Stability and error analysis of the anti-causal realization of periodic inverse systems

Xiaofang Chen
School of Electrical and Electronic Engineering
Nanyang Technological University
Nanyang Avenue, Singapore 639798
Email: chen0083@ntu.edu.sg

Cishen Zhang
School of Electrical and Electronic Engineering
Nanyang Technological University
Nanyang Avenue, Singapore 639798
Email: ecszhang@ntu.edu.sg

Jingxin Zhang
Department of Electrical and Computer Systems Engineering
Monash University,
VIC 3800, Australia
jingxin.zhang@eng.monash.edu.au

***Abstract*— A causal realization of an inverse system can be unstable and an anti-casual realization is to deal with this problem to provide a numerically stable procedure to inverse the system and compute its input signal. In this paper, we consider the anti-causal realization of the inverse of discrete time linear periodic systems obtained by an outer-inner factorization approach. It is shown that the outer-inner factorization can result in a stable anti-causal realization. It also derives a formula of the inversion error, which can show that the inversion error is inevitable due to the anti-causal reversal operation.**

I. Introduction

Inverse systems have important applications in signal processing, control, filtering and data coding, which may also be called equalization or deconvolution systems in different circumstance. The problem of inverting a linear periodic time-varying (LPTV) system has attracted considerable attentions, see for example, [1]-[5]. The anti-causal realization of the inverse system is to deal with the stability problem in the inverse system, which was earlier studied and applied to multirate filter banks and its implementation [6], [7]. In these works, a state space representation is employed to induce the anti-causal inverse.

The inner-outer factorization is a useful technique to deal with the system inverse problem. In [8], a method is proposed to construct optimal causal approximate inverse for discrete time single-input single-output (SISO) causal periodic filters in the presence of measurement noise. The method is based on an inner-outer factorization of the plant rational matrix. In [9]-[11], van der Veen and Dewilde present an inner outer factorization method for linear discrete time-varying systems. Instead of factorizing the transfer function matrix, this method factorizes the system time domain operator. It has an computational advantage that all computations are done by using the state space realization.

While [9] presents the concept and theory necessary for the development of the inner-outer factorization method for linear time varying systems, it does not provide explicit stability analysis of the factorized systems.

In this paper, we will analyze the stability of the anti-causal realization of the periodic inverse system which is obtained from the outer-inner factorization of the inverse system in the state space realization. A periodic Lyapunov equation is introduced to the stability analysis. We will further examine the error introduced by using the anti-causal realization of the system inverse to compute the system input signal. Such an error analysis has not been presented in the existing literature.

The rest of the paper is organized as follows. Section II presents necessary preliminaries. The anti-causal realization of the inverse system is presented in III. Section IV analyzes the stability of the inner factor and its anti-causal inverse. And the system inversion error is analyzed in section V. Section VI illustrates the system error by examples before conclusion in section VII.

II. Preliminaries

The following notations are used in this paper. I denotes the identity matrix of an appropriate dimension, $*$ denotes the hermitian conjugate, $\mathcal{R}^{m\times n}$ denotes the real space of dimension $(m \times n)$. The factorization of an operator T into $T = T_oV$ is called an outer-inner factorization, where T_o is the left outer factor and V is an isometry satisfying $VV^* = I$. The matrix V is inner if and only if T^* is full column rank.

Let u and y be the input and output of the system T such that $y = uT$. The idealized inverse problem is to find the inverse system T^{-1} of T, such that the input u is computed and obtained from the measurement of y, i.e. $u = yT^{-1}$. This idealized case is often practically not feasible since there is no guarantee that T^{-1} is a stable system. The anti-causal realization of the inverse is a method for stably computing the input u and its restriction is that u should be of finite length.

