vartheta))$.

Let's the convex functional $\gamma_{\overline{0,\mathrm{T}}}$, determined for the Problem 1 by the relation (3) such that its values we can calculate using the functional F, and it estimate the quality of the control process in dynamical system (1), (2) on the time interval $\overline{0,\mathrm{T}}$.

And let's the set $\mathbf{X}^{(+)}(0, \{x_0\}, \mathrm{T}) \in \mathbf{2}^{\mathbf{R}^n}$ is the direct attainability domain of the object at the time moment T, which correspond the collection $(0, x_0)$.

On the base of the above-mentioned constructions one can proof that the solution of the Problem 1 can be represented in the form of the solutions the following three problems:

1) the construction of the direct attainability domain $\mathbf{X}^{(+)}(0, \{x_0\}, \ \mathrm{T})$ (it's solved with using of a direct construction which is realizing as a finite sequence of one-step operations only);

2) the minimization of the convex functional $\mathrm{F} : \mathbf{R}^n \longrightarrow \mathbf{R}^1$ determined by the relation (3) on the set $\mathbf{X}^{(+)}(0, \{x_0\}, \mathrm{T})$, i.e. the construction of the set $X^{(e)}_{\mathrm{F}}(\mathrm{T})$ of all terminal phase vectors $x^{(e)}(\mathrm{T})$ of the object and which is determine by formula

$$X^{(e)}_{\mathrm{F}}(\mathrm{T}) = \{x^{(e)}(\mathrm{T}) : \ x^{(e)}(\mathrm{T}) \in \mathbf{X}^{(+)}(0, \{x_0\}, \mathrm{T}),$$

$$\mathrm{F}(x^{(e)}(\mathrm{T})) = \min_{x(\mathrm{T}) \in \mathbf{X}^{(+)}(0,\{x_0\},\mathrm{T})} \mathrm{F}(x(\mathrm{T})) =$$

$$= \mathrm{F}^{(e)}(x_0)\} \qquad (7)$$

(it reduce to solving of the convex programming problem which is realizing as a finite sequence of one-step operations only, this problem is solved by using, for example, the algorithm of the Zoutendijk method, the case of linear constraints in the form of inequalities, see, for example, [7]);

3) the solving on time interval $\overline{0,\mathrm{T}}$ the two-point boundary value problem for the discrete-time dynamical system (1), (2) under the boundary conditions $x(0) = x_0$ and $x(\mathrm{T}) =$

$x^{(e)}(\mathrm{T}) \in X^{(e)}_{\mathrm{F}}(\mathrm{T})$, namely, the construction the following set

$$\mathbf{U}^{(e)}(\overline{0,\mathrm{T}}, \{x_0\}, X^{(e)}_{\mathrm{F}}(\mathrm{T})) = \{u^{(e)}(\cdot):\ u^{(e)}(\cdot) \in \mathbf{U}(\overline{0,\mathrm{T}}),$$
$$x^{(e)}(\mathrm{T}) = \tilde{x}(\mathrm{T}; \overline{0,\mathrm{T}}, x_0, u^{(e)}(\cdot)) \in X^{(e)}_{\mathrm{F}}(\mathrm{T})\} =$$
$$= \mathbf{U}^{(e)}_{\gamma_{\overline{0,\mathrm{T}}}}(x_0) \tag{8}$$

of all optimal controls on its time interval (it's solved with the combine of the direct and inverse recurrent procedures which is realizing as a finite sequence of one-step operations only).

Below we will make analysis of the general elements and possible a directions of solves these subproblems.

We fix the time interval $\overline{\tau,\vartheta} \subseteq \overline{0,\mathrm{T}}$ $(\tau < \vartheta)$ and collection $(t-1, X(t-1)) \in \overline{\tau,\vartheta-1} \times \mathbf{2}^{\mathbf{R}^n}$, where $X(t-1)$ is a convex, closed, and bounded polyhedron (with a finite number of vertices) in the space $\mathbf{R}^n$. Let the set $X^{(+)}(t) = \mathbf{X}^{(+)}(t-1, X(t-1), t)$ correspond the definition (5), i.e. is the direct attainability domain of the object corresponding the collection $(t-1, X(t-1))$ at time moment t. Then, it follows from the definition (5) and the above assumptions that for all $t \in \overline{\tau+1,\vartheta}$ the set $X^{(+)}(t)$ is a convex, closed, and bounded polyhedron (with a finite number of vertices) in the space $\mathbf{R}^n$. It is also known (see, for example, [6]), that any point x of the convex, closed, and bounded polyhedron $X^{(+)}(t)$ is represented as

