Confidence Regions for Systems with Random and Uncertain Perturbations

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***Abstract*— The state estimation problem for statistically uncertain systems with observation is investigated. A system is called statistically uncertain one if it contains random perturbations with incompletely known distributions, or it contains both random and nonrandom uncertain perturbations. Confidence estimates for the system states are studied. It is shown that linear estimates are not optimal even for linear systems depended on Gaussian random perturbations with uncertain mean values. The nonlinear confidence estimates for the system state are constructed using a notion of a random information set. The properties of the estimates are studied.**

I. Problem Statement

The state estimation problems for statistically uncertain systems were studied by many authors. Kats and Kurzhanski [1] formulated the state estimation problem for multistage statistically uncertain systems and obtained the recurrent equations for the sets of the possible mean values. Verdu and Poor [2] suggested the minimax approach to the statistically uncertain estimation problem. Milanese, Vicino [3], Matasov [4] investigated different approaches to the estimation problem in condition of incomplete information.

In the paper [5] a new method for confidence estimation in statistically uncertain system was suggested.

Let consider an ordinary statistically uncertain problem

$$\begin{aligned} x &= \quad\ x_0 + Q_1\xi_1, \\ y &= Gx + v + Q_2\xi_2. \end{aligned} \tag{1}$$

Here x is unknown n-vector, $y \in \mathbf{R}^m$ is known observation. Perturbations ξ_1, ξ_2 are independent random vectors with normal distributions and

$$E\xi_1 = 0, \quad E\xi_2 = 0, \quad E\xi_1\xi_1^T = I_{(n)}, \quad E\xi_2\xi_2^T = I_{(m)}, \tag{2}$$

where $I_{(n)}$ is identity $n \times n$ matrix, Q_1, Q_2 are nonsingular $n \times n$ and $m \times m$ matrixes.

The vectors $x_0 \in \mathbf{R}^n$, $v \in \mathbf{R}^m$ are nonrandom and uncertain. It is supposed that information about them is given by the membership:

$$x_0 \in X_0, \quad v \in V, \tag{3}$$

where $X_0 \subset \mathbf{R}^n$, $V \subset \mathbf{R}^m$ are given convex compact sets.

As a rule the linear estimates are used for estimation and for confidence estimation in problem (1)–(3).

Statement 1: Let Λ is an arbitrary matrix $n \times m$ and B_α is a confidence set for the random vector

$$e = e(\xi_1, \xi_2) = (I - \Lambda G)Q_1\xi_1 - \Lambda Q_2\xi_2.$$

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Then

$$\hat{X}_\alpha = \hat{X}(\Lambda) + B_\alpha$$

is a confidence set for vector x of probability not less then α, i.e. $P\{x \in \hat{X}_\alpha\} \geq \alpha$. Here

$$\hat{X} = \hat{X}(\Lambda) = (I - \Lambda G)X_0 + \Lambda(y - V), \tag{4}$$

Λ is an arbitrary $n \times m$ matrix.

Statement 1 follows immediately from (1)–(3).

Example 1: Let us consider a simple example of the linear estimation in $\mathbf{R}^1$:

$$\begin{aligned} x &= x_0 + \sigma\xi_1, \\ y &= x + v + \sigma\xi_2. \end{aligned} \tag{5}$$

Here ξ_1, ξ_2 are independent Gaussian random values and

$$E\xi_1 = E\xi_2 = 0, \quad E\xi_1^2 = E\xi_2^2 = 1. \tag{6}$$

It is known that

$$x_0 \in X_0 = [-\Delta; \Delta], \quad v \in V = [-\Delta; \Delta]. \tag{7}$$

Let $\Delta = 1$, $\sigma = 0.1$, and the observation $y = 2$.

The linear confidence estimate has a form

$$\hat{X}_\alpha = \hat{X} + B_\alpha, \quad B_\alpha = [-t_\alpha\sigma/\sqrt{2};\ t_\alpha\sigma/\sqrt{2}],$$
$$\hat{X} = 0.5X_0 + 0.5(y - V) = [0; 2],$$

where t_α is the normal distribution two-sided quantile:

$$P\{|\xi| < t_\alpha\} = \alpha.$$

For $\alpha = 0.9$ we have $t_\alpha = 1.65$ and $\hat{X}_\alpha = [-0.165; 2.165]$.