The state realization of a SISO LPTV causal system T is written as:

$$\begin{aligned} x_{k+1} &= x_kA_k + u_kB_k, \\ y_k &= x_kC_k + u_kD_k, \end{aligned} \tag{1}$$

where $u_k \in \mathcal{R}^{1\times 1}$ is the input and $y_k \in \mathcal{R}^{1\times 1}$ is the output. If the state row vector x_k has a time varying dimension $\mathcal{R}^{1\times b_k}$, then the dimensions of A_k, B_k, C_k and D_k are respectively $\mathcal{R}^{b_k\times b_{k+1}}$, $\mathcal{R}^{1\times b_{k+1}}$, $\mathcal{R}^{b_k\times 1}$ and $\mathcal{R}^{1\times 1}$. In this paper, we consider the constant state dimension $\mathcal{R}^{1\times n}$. For the N-periodic time varying system, the state matrices satisfy $A_{k+N} = A_k$, $B_{k+N} = B_k$, $C_{k+N} = C_k$, and $D_{k+N} = D_k$. It follows from the state equation (1) that entries of the system operator T are given by

$$T_{i,j} = \begin{cases} D_i, & i = j, \\ B_i A_{i+1} \cdots A_{j-1} C_j, & i < j. \end{cases} \quad (2)$$

The state realization of an N-periodic LPTV anti-causal system $T^{'}$ is written as:

$$\begin{aligned} x^{'}_{k-1} &= x^{'}_k A^{'}_k + u^{'}_k B^{'}_k, \\ y^{'}_k &= x^{'}_k C^{'}_k + u^{'}_k D^{'}_k. \end{aligned} \quad (3)$$

The dimensions of all the matrices in the realization must be compatible. If the input $u^{'}_k \in \mathcal{R}^{1\times 1}$, the output $y^{'}_k \in \mathcal{R}^{1\times 1}$ and the state row vector $x^{'}_k$ has a time varying dimension $\mathcal{R}^{1\times b^{'}_k}$, the dimensions of A_k, B_k, C_k and D_k are respectively $\mathcal{R}^{b^{'}_k \times b^{'}_{k-1}}$, $\mathcal{R}^{1\times b^{'}_{k+1}}$, $\mathcal{R}^{b^{'}_k \times 1}$ and $\mathcal{R}^{1\times 1}$. The state matrices are also in period of N and entries of the system operator $T^{'}$ are given by

$$T^{'}_{i,j} = \begin{cases} D^{'}_i, & i = j, \\ B^{'}_i A^{'}_{i-1} \cdots A^{'}_{j+1} C^{'}_j, & i > j. \end{cases} \quad (4)$$

Definition 1: The causal system T with the state equation (1) is stable if all the eigenvalues of the lifted matrix $\bar{A} = A_k \cdots A_{k+N-1}$ are inside the unit disk, or the greatest eigenvalue of $\bar{A}$ is small than unity, i.e.

$$l_A = \max_i \lambda_i(\bar{A}) < 1,$$

where $\lambda_i(\bar{A})$ for $i = 0, 1, \cdots, N-1$ are the eigenvalues of the lifted matrix $\bar{A}$.

The stability of the anti-causal system $T^{'}$ with state matrices $\{A^{'}_k, B^{'}_k, C^{'}_k, D^{'}_k\}$ is defined as follows:

Definition 2: The anti-causal realization system $T^{'}$ is stable if

$$l_{A'} = \max_i \lambda_i(\bar{A}^{'}) < 1,$$

where $\bar{A}^{'} = A^{'}_{k+N-1} \cdots A^{'}_k$ is the lifted matrix, and $\lambda_i(\bar{A}^{'})$ for $i = 0, 1, \cdots, N-1$ are the eigenvalues of $\bar{A}^{'}$.