$$x = \sum_{i=1}^{n_t} \alpha_i x^{(i)}_*,\ \sum_{i=1}^{n_t} \alpha_i = 1,\ \alpha_i \geq 0,\ i \in \overline{1,n_t},$$

where $\{x^{(i)}_*\}_{i\in\overline{1,n_t}}$ is the set of all vertices of the polyhedron $X^{(+)}(t)$ and n_t is the number of its vertices. Then, if we know the vertices of the set $X^{(+)}(t)$, this set is constructed.

Now we assume that the set $X(t-1)$ is already constructed $(t \in \overline{\tau+1,\vartheta})$. Then one can easily prove the following auxiliary assertion.

Lemma 1: Let on the fix time interval $\overline{\tau,\vartheta} \subseteq \overline{0,\mathrm{T}}$ $(\tau < \vartheta)$ for $t \in \overline{\tau+1,\vartheta}$ the set $X(t-1)$ is a convex, closed, and bounded polyhedron (with a finite number of vertices) in the space $\mathbf{R}^n$ and on the base of the equation (1) is constructing the following sets:

$$\bar{X}^{(+)}_n(t) = \{\bar{x}(t):\ \bar{x}(t) \in \mathbf{R}^n,$$
$$\bar{x}(t) = A(t-1)x(t-1),\ x(t-1) \in X(t-1)\};$$
$$\hat{X}^{(+)}_n(t) = \{\hat{x}(t):\ \hat{x}(t) \in \mathbf{R}^n,$$
$$\hat{x}(t) = A(t-1)x(t-1),\ x(t-1) \in \mathbf{\Gamma}_n(X(t-1))\}.$$

Then $\mathbf{\Gamma}_n(\bar{X}^{(+)}_n(t)) = \mathbf{\Gamma}_n(\mathrm{co}_n\hat{X}^{(+)}_n(t))$.

Here and below, $\mathbf{\Gamma}_m(M)$ is the set of all vertices of the polyhedron $M \subset \mathbf{R}^m,\ m \in \mathbf{N}$, and $\mathrm{co}_m M$ is the convex hull of this set.

We formulate a similar assertion for the controlling part of the equation (1).

Lemma 2: Let on the fix time interval $\overline{\tau,\vartheta} \subseteq \overline{0,\mathrm{T}}$ $(\tau < \vartheta)$ for $t \in \overline{\tau+1,\vartheta}$ on the base (1), (2) are constructing the following sets:

$$\bar{Y}^{(+)}_n(t) = \{\bar{y}(t):\ \bar{y}(t) \in \mathbf{R}^n,$$
$$\bar{y}(t) = B(t-1)u(t-1),\ u(t-1) \in \mathrm{U}_1\};$$
$$\hat{Y}^{(+)}_n(t) = \{\hat{y}(t):\ \hat{y}(t) \in \mathbf{R}^n,$$
$$\hat{y}(t) = B(t-1)u(t-1),\ u(t-1) \in \mathbf{\Gamma}_p(\mathrm{U}_1)\}.$$

Then $\mathbf{\Gamma}_n(\bar{Y}^{(+)}_n(t)) = \mathbf{\Gamma}_n(\mathrm{co}_n\hat{Y}^{(+)}_n(t))$.

On the base of Lemmas 1 and 2 one can prove, that the following assertion is valid (see [4]).

Theorem 1: Let on the fix time interval $\overline{\tau,\vartheta} \subseteq \overline{0,\mathrm{T}}$ $(\tau < \vartheta)$ for $t \in \overline{\tau+1,\vartheta}$ the set $X(t-1)$ is a convex, closed, and bounded polyhedron (with a finite number of vertices) in the space $\mathbf{R}^n$ and the set $\tilde{X}^{(+)}(t)$ is constructing in the form

$$\tilde{X}^{(+)}(t) = \{\tilde{x}(t):\ \tilde{x}(t) \in \mathbf{R}^n,\ \tilde{x}(t) = \hat{x}(t) + \hat{y}(t),$$
$$\hat{x}(t) \in \hat{X}^{(+)}_n(t),\ \hat{y}(t) \in \hat{Y}^{(+)}_n(t)\}.$$

Then it's valid the following equality

$$X^{(+)}(t) = \mathbf{X}^{(+)}(t-1, X(t-1), t) = \mathrm{co}_n\tilde{X}^{(+)}(t).$$

By analogy, one can prove, that the following assertion is valid.