If $\sigma = 0.01$ then $\hat{X}_\alpha = [-0.0165; 2.0165] \supset \hat{X} = [0; 2]$.

But if $\sigma = 0$, i.e. there are not any random perturbation in the system, then the estimation of unknown vector x is an information set:

$$X^{det} = X_0 \cap (y - V) = \{1\}.$$

And it is obvious that linear confidence estimates $\hat{X}_\alpha$ do not approach to the information set if σ tends to 0.

Let us obtain confidence estimate for x using intersection of the confidence regions.

For $\beta = \sqrt{\alpha}$ the confidence sets for ξ_1 and ξ_2 are $[-t_\beta; t_\beta]$. Then we have an estimate in case $\alpha = 0.9$, $\sigma = 0.1$:

$$\breve{X}_\alpha = (X_0 + [-0.195; 0.195]) \cap (y - V - [-0.195; 0.195]) =$$
$$= [0.805; 1.195].$$

But $y = 2.4$, we obtain an empty confidence estimate: $\breve{X}_\alpha = [-1.195; 1.195] \cap [1.205; 3.595] = \emptyset$. It means that the approach should be modified.

II. Random Information Sets and Permissible Sets for Random Perturbation

Along system (1)–(3) let us consider a system with different nonrandom perturbations

$$\begin{aligned} x = & \quad x_0 + Q_1 d_1, \\ y = & \quad Gx + v + Q_2 d_2. \end{aligned} \tag{8}$$

Here x_0, v and d_1, d_2 are given by (3) and by the relation

$$d = \{d_1, d_2\} \in D, \tag{9}$$

D is a given set in $\mathbf{R}^{n+m}$.

Definition 1. [6] The set $\tilde{X}(D) = \tilde{X}(y, X_0, V, D) \subset \mathbf{R}^n$ is called *the information set* for system (8)–(9), (3) if for every $x \in \tilde{X}(D)$ there are exist vectors $x_0 \in X_0, v \in V$, $d = \{d_1, d_2\} \in D$ such that equations (8) hold for the given observation y.

Definition 2. [7] The set $\tilde{X}(\xi^*) = \tilde{X}(y, X_0, V, \xi^*)$ of all states of system (1)–(3) consistent with the observation y for a given value ξ^* of the random perturbation $\xi = \{\xi_1, \xi_2\}$ is called *a random information set*.

It is obvious that

$$\tilde{X}(\xi^*) = (X_0 + Q_1\xi_1^*) \cap G^+(y - V - Q_2\xi_2^*),$$

here and further G^+ is the inverse operator

$$G^+ z = \{u \in \mathbf{R}^n : z = Gu\}.$$

From the Definitions it follows that the random information set is the information set for system (8)–(9), (3) with $D = \{\xi^*\}$.

Definition 3. A set $D^0 = D^0(y, X_0, V) \subset \mathbf{R}^{n+m}$ of values of the random vectors ξ consistent with an observation y in system (1)–(3) is called *a permissible set* for the random parameters ξ corresponding to the given observation:

$$\begin{aligned} & D^0(y, X_0, V) = \\ & = \{d = \{d_1, d_2\} \in \mathbf{R}^{n+m} : \tilde{X}(y, X_0, V, \{d\}) \neq \emptyset\}. \end{aligned} \tag{10}$$

Lemma 1: If y is an observation for system (1)–(3), then the permissible set for the random parameters is nonempty, i.e. $D^0(y, X_0, V) \neq \emptyset$.

Lemma 1 follows from the Definition of the $D^0(y, X_0, V)$.

Let us take a measurable set $X \subset \mathbf{R}^n$ and consider the following random events:

$$A^-(X) = \{\tilde{X}(y, X_0, V, \xi) \subset X\},$$

$$A^+(X) = \{\tilde{X}(y, X_0, V, \xi) \cap X \neq \emptyset\}.$$

Definition 4. Let X be a measurable set in $\mathbf{R}^n$. A set

$$D^-(X) = D^-(X; y, X_0, V) \subset \mathbf{R}^{n+m}$$

of all values of the random parameter ξ consistent with the observation y for system (1)–(3) and such that the vector x belongs to X for any possible $x_0 \in X_0$, $v \in V$, is called *the minimal permissible set* for the random parameters ξ corresponding to the set X:

$$D^-(X) = \{d \in \mathbf{R}^{n+m} : \emptyset \neq \tilde{X}(y, X_0, V, \{d\}) \subset X\}. \tag{11}$$