In this paper, we consider the stable causal-anticausal realization of the inverse of the causal SISO LPTV system T with the state equation (1) and entries represented in (2). It is assume that T is a stable system. In general and following from the method in [9], the system T can be factorized into a left outer factor T_o and the inner factor V, such that

$$T = T_o V. \quad (5)$$

The above outer-inner factorization is computed by the backward recursive LQ factorization

$$\begin{bmatrix} D_k & B_k Y_{k+1} \\ C_k & A_k Y_{k+1} \end{bmatrix} = \begin{bmatrix} D_{o,k} & 0 & 0 \\ C_{o,k} & Y_k & 0 \end{bmatrix} W_k, \quad (6)$$

where the initial starting point Y_{k+1} is obtained by the the periodic Riccati equation $M_{k+1} = Y_{k+1} Y^*_{k+1}$, which is written as

$$\begin{aligned} M_k = \ & A_k M_{k+1} A^*_k + C_k C^*_k \\ & -[A_k M_{k+1} B^*_k + C_k D^*_k](D_k D^*_k + B_k M_{k+1} B^*_k)^\dagger \\ & \times [D_k C^*_k + B_k M_{k+1} A^*_k]. \end{aligned}$$

The numerical solution of the periodic Riccati equation is presented in [13]. The partitioning of the outer-inner factorization is such that $D_{o,k}$ and Y_k have full column rank, and W_k is unitary matrix with partitioning in the form

$$W_k = \begin{bmatrix} D_{v,k} & B_{v,k} \\ C_{v,k} & A_{v,k} \\ d & d \end{bmatrix},$$

where d can be any matrix.

Since W_k is a unitary matrix, we can obtain the following equations.

$$\begin{aligned} &A_{v,k} A^*_{v,k} + C_{v,k} C^*_{v,k} = I, \quad A_{v,k} B^*_{v,k} + C_{v,k} D^*_{v,k} = 0, \\ &B_{v,k} A^*_{v,k} + D_{v,k} C^*_{v,k} = 0, \quad B_{v,k} B^*_{v,k} + D_{v,k} D^*_{v,k} = I. \end{aligned} \quad (7)$$

The state-space realization of the outer factor T_o is given by $\{A_k, B_k, C_{o,k}, D_{o,k}\}$. And the realization of the inner factor V is given by $\{A_{v,k}, B_{v,k}, C_{v,k}, D_{v,k}\}$. The outer-inner factorization as given in the above will be the basis for the stable anti-causal realization of the inverse system of T.

III. Anti-causal realization of the inverse system

We now derive the stable anti-causal realization of the inverse system of the causal stable LPTV system T. It is noted that the anti-casual realization can only permit practically finite step of computation and we assume that the number of the input data in u is $L + 1$. Thus the input sequence of u is a collection of u_k for $0 \le k \le L$, as follows.

$$u = \begin{bmatrix} u_0 & u_1 & u_2 & \cdots & u_L \end{bmatrix}.$$

Mathematically, the inverse of the transfer operator T is a combination of the inverse of the inner factor and the inverse of the outer factor, resulting in

$$\begin{aligned} \hat{u} = & \ y V^{-1} T_o^{-1}, \\ = & \ u T_o V V^{-1} T_o^{-1}. \end{aligned}$$

However, the direct casual inverse V^{-1} of the inner factor V is an unstable system and practically unfeasible, which will be shown in the next section. In real computation, the unstable causal factor of the inverse system can be realized anti-causally to deal with the computational divergence problem. A result on the anti-causal realization is stated in the following.

Theorem 1: The anti-causal realization of the inverse of the inner factor is $\hat{V} = V^*$, where V^* is anti-causal and is the Hermitian conjugates V with a state space realization $\{A^*_{v,k}, C^*_{v,k}, B^*_{v,k}, D^*_{v,k}\}$.