Theorem 2: Let on the fix time interval $\overline{\tau,\vartheta} \subseteq \overline{0,\mathrm{T}}$ $(\tau < \vartheta)$ for $t \in \{\vartheta, \vartheta-1, \cdots, \tau+2, \tau+1\}$ the set $X(t)$ is a convex, closed, and bounded polyhedron (with a finite number of vertices) in the space $\mathbf{R}^n$ and the set $\tilde{X}^{(-)}(t-1)$ is constructing in the form

$$\tilde{X}^{(-)}(t-1) = \{\tilde{x}(t-1):\ \tilde{x}(t-1) \in \mathbf{R}^n,$$
$$\tilde{x}(t-1) = \hat{x}(t-1) + \hat{y}(t-1),$$
$$\hat{x}(t-1) \in \hat{X}^{(-)}_n(t-1),\ \hat{y}(t-1) \in \hat{Y}^{(-)}_n(t-1)\},$$

where the sets $\hat{X}^{(-)}_n(t-1)$ and $\hat{Y}^{(-)}_n(t-1)$ are constructing by the following formulas

$$\hat{X}^{(-)}_n(t-1) = \{\hat{x}(t-1):\ \hat{x}(t-1) \in \mathbf{R}^n,$$
$$\hat{x}(t-1) = A^{-1}(t-1)x(t),\ x(t) \in \mathbf{\Gamma}_n(X(t))\},$$
$$\hat{Y}^{(-)}_n(t-1) = \{\hat{y}(t-1):\ \hat{y}(t-1) \in \mathbf{R}^n,$$
$$\hat{y}(t-1) = -A^{-1}(t-1)\cdot B(t-1)u(t-1),\ u(t-1) \in \mathbf{\Gamma}_p(\mathrm{U}_1)\}.$$

Then it's valid the following equality

$$X^{(-)}(t-1) = \mathbf{X}^{(-)}(t, X(t), t-1) = \mathrm{co}_n\tilde{X}^{(-)}(t-1).$$

Note that on the base of the Theorems 1 and 2, and granting that the recurrent system (1), (2), which is described by the motion of the object, is linear, for the direct attainability domain $\mathbf{X}^{(+)}(\tau, X(\tau), \vartheta) \in \mathbf{2}^{\mathbf{R}^n}$ of the object and its inverse attainability domain $\mathbf{X}^{(-)}(\vartheta, X(\vartheta), \tau) \in \mathbf{2}^{\mathbf{R}^n}$, which are defined by the relations (5) and (6) respectively, the following assertion is valid (see [4]).

Lemma 3: Let on the fix time interval $\overline{\tau, \vartheta} \subseteq \overline{0, \mathrm{T}}$ ($\tau < \vartheta$) for collections $(\tau,\ X(\tau)) \in \{\tau\} \times \mathbf{2}^{\mathbf{R}^n}$ and $(\vartheta,\ X(\vartheta)) \in \{\vartheta\} \times \mathbf{2}^{\mathbf{R}^n}$, where the sets $X(\tau)$ and $X(\vartheta)$ are a convex, closed, and bounded polyhedrons (with a finite number of vertices) in the space $\mathbf{R}^n$, the attainability domains $\mathbf{X}^{(+)}(\tau, X(\tau), \vartheta) \in \mathbf{2}^{\mathbf{R}^n}$ and $\mathbf{X}^{(-)}(\vartheta, X(\vartheta), \tau) \in \mathbf{2}^{\mathbf{R}^n}$ of the object are defined by the relations (5) and (6) respectively. Then from the above assumptions for elements of the discrete-time dynamical system (1), (2) one can easily prove the following properties for these domains:

1) for all $t \in \overline{\tau + 1, \vartheta}$ the set $\mathbf{X}^{(+)}(\tau, X(\tau), t) = X^{(+)}(t)$ is a convex, closed, and bounded polyhedron (with a finite number of vertices) in the space $\mathbf{R}^n$;