Definition 5. Let X be a measurable set in $\mathbf{R}^n$. A set

$$D^+(X) = D^+(X; y, X_0, V) \subset \mathbf{R}^{n+m}$$

of all values of random perturbations ξ consistent with a given observation y for system (1)–(3) with some $x_0 \in X_0$, $v \in V$ and $x \in X$ is called *the maximal set* for the random parameters ξ corresponding to the set X:

$$D^+(X) = \{d \in \mathbf{R}^{m+n} : \tilde{X}(y, X_0, V, \{d\}) \cap X \neq \emptyset\}. \tag{12}$$

Lemma 2: For any measurable set $X \in \mathbf{R}^n$ and any observation y the following relations hold

$$\begin{aligned} & D^-(X; y, X_0, V) \subset D^+(X; y, X_0, V) \subset \\ & \subset D^0(y, X_0, V). \end{aligned}$$

Lemma 3: Let $\breve{X}$ be the complement of the set X to $\mathbf{R}^n$, i.e. $\breve{X} = \mathbf{R}^n \backslash X$. Then

$$D^-(\breve{X}; y, X_0, V) = D^0(y, X_0, V) \backslash D^+(X; y, X_0, V),$$

$$D^+(\breve{X}; y, X_0, V) = D^0(y, X_0, V) \backslash D^-(X; y, X_0, V).$$

The lemmas follow from Definitions 3–5.

III. Nonlinear Confidence Regions

Definition 6. The measurable set X_α is called *a confidence region* of a level α for system (1)–(3), if the conditional probability

$$\begin{aligned} & \mathcal{P}\{x \in X_\alpha \mid y, x_0 \in X_0, v \in V\} \triangleq \\ & = P\{\tilde{X}(y, X_0, V, \xi) \subset X_\alpha \mid \tilde{X}(y, X_0, V, \xi) \neq \emptyset\} = \alpha. \end{aligned}$$

It should be noted that the conditional probability cannot be substituted by unconditional one in the considered problem.

The Definition of the confidence region results in

$$\begin{aligned} & \mathcal{P}\{x \in \tilde{X}_\alpha \mid y, x_0 \in X_0, v \in V\} = \\ & = \frac{P\{\xi \in D^-(X_\alpha; y, X_0, V)\}}{P\{\xi \in D^0(y, X_0, V)\}}. \end{aligned} \tag{13}$$

Lemma 4: Let $\tilde{X} = \tilde{X}(y, X_0, V, D)$ be the information set for system (8)–(9). Then

$$D \cap D^0(y, X_0, V) \subseteq D^-(\tilde{X}; y, X_0, V). \tag{14}$$

Proof: From the Definition of the information set it follows that

$$\tilde{X}(y, X_0, V, \{d\}) \subset \tilde{X}$$

for any $d \in D$. From the Definition of the minimal permissible set $D^-(\tilde{X}, y, X_0, V)$ we get (14). ■

Theorem 1. Let a set $D_\beta \subset \mathbf{R}^n$ be a confidence set of a level β for the random value ξ, then an information set $\tilde{X}(D_\beta) = \tilde{X}(y, X_0, V, D_\beta)$ for system (8)–(9) with $D = D_\beta$ is a confidence region for x of a level not less than α.

Here

$$\alpha = 1 - (\alpha_1)^{-1}(1 - \beta),$$

$$\alpha_1 = P\{\tilde{X}(y, X_0, V, \xi) \neq \emptyset\} = P\{\xi \in D^0(y, X_0, V)\}.$$

Proof: Lemma 4 results in

$$D_\beta \cap D^0(y, X_0, V) \subseteq D^-(\tilde{X}(D_\beta); y, X_0, V).$$

From equality (13) it follows that

$$\mathcal{P}\{x \in \tilde{X}(D_\beta) \mid y\} = \frac{P\{\xi \in D^-(\tilde{X}(D_\beta); y, X_0, V)\}}{P\{\xi \in D^0(y, X_0, V)\}}.$$