Proof: The casual realization of the N-periodic inner factor can be rewritten into an operator form by assembling the state matrices $\{A_{v,k}, B_{v,k}, C_{v,k}, D_{v,k}\}$ as diagonal operators:

$$A_v = \begin{bmatrix} \ddots & & \mathbf{0} \\ & A_{v,k} & \\ \mathbf{0} & & \ddots \end{bmatrix} \quad C_v = \begin{bmatrix} \ddots & & \mathbf{0} \\ & C_{v,k} & \\ \mathbf{0} & & \ddots \end{bmatrix}$$

$$B_v = \begin{bmatrix} \ddots & & \mathbf{0} \\ & B_{v,k} & \\ \mathbf{0} & & \ddots \end{bmatrix} \quad D_v = \begin{bmatrix} \ddots & & \mathbf{0} \\ & D_{v,k} & \\ \mathbf{0} & & \ddots \end{bmatrix}$$

Hence, the transfer function of the inner factor V is given as

$$V(z) = D_v + B_v z(I - A_v z)^{-1} C_v,$$

and the transfer function of the anti-causal realization of the inverse of the inner factor is shown as

$$V^*(z) = D_v^* + C_v^*(I - z^*)A_v^*)^{-1} z^* B_v^*.$$

In view of (7), we can obtain $D_v D_v^* = I - B_v B_v^*$, $C_v D_v^* = -A_v B_v^*$, $D_v C_v^* = -B_v A_v^*$, and $A_v A_v^* + C_v C_v^* = I$. Using these, we have

$$\begin{aligned}
&V(z)V^*(z)\\
&= [D_v + B_v z(I - A_v z)^{-1} C_v][D_v^* + C_v^*(I - z^* A_v^*)^{-1} z^* B_v^*]\\
&= D_v D_v^* + D_v C_v^*(I - z^* A_v^*)^{-1} z^* B_v^*\\
&\quad + B_v z(I - A_v z)^{-1} C_v D_v^*\\
&\quad + B_v z(I - A_v z)^{-1} C_v C_v^*(I - z^* A_v^*)^{-1} z^* B_v^*\\
&= D_v D_v^* - B_v z(I - A_v z)^{-1} A_v B_v^*\\
&\quad - B_v A_v^*(I - z^* A_v^*)^{-1} z^* B_v^*\\
&\quad + B_v z(I - A_v z)^{-1} C_v C_v^*(I - z^* A_v^*)^{-1} z^* B_v^*\\
&= D_v D_v^* + B_v z(I - A_v z)^{-1}[-A_v z(I - z^* A_v^*)\\
&\quad -(I - A_v z) z^* A_v^* + C_v C_v^*](I - z^* A_v^*)^{-1} z^* B_v^*\\
&= D_v D_v^* + B_v z(I - A_v z)^{-1}(I - A_v z)\\
&\quad (I - z^* A_v^*)(I - z^* A_v^*)^{-1} z^* B_v^*\\
&= I
\end{aligned}$$

Hence, V^* with the state matrices $\{A_{v,k}^*, C_{v,k}^*, B_{v,k}^*, D_{v,k}^*\}$ is the anti-causal inverse of V. □

In the rest of this paper, we use the notations $\hat{V}$ and $\hat{T}_o$ as the anti-casual inverse of the inner factor and the causal inverse of the outer factor, respectively.

With u being the input to the LPTV system T and with the anti-casual realization of the inverse system, the output $\hat{u}$ the inverse system is written as

$$\hat{u} = u T_o V \hat{V} \hat{T}_o$$

We define the error of the inverse system as follows.

Definition 3: The system error is defined as the difference of the the reconstructed signal (output signal from the inverse system) and the input signal, that is,

$$e = \hat{u} - u.$$

IV. Stability analysis of the inner factor

The stability of an LPTV system can be examined by using the periodic Lyapunov equation technique presented in [12]. The following lemma is deduced from [12] for the stability analysis of the LPTV inverse system.

Lemma 1: The N-periodic LPTV system T in the form (1) is stable if and only if there exists an N-periodic system matrix $Q_k \in \mathcal{R}^{n \times n}$ with $Q_k = Q_k^* \geq 0$ and $Q_{k+N} = Q_k$ for $k = 0, 1, \cdots, N-1$ such that the following periodic Lyapunov equation is satisfied:

$$A_k Q_{k+1} A_k^* + C_k C_k^* - Q_k = 0, k = 0, 1, \cdots, N-1. \quad (8)$$

□

The existence of $Q_k = Q_k^* \geq 0$ implies that the limit

$$\lim_{M \to \infty} \sum_{j=0}^{M} A_k A_{k+1} \cdots A_{k+j-1} C_{k+j} C_{k+j}^* A_{k+j-1}^* \cdots A_{k+1}^* A_k^*$$

exists as M approaches infinity. Further more, it implies that

$$l_A = \max_i \lambda_i(\bar{A}) < 1,$$

hence the system (1) is stable.