2) for all $t \in \overline{\tau, \vartheta - 1}$ and $X^{(+)}(\tau) = X(\tau)$ it's true the following recurrent relation

$$\mathbf{X}^{(+)}(\tau, X(\tau), t + 1) = \mathbf{X}^{(+)}(t, X^{(+)}(t), t + 1); \quad (9)$$

3) for all $t \in \overline{\tau, \vartheta - 1}$ the set $\mathbf{X}^{(-)}(\vartheta, X(\vartheta), t) = X^{(-)}(t)$ is a convex, closed, and bounded polyhedron (with a finite number of vertices) in the space $\mathbf{R}^n$;

4) for all $t \in \overline{\tau + 1, \vartheta}$ and $X^{(-)}(\vartheta) = X(\vartheta)$ it's true the following recurrent relation

$$\mathbf{X}^{(-)}(\vartheta, X(\vartheta), t - 1) = \mathbf{X}^{(-)}(t, X^{(-)}(t), t - 1). \quad (10)$$

Then from the relations (9) and (10) of this assertion it follows that multistep problems of the construction the direct attainability domain $\mathbf{X}^{(+)}(\tau, X(\tau), \vartheta) \in \mathbf{2}^{\mathbf{R}^n}$ of the object and it inverse attainability domain $\mathbf{X}^{(-)}(\vartheta, X(\vartheta), \tau) \in \mathbf{2}^{\mathbf{R}^n}$ one can realize as a finite recurrent sequences only one-step problems constructing of the corresponding following attainability domains:

$$X^{(+)}(t + 1) = \mathbf{X}^{(+)}(t, X^{(+)}(t), t + 1),$$
$$t \in \overline{\tau, \vartheta - 1},\ X^{(+)}(\tau) = X(\tau); \quad (11)$$
$$X^{(-)}(t - 1) = \mathbf{X}^{(-)}(t, X^{(-)}(t), t - 1),$$
$$t \in \{\vartheta, \vartheta - 1, \cdots, \tau + 2, \tau + 1\},\ X^{(-)}(\vartheta) = X(\vartheta). \quad (12)$$

Now on the base of the definitions (5) and (6) for all $t \in \overline{\tau + 1, \vartheta - 1}$ we may construct the following sets:

$$\mathbf{X}_t^{(e)}(\overline{\tau, \vartheta}, X(\tau), X(\vartheta)) =$$
$$= \mathbf{X}^{(+)}(\tau, X(\tau), t) \bigcap \mathbf{X}^{(-)}(\vartheta, X(\vartheta), t), \quad (13)$$

where the sets $X(\tau)$ and $X(\vartheta)$ are a convex, closed, and bounded polyhedrons (with a finite number of vertices) in the space $\mathbf{R}^n$.

Then on the base of the Theorems 1 and 2, and Lemma's 3, one can prove that the following auxiliary assertion is valid.

Lemma 4: Let on the fix time interval $\overline{\tau, \vartheta} \subseteq \overline{0, \mathrm{T}}$ ($\tau < \vartheta$) the sets $X(\tau)$ and $X(\vartheta)$ are a convex, closed, and bounded polyhedrons (with a finite number of vertices) in the space $\mathbf{R}^n$, then the set $\mathbf{X}_t^{(e)}(\overline{\tau, \vartheta}, X(\tau), X(\vartheta))$ defined for all $t \in \overline{\tau + 1, \vartheta - 1}$ by the formula (13) is a convex, closed, and bounded polyhedron (with a finite number of vertices) in the space $\mathbf{R}^n$ and can be constructing as the realization of a finite sequence of one-step operations only.

From the above definitions and constructions (5)–(13), and assertions it follows that is true the following assertion — necessary and sufficient condition for solving of the Problem 1.

Theorem 3: For a fixed time interval $\overline{0, \mathrm{T}}$ ($0 < \mathrm{T}$), and initial phase state x_0 of the controlled discrete-time dynamical system (1)–(3), and the set $X_{\mathrm{F}}^{(e)}(\mathrm{T}) \in \mathbf{2}^{\mathbf{R}^n}$, which is defined by the relation (8) and is a convex, closed, and bounded polyhedron (with a finite number of vertices) in the space $\mathbf{R}^n$, and therefore it's the set of the optimal terminal states of the object, let the set $\tilde{\mathbf{U}}^{(e)}(\overline{0, \mathrm{T}}, \{x_0\}, X_{\mathrm{F}}^{(e)}(\mathrm{T})) \subseteq \mathbf{U}(\overline{0, \mathrm{T}})$ determined on the base (13), and constructed by the following recurrent relation