Therefore

$$P\{x \in \tilde{X}(D_\beta) \mid y\} \geq \frac{P\{\xi \in D_\beta \cap D^0(y, X_0, V)\}}{P\{\xi \in D^0(y, X_0, V)\}} =$$

$$= P\{\xi \in D_\beta \mid \xi \in D^0(y, X_0, V)\}.$$

Denote by $\breve{D}_\beta$ the compliment of the set D_β to the space $\mathbf{R}^{n+m}$:

$$\breve{D}_\beta = \mathbf{R}^{n+m} \setminus D_\beta.$$

Since

$$P\{\xi \in \breve{D}_\beta \cap D^0(y, X_0, V)\} \geq P\{\xi \in \breve{D}_\beta\} = 1 - \beta,$$

then

$$P\{\xi \in D_\beta \mid \xi \in D^0(y, X_0, V)\} =$$

$$= 1 - P\{\xi \in \breve{D}_\beta \mid \xi \in D^0(y, X_0, V)\} =$$

$$= 1 - \frac{P\{\xi \in \breve{D}_\beta \cap D^0(y, X_0, V)\}}{P\{\xi \in D^0(y, X_0, V)\}} \leq 1 - (\alpha_1)^{-1}(1-\beta).$$

■

Lemma 5:

$$P\{\tilde{X}(y, X_0, V, \xi) \neq \emptyset\} = P\{H_2\xi \in y - V - GX_0\},$$

where

$$H_2\xi = GQ_1\xi_1 + Q_2\xi_2. \qquad (15)$$

Proof: Show that

$$\{\tilde{X}(y, X_0, V, \xi) \neq \emptyset\} \Leftrightarrow \{GQ_1\xi_1 + Q_2\xi_2 \in y - V - GX_0\}.$$

Indeed, let for $\xi = d = \{d_1, d_2\} \in \mathbf{R}^{n+m}$ the information set is nonempty: $\tilde{X}(y, X_0, V, d) \neq \emptyset$. Then there are $x_0^* \in X_0,\ v^* \in V$ such that $x_0^* + Q_1 d_1 = G^+(y - v^* - Q_2 d_2)$, thus

$$G(x_0^* + Q_1 d_1) = y - v^* - Q_2 d_2,$$

and

$$G(X_0 + Q_1 d_1) \cap (y - V - Q_2 d_2) \neq \emptyset.$$

It results in $GQ_1 d_1 + Q_2 d_2 = y - v^* - Gx_0^*$, i. e.

$$GQ_1 d_1 + Q_2 d_2 \in y - V - GX_0. \qquad (16)$$

Conversely, let for $\xi = d = \{d_1, d_2\}$ inclusion (16) holds. Then there is $x_0^* \in X_0,\ v^* \in V$ for which

$$GQ_1 d_1 + Q_2 d_2 = y - v^* - Gx_0^*,$$

i.e.

$$G(x_0^* + Q_1 d_1) = y - v^* - Q_2 d_2.$$

Moreover $x_0^* + Q_1 d_1 = G^+(y - v^* - Q_2 d_2)$ implies an inequality

$$(X_0 + Q_1 d_1) \cap G^+(y - V - Q_2 d_2) \neq \emptyset$$

and $\tilde{X}(y, X_0, V, d) \neq \emptyset$. ■

The confidence regions $\tilde{X}_\alpha$ approximate to the information set X^{det} if variances of the random perturbations in system (1)–(3) tend to zero.

Theorem 2. Let matrices of the coefficients in (1)–(3) tend to 0: $Q_1(\varepsilon) = \varepsilon Q_1$, $Q_2(\varepsilon) = \varepsilon Q_2$ and $\varepsilon \to 0$. If the sets X_0 and V have interior points and for a given observation y the information set is not empty:

$$X^{det} = X_0 \cap G^+(y - V) \neq \emptyset,$$

then for any probability $\alpha \in (0.5; 1)$ there are confidence sets $\tilde{X}_\alpha^\varepsilon$ such that:

1) $\tilde{X}_\alpha^{\varepsilon_1} \supseteq \tilde{X}_\alpha^{\varepsilon_2} \supseteq X^{det}$ if $\varepsilon_1 > \varepsilon_2$;
2) $\tilde{X}_\alpha^\varepsilon \to X^{det}$ if $\varepsilon \to 0$, i.e. $\lim\limits_{\varepsilon \to 0} \rho(\tilde{X}_\alpha^\varepsilon, X^{\det}) = 0$, where $\rho(\tilde{X}_\alpha^\varepsilon, X^{\det})$ is the Hausdorff distance between two sets

$$\rho(X, Y) \triangleq \max\{\sup_{x \in X} \inf_{y \in Y} ||x - y||,\ \sup_{y \in Y} \inf_{x \in X} ||x - y||\}.$$

Proof: Denote by $\tilde{X}^\varepsilon(y, X_0, V, \xi)$ a random confidence set for system (1)–(3) with $Q_i(\varepsilon) = \varepsilon Q_i$, $\varepsilon \in (0; 1)$.