Given T with the state-space realization (1), the left-outer-inner factorization is computed in (6), where W_k is an unitary matrix yielding (7). Thus, we have

$$C_{v,k} C_{v,k}^* + A_{v,k} A_{v,k}^* = I. \quad (9)$$

It is clear that (9) is exactly the same as (8) with $Q_k = I$, for $k = 0, 1, \cdots, N-1$. Thus $Q_k = I$, for $k = 0, 1, \cdots, N-1$, are solutions for (8), resulting in

$$l_{A_v} = \max_i \lambda_i(\bar{A}_v) < 1,$$

where $\bar{A}_v = A_{v,k} \cdots A_{v,k+N-1}$ is the lifted state matrix for the inner part. From the stability of causal system in *Definition 1*, we can conclude that the inner factor with state realization $\{A_{v,k}, B_{v,k}, C_{v,k}, D_{v,k}\}$ is stable. This proves that the inner factor derived by the outer-inner factorization is stable. The stability of the inner factor derived by the inner-outer factorization in [9] can be checked similarly. And it can be proved that the inner factor derived by the inner-outer factorization is also stable.

The inverse of the inner factor is anti-causally realized and has the state matrices $\{A_{v,k}^*, C_{v,k}^*, B_{v,k}^*, D_{v,k}^*\}$. The existence of $Q_k = Q_k^* = I \geq 0$ also implies that

$$l_{A_v^*} = \max_i \lambda_i(\bar{A}_v^*) < 1,$$

where $\bar{A}_v^* = A_{v,k+N-1}^* \cdots A_{v,k}^*$ is the lifted state matrix for the anti-causal inverse of the inner part. Hence, we can make the following statement based on the stability of the anti-casual system defined in *Definition 2*.

Theorem 2: The anti-causal inverse system $\hat{V} = V^*$ of the inner factor V is stable. □

The stability of V^* implies that the direct inverse V^{-1} of the inner factor V is unstable.

V. Error analysis

A. Zero-input and zero-state responses

The output of the LPTV system T is a combination of two parts, the zero-input response and the zero-state response and can be written as:

$$y = x_0 T_x + uT, \quad (10)$$

where x_0 is the unknown initial state value and T_x is the system zero-input operator, which can be expressed in terms of the state matrices as follows

$$T_x = \begin{bmatrix} C_0 & A_0 C_1 & A_0 A_1 C_2 & \cdots & A_0 A_1 \cdots A_{L-1} C_L \end{bmatrix}.$$

For the system T being factorized into the outer and inner factors, i.e. $T = T_o V$, we consider the system outputs from the outer factor and from the inner factor, respectively, in the following.

1). The output y_o from the outer part

$$y_o = x_{o,0} T_{o,x} + u T_o, \tag{11}$$

where $x_{o,0}$ is the unknown initial state value for the outer part, and $T_{o,x}$ is the zero-input operator, which can be written as

$$T_{o,x} = \begin{bmatrix} C_{o,0} & A_{o,0}C_{o,1} & \cdots & A_{o,0}A_{o,1}\cdots A_{o,L-1}C_{o,L} \end{bmatrix}.$$

And T_o is the outer factor of the transfer operator, which is given by

$$T_{o,i,j} = \begin{cases} D_{o,i}, & i = j, \\ B_{o,i}A_{o,i+1}\cdots A_{o,j-1}C_{o,j}, & i < j. \end{cases}$$