$$\tilde{\mathbf{U}}^{(e)}(\overline{0, \mathrm{T}}, \{x_0\}, X_{\mathrm{F}}^{(e)}(\mathrm{T})) = \{u^{(e)}(\cdot) :$$
$$u^{(e)}(\cdot) \in \mathbf{U}(\overline{0, \mathrm{T}}),\ \forall\ t \in \overline{0, \mathrm{T} - 1},$$
$$x^{(e)}(t + 1) = A(t)x^{(e)}(t) + B(t)u^{(e)}(t) \in$$
$$\in \mathbf{X}_{t+1}^{(e)}(\overline{0, \mathrm{T}}, \{x_0\}, X_{\mathrm{F}}^{(e)}(\mathrm{T})),$$
$$x^{(e)}(0) = x_0,\ x^{(e)}(\mathrm{T}) \in X_{\mathrm{F}}^{(e)}(\mathrm{T})\}.$$

Then the control $u^{(e)}(\cdot) \in \tilde{\mathbf{U}}^{(e)}(\overline{0, \mathrm{T}}, \{x_0\}, X_{\mathrm{T}}^{(e)}(\mathrm{T}))$ is the solution of the two-point boundary value problem for the recurrent equation (1) on the time interval $\overline{0, \mathrm{T}}$ under the boundary conditions $x(0) = x_0$ and $x(\mathrm{T}) \in X_{\mathrm{F}}^{(e)}(\mathrm{T})$, therefore it's optimal control for the Problem 1, and i.e. it exist and satisfy the following inclusion

$$u^{(e)}(\cdot) \in \mathbf{U}^{(e)}(\overline{0, \mathrm{T}}, \{x_0\}, X_{\mathrm{F}}^{(e)}(\mathrm{T})) \neq \emptyset,$$

then and only then, when it satisfy the inclusion

$$u^{(e)}(\cdot) \in \tilde{\mathbf{U}}^{(e)}(\overline{0, \mathrm{T}}, \{x_0\}, X_{\mathrm{F}}^{(e)}(\mathrm{T})) \neq \emptyset,$$

and therefore the following equality is true

$$\mathbf{U}^{(e)}_{\gamma_{\overline{0,\mathrm{T}}}}(x_0) = \mathbf{U}^{(e)}(\overline{0, \mathrm{T}}, \{x_0\}, X_{\mathrm{F}}^{(e)}(\mathrm{T})) =$$
$$= \tilde{\mathbf{U}}^{(e)}(\overline{0, \mathrm{T}}, \{x_0\}, X_{\mathrm{F}}^{(e)}(\mathrm{T})),$$

and this set and the number $\mathrm{F}^{(e)}(x_0)$, which is the optimal value of the result for the Problem 1, and is defined by the relation (4), are calculate as realizations of a finite sequences of one-step operations only.

Note, that the solving of the Problem 1, i.e. the solving of the problem of the constructing the set $\tilde{\mathbf{U}}^{(e)}(\overline{0,\mathrm{T}},\{x_0\},\underline{X_{\mathrm{F}}^{(e)}}(\mathrm{T}))$ of optimal program controls on the time interval $\overline{0,\mathrm{T}}$, is formed as constructive procedure — as realization of the finite recurrent sequence of the one-step operations only.

On the above constructions and assertions we can make the following conclusion, that the solving of the Problem 1 — optimal terminal control for discrete-time dynamical system (1)–(3), we can reduce to realizing of solving of the subproblems 1)–3). Note, that on the base of the Theorem 3 the realization of the solving of these subproblems is a finite recurrent sequence of the one-step operations only, and linear and convex programming problems.

IV. Conclusion

In this report for solving the Problem 1 of optimal terminal control for discrete-time dynamical system (1)–(3), we propose a recurrent algorithm, which reduces the multistep Problem 1 to realization of the sequence of one-step optimization problems of linear and convex programming, i.e. to realization of the finite recurrent sequence of one-step operations only. We note that for this algorithm the dimension of the discrete-time dynamical system (1)–(3) and the number of the control process steps are limited only by the memory and speed of the computer. The results obtained in this report are based on [2]–[4] and can be used for computer simulation of an actual economics processes and for designing of optimal digital controlling and navigation systems for technological and transportation systems.

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