A probability

$$\alpha_1^{(\varepsilon)} = P\{\tilde{X}^\varepsilon(y, X_0, V, \xi) \neq \emptyset\} =$$

$$= P\{\varepsilon H_2 \xi \in y - V - GX_0\} > 0$$

since the set V contains exterior points.

Let show that $0 \in y - V - GX_0$.

From equality $X^{\det} = X_0 \cap G^+(y - V) \neq \emptyset$ it follows that there are $x^* \in X_0,\ v^* \in V$ such that $y - v^* = Gx^*$, therefore $y - v^* - Gx^* = 0$, i.e. $0 \in y - V - GX_0$.

The convexity of the set $y - V - GX_0$ and the condition $0 \in y - V - GX_0$ imply that

$$\varepsilon_1^{-1}(y - V - GX_0) \subset \varepsilon_2^{-1}(y - V - GX_0) \ \text{ if } \ \varepsilon_1 > \varepsilon_2.$$

Therefore $\alpha_1^{(1)} \leq \alpha_1^{(\varepsilon_1)} \leq \alpha_1^{(\varepsilon_2)}$ if $1 > \varepsilon_1 > \varepsilon_2 > 0$.

Let $D_\beta = B_{1\beta} \times B_{2\beta} \subset \mathbf{R}^{n+m}$ be a compact convex confidence set for ξ and the following conditions hold:

$$0 \in D_\beta \ \text{ and } \ P\{\xi_1 \in B_{1\beta},\ \xi_2 \in B_{2\beta}\} = \beta.$$

Here $\beta = 1 - (1 - \alpha)\alpha_1$ and

$$\alpha_1 = P\{H_2\xi \in y - V - GX_0\}.$$

Let us construct sets

$$\tilde{X}^\varepsilon(D_\beta) = (X_0 + \varepsilon Q_1 B_{1\beta}) \cap G^+(y - V - \varepsilon Q_2 B_{2\beta}).$$

From Theorem 1 the set $\tilde{X}^\varepsilon(D_\beta)$ is a confidence region of level not less than $\beta_\varepsilon = 1 - \left(\alpha_1^{(\varepsilon)}\right)^{-1}(1 - \beta)$.

For any $\varepsilon \in (0; 1)$ an inequality $\alpha_1^{(1)} \leq \alpha_1^{(\varepsilon)}$ holds, hence

$$\beta_\varepsilon = 1 - \frac{1}{\alpha_1^{(\varepsilon)}}(1 - \beta) \geq 1 - \frac{1}{\alpha_1^{(1)}}(1 - \beta) = \alpha.$$

Thus the set $\tilde{X}_\alpha^\varepsilon = \tilde{X}^\varepsilon(D_\beta)$ is a confidence region of level not less than α for system (1)-(3) if $Q_i(\varepsilon) = \varepsilon Q_i$

If $1 > \varepsilon_1 > \varepsilon_2 > 0$ then $\tilde{X}_\alpha^{\varepsilon_2} \supset \tilde{X}_\alpha^{\varepsilon_1} \supset X^{\det}$.

Indeed,

$$X^{\det} = \bigcap_{\varepsilon \in (0,1)} \tilde{X}_\alpha^\varepsilon.$$

Since $\tilde{X}_\alpha^\varepsilon$ are embedded compact sets then $\tilde{X}_\alpha^\varepsilon \to X^{\det}$ in Hausdorff metric if $\varepsilon \to 0$. ■

IV. Simulation

Let us illustrate Theorem 2 by the estimation problem in $\mathbf{R}^1$ (Example 1). Let consider model (5)–(7) with $\Delta = 1$, $x_0 = 0$ and $y = 1$.

We construct the standard linear confidence estimates $\hat{X}_\alpha$ of level $\alpha = 0.95$ for different values $\sigma = 0.5; 0.45; ...; 0.05$:

$$\hat{X}_\alpha = \hat{X} + \frac{\sigma}{\sqrt{2}}[-t_\alpha; t_\alpha], \qquad (17)$$

where $\hat{X} = 0.5[0; 2] + 0.5[-1; 1] = [-0.5; 1.5]$ is the set of possible mean values.