2). The output y_o from the outer part is the input to the inner part, resulting in the following system output

$$y = x_{v,0} T_{v,x} + y_o V, \tag{12}$$

where $x_{v,0}$ is the unknown initial state value for the inner part, and $T_{v,x}$ is the zero-input operator of V represented by

$$T_{v,x} = \begin{bmatrix} C_{v,0} & A_{v,0}C_{v,1} & \cdots & A_{v,0}A_{v,1}\cdots A_{v,L-1}C_{v,L} \end{bmatrix}.$$

And V is the inner factor given by

$$V_{i,j} = \begin{cases} D_{v,i}, & i = j, \\ B_{v,i}A_{v,i+1}\cdots A_{v,j-1}C_{v,j}, & i < j. \end{cases}$$

In applying the inverse system with the anti-causal stable realization to compute the system input u, the output signal y from the stable LPTV system T will pass through the anti-causal inverse of the inner factor first. It follows that the output of the anti-causal inverse inner factor is the input of the casual inverse outer factor. In the following, we will discuss, respectively, errors raised in the anti-causal inverse of the inner factor and in the causal inverse of the outer factor.

B. Error in the anti-causal stable inverse of the inner factor

The state matrices of the anti-causal inverse of the inner factor are written as $\{\hat{A}_{v,k}, \hat{B}_{v,k}, \hat{C}_{v,k}, \hat{D}_{v,k}\} = \{A^*_{v,k}, C^*_{v,k}, B^*_{v,k}, D^*_{v,k}\}$. The following equations can then be obtained following from (7):

$$\begin{array}{ll} A_{v,k}\hat{A}_{v,k} + C_{v,k}\hat{B}_{v,k} = I, & A_{v,k}\hat{C}_{v,k} + C_{v,k}\hat{D}_{v,k} = 0, \\ B_{v,k}\hat{A}_{v,k} + D_{v,k}\hat{B}_{v,k} = 0, & B_{v,k}\hat{C}_{v,k} + D_{v,k}\hat{D}_{v,k} = I. \end{array} \tag{13}$$

The periodic Lyapunov equation with $Q_k = I$ can be formed by the first equation in (13), which is shown as

$$A_{v,k}Q_{k+1}\hat{A}_{v,k} + C_{v,k}\hat{B}_{v,k} - Q_k = 0, k = 0, 1, \cdots, N. \tag{14}$$

Recursively using (14), we can get

$$\begin{aligned} Q_k &= C_{v,k+1}\hat{B}_{v,k+1} + \textstyle\sum_{i=k+2}^{\infty} A_{v,k+1}A_{v,k+2} \\ &\quad \cdots A_{v,i-1}C_{v,i}\hat{B}_{v,i}\hat{A}_{v,i-1}\cdots\hat{A}_{v,k+2}\hat{A}_{v,k+1} \\ &= I \end{aligned} \tag{15}$$

We now express the anti-causal inverse output sequence $\hat{y}_o$ as

$$\hat{y}_o = \hat{x}_{v,L}\hat{T}_{v,x} + y\hat{V}, \tag{16}$$

where $\hat{x}_{v,L}$ is an unknown initial state value of the anti-causal inverse of the inner part, entries of the zero input response $\hat{T}_{v,x}$ are

$$\hat{T}_{v,x} = \begin{bmatrix} \hat{A}_{v,L}\hat{A}_{v,L-1}\cdots\hat{A}_{v,1}\hat{C}_{v,0} & \cdots & \hat{A}_{v,L}\hat{C}_{v,L-1} & \hat{C}_{v,L} \end{bmatrix},$$

and entries of $\hat{V}$ are

$$\hat{V}_{i,j} = \begin{cases} \hat{D}_{v,i}, & i = j, \\ \hat{B}_{v,i}\hat{A}_{v,i-1}\cdots\hat{A}_{v,j+1}\hat{C}_{v,j}, & i > j. \end{cases}$$

The error in $\hat{y}_o$ is its difference from y_o in (11) which is obtained by using (11) and (12) as follows.