We find nonlinear confidence regions $\tilde{X}_\alpha$ for the same values of σ using Theorem 1.

The information set in this case has a form: $X^{\det} = [0; 1]$.

On Fig.1 it is shown how different kinds of estimates depend on variances of the random perturbation: the linear standard estimates $\hat{X}_\alpha$ – dash lines, the nonlinear confidence regions $\tilde{X}_\alpha$ – solid lines, the values of σ are marked on the horizontal axe.

The dotted lines is the information sets $X^{\det}$. The information set is not a confidence estimate, it is an estimate in case of $\sigma = 0$.

One can see that the nonlinear confidence regions $\tilde{X}_\alpha$ approximate to $X^{\det} = [0; 1]$ if $\sigma \to 0$ unlike to the linear estimates which tend to $\hat{X} = [-0.5; 1.5]$.

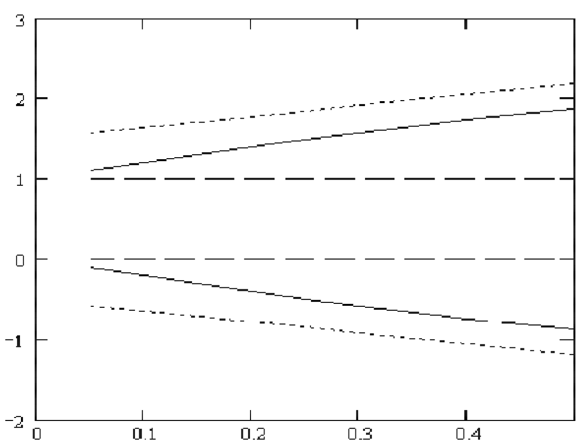

Fig.1. Dependence of confidence regions on σ.

A main peculiarity of the constructed confidence region is their dependence on y.

On Fig.2 it is shown how radiuses of the confidence regions for system (5)–(7) depend on $|y - x_0|$. The dash line is a radius of the optimal linear estimate for the system, the curve with markers is a radius of the nonlinear confidence region, values of $|y - x_0|$ are marked on the horizontal axe. Here $\alpha = 0.95$, $\sigma = 0.1$, $\Delta = 1$.

The solid line is a radius of a confidence region in case without uncertainty, i.e. if $\Delta = 0$ and $X_0 = \{x_0\}$, $V = \{0\}$.

One can see that in case $y = x_0$ the proposed approach do not improve the linear estimate, but if $|y - x_0|$ increases the nonlinear estimates $\tilde{X}_\alpha$ becomes more precise.

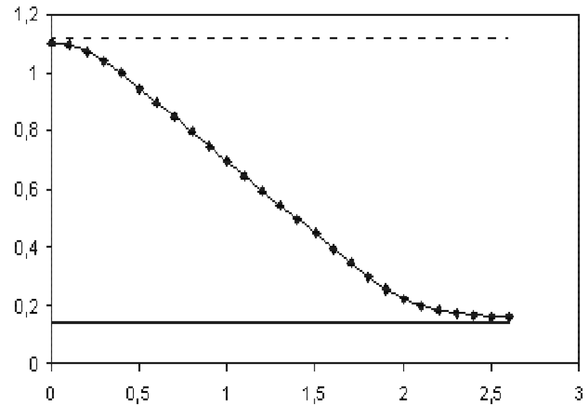

Fig.2. Dependence on $|y - x_0|$.

V. Dependence of Confidence Regions on Uncertainty Sets

Let us consider the how the confidence regions change if the radiuses of the sets of uncertain parameters tens to zero.

Lemma 6: Let

1) $0 \in V$, $x_0 \in X_0$;
2) B_α is a confidence set of the level α for a random vector
$$e = (I - \Lambda G)Q_1\xi_1 - \Lambda Q_2\xi_2; \qquad (18)$$
3) the following condition holds:
$$P\{e \in B_\alpha + b\} \le P\{e \in B_\alpha\} = \alpha \ \text{ for all } \ b \in \mathbf{R}^n. \qquad (19)$$

Then a set $\hat{x} + B_\alpha$ is a confidence region of the level not greater then α for system state (1)–(3), i.e.