$$\begin{aligned} e_o &= \hat{y}_o - y_o, \\ &= \hat{x}_{v,L}\hat{T}_{v,x} + x_{v,0}T_{v,x}\hat{V} + y_o[V\hat{V} - I]. \end{aligned}$$

The inversion computation can yield $V\hat{V} = I$ for an infinitely long dada length. But with a real finite length computation, $V \in \mathcal{R}^{(L+1)\times(L+1)}$, then $V\hat{V} \neq I$. With the expressions of the entries of the matrices V and $\hat{V}$, we can obtain entries of the matrix $V\hat{V}$ as follows:

For $i = j$:

$$\begin{aligned} &[V\hat{V}]_{i,i} = D_{v,i}\hat{D}_{v,i} + B_{v,i}[C_{v,i+1}\hat{B}_{v,i+1} + \textstyle\sum_{k=i+2}^{L} A_{v,i+1} \\ &A_{v,i+2}\cdots A_{v,k-1}C_{v,k}\hat{B}_{v,k}\hat{A}_{v,k-1}\cdots\hat{A}_{v,i+2}\hat{A}_{v,i+1}]\hat{C}_{v,i}. \end{aligned}$$

If $L \to \infty$, (15) can be applied, leading to

$$[V\hat{V}]_{i,i} = D_{v,i}\hat{D}_{v,i} + B_{v,i}\hat{C}_{v,i} = I.$$

For $i > j$:

$$\begin{aligned} &[V\hat{V}]_{i,j} = B_{v,i}[C_{v,i+1}\hat{B}_{v,i+1} + \textstyle\sum_{k=i+2}^{L} A_{v,i+1}\cdots A_{v,k-1} \\ &C_{v,k}\hat{B}_{v,k}\hat{A}_{v,k-1}\cdots\hat{A}_{v,i+1} - I]\hat{A}_{v,i}\cdots\hat{A}_{v,j+1}\hat{C}_{v,j}. \end{aligned}$$

If $L \to \infty$, (15) can be applied, leading to

$$[V\hat{V}]_{i,j} = 0.$$

For $i < j$:

$$\begin{aligned} &[V\hat{V}]_{i,j} = B_{v,i}A_{v,i+1}\cdots A_{v,j}[C_{v,j+1}\hat{B}_{v,j+1} + \textstyle\sum_{k=j+2}^{L} A_{v,j+1} \\ &\cdots A_{v,k-1}C_{v,k}\hat{B}_{v,k}\hat{A}_{v,k-1}\cdots\hat{A}_{v,j+1} - I]\hat{C}_{v,j}. \end{aligned}$$

If $L \to \infty$, (15) can be applied, leading to

$$[V\hat{V}]_{i,j} = 0.$$

For practically feasible computing with the anti-causal realization in finite L computation steps, $V \in \mathcal{R}^{(L+1)\times(L+1)}$ and $V\hat{V} \neq I$. For small values of i, j, $[V\hat{V}]_{i,j}$ is approaching zero since $l_{A_v} < 1$. And for large values of i, j, $[V\hat{V}]_{i,j}$ becomes non-negligible.

Even thought we force the initial state values of the inner part and the anti-causal inverse of the inner part, $x_{v,0}$ and $\hat{x}_{v,L}$ respectively, to be zero, the system error is not equal to zero. The error is then propagated to the causal inverse of the outer factor and can be amplified in this part.