$$P\{\tilde{X}(y, X_0, V, \xi) \subset \hat{x} + B_\alpha \mid \tilde{X}(y, X_0, V, \xi) \ne \emptyset\} \le \alpha.$$

Here and further

$$\hat{x} = (I - \Lambda G)x_0 + \Lambda y, \qquad (20)$$

a matrix Λ is defined as for the standard stochastic linear estimate:

$$\Lambda = P_1 G^T R^{-1}, \quad P_1 = ((Q_1Q_1^T)^{-1} + G^T(Q_2Q_2^T)^{-1}G)^{-1}. \qquad (21)$$

Lemma 7: [8] Let $X_\alpha = \hat{X} + B_\alpha$, where B_α is a confidence set of the level α for a random vector e, $\hat{X}$ is the set of a posteriori mean values defined by (4), (21).

Then X_α is a confidence region of the level not less then α in sense of Definition 6, i.e.

$$P\{\tilde{X}(y, X_0, V, \xi) \subset X_\alpha \mid \tilde{X}(y, X_0, V, \xi) \ne \emptyset\} \ge \alpha.$$

Theorem 3. Let the following conditions hold:

1) $X_0 = x_0 + U$, where $U \subset \mathbf{R^n}$ is a given convex compact set;
2) $0 \in V$, $0 \in V$;
3) B_α is a confidence set of the level α for the random vector e defined (18),(21);
4) condition (19) holds.

Then there is a confidence set $\tilde{X}_\alpha$ of level not less than α in sense Definition 6 such that

$$\hat{x} + B_\alpha \subseteq \tilde{X}_\alpha \subseteq \hat{X} + B_\alpha. \tag{22}$$

Here $\hat{x}$ and $\hat{X}$ are defined by (20) and (4) respectively.

If the sets X_0, V of possible values of uncertain perturbation in (1)–(3) are reduced to a point then the confidence regions tend to the standard confidence estimates for a stochastic linear system.

Corollary 1. Let for the system (1)–(3) the following conditions hold

1) $V(\gamma) = \gamma V,\ X_0(\gamma) = x_0 + \gamma U;$
2) V and U are given compact convex sets containing 0,
3) B_α is a confidence set of the level α for the random vector e;
4) condition (19) holds.

Then there are confidence regions $\tilde{X}_\alpha(\gamma)$ of level not less then α for system (1)–(3) such that

$$\tilde{X}_\alpha(\gamma) \to \hat{x} + B_\alpha \quad \text{if} \quad \gamma \to 0,$$

i.e. $\lim\limits_{\gamma \to 0} \rho(\tilde{X}_\alpha(\gamma), \hat{x} + B_\alpha) = 0$, here $\hat{x}$ is defined by (20), $\rho(X, Y)$ is the Hausdorff distance between two sets.

Difference between of confidence regions of probability $\alpha = 0.95$ for linear and nonlinear estimates in system (5)–(7) is shown on the Fig.3. Here the solid line is the standard linear confidence estimates (17) , the dash line is the nonlinear confidence regions, the values of Δ are marked on the horizontal axe. Here $\sigma = 0.1$, $x_0 = 0$, $y = 2$.

The dotted line is confidence sets for the system without uncertainty, i.e. in case $\Delta = 0$.

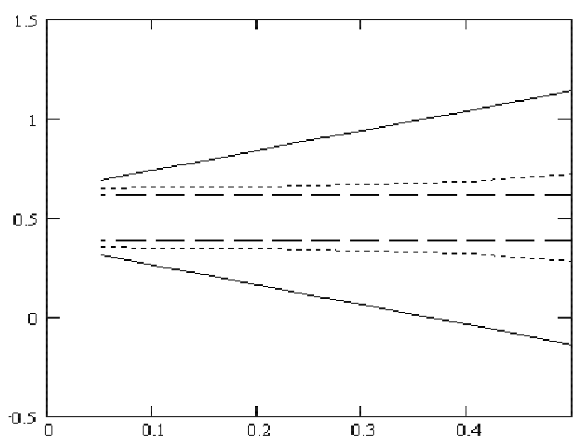

Fig.3. Dependence on Δ.

VI. Conclusion

In this paper different approaches to the confidence estimation for statistically uncertain problem with observation are analyzed. It is shown that the optimal confidence estimate is not linear. The calculation algorithm for the confidence estimates is more complicated than the linear procedure. But it allows to improve significantly the confidence estimate in the case of small dispersions of the random perturbations.

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