C. Error analysis for causal stable inverse of the outer factor

For the causal stable inverse of the outer factor, the state matrices are given as

$$\begin{array}{ll} \hat{A}_{o,k} = A_{o,k} - C_{o,k}D_{o,k}^{-1}B_{o,k}, & \hat{B}_{o,k} = D_{o,k}^{-1}B_{o,k}, \\ \hat{C}_{o,k} = -C_{o,k}D_{o,k}^{-1}, & \hat{D}_{o,k} = D_{o,k}^{-1}, \end{array}$$

yielding,

$$\begin{aligned} \hat{A}_{o,k} + C_{o,k}\hat{B}_{o,k} &= A_{o,k} - C_{o,k}D_{o,k}^{-1}B_{o,k} + C_{o,k}D_{o,k}^{-1}B_{o,k} \\ &= A_{o,k} \end{aligned} \tag{17}$$

The transfer operator $\hat{T}_o$ of the causal inverse of the outer factor has the entries shown as:

$$\hat{T}_{o,i,j} = \begin{cases} \hat{D}_{o,i}, & i = j, \\ \hat{B}_{o,i}\hat{A}_{o,i+1}\cdots\hat{A}_{o,j-1}\hat{C}_{o,j}, & i < j. \end{cases}$$

Hence, it can be shown that for the finite length computing, $T_o\hat{T}_o = I$ is satisfied. Entries of $T_o\hat{T}_o$ satisfy the following results.

For $i = j$:

$$[T_o\hat{T}_o]_{i,i} = D_{o,i}\hat{D}_{o,i} = I,$$

For $i > j$:

$$[T_o\hat{T}_o]_{i,j} = 0,$$

For $i < j$:

$$\begin{aligned} [T_o\hat{T}_o]_{i,j} = \; & D_{o,i}\hat{B}_{o,i}\hat{A}_{o,i+1}\cdots\hat{A}_{o,j-1}\hat{C}_{o,j} \\ & + B_iC_{o,i+1}\hat{B}_{o,i+1}\hat{A}_{o,i+2}\cdots\hat{A}_{o,j-1}\hat{C}_{o,j} \\ & + B_{o,i}A_{o,i+1}C_{o,i+2}\hat{B}_{o,i+2}\hat{A}_{o,i+3}\cdots\hat{A}_{o,j-1}\hat{C}_{o,j} \\ & \vdots \\ & + B_{o,i}A_{o,i+1}A_{o,i+2}\cdots A_{o,j-1}C_{o,j}\hat{D}_{o,j}. \end{aligned}$$

Recursively using (17) can result in $[T_o\hat{T}_o]_{i,j} = 0$, for $i > j$.

The output from the inverse of the outer part is written as

$$\begin{aligned} \hat{u} &= \hat{x}_{o,0}\hat{T}_{o,x} + (y_o + e_o)\hat{T}_o, \\ &= \hat{x}_{o,0}\hat{T}_{o,x} + x_{o,0}T_{o,x}\hat{T}_o + uT_o\hat{T}_o + e_1\hat{T}_o, \end{aligned} \tag{18}$$

where $\hat{x}_{o,0}$ is the unknown initial state value for the inverse of the outer part, and the zero input response $\hat{T}_{o,x}$ can be expressed as

$$\hat{T}_{o,x} = \begin{bmatrix} \hat{C}_{o,0} & \hat{A}_{o,0}\hat{C}_{o,1} & \cdots & \hat{A}_{o,0}\hat{A}_{o,1}\cdots\hat{A}_{o,L-1}\hat{C}_{o,L} \end{bmatrix}.$$

The system error e is the difference between the input signal and the reconstructed signal $\hat{u}$, which can be presented as follows.

$$\begin{aligned} e = \; & \hat{u} - u, \\ = \; & \hat{x}_{o,0}\hat{T}_{o,x} + x_{o,0}T_{o,x}\hat{T}_o + u[T_o\hat{T}_o - I] + e_1\hat{T}_o, \\ = \; & \hat{x}_{o,0}\hat{T}_{o,x} + x_{o,0}T_{o,x}\hat{T}_o + u[T_o\hat{T}_o - I] \\ & + \hat{x}_{v,L}\hat{T}_{v,x}\hat{T}_o + x_{v,0}T_{v,x}\hat{V}\hat{T}_o + y_o[V\hat{V} - I]\hat{T}_o \\ = \; & \hat{x}_{o,0}\hat{T}_{o,x} + \hat{x}_{v,L}\hat{T}_{v,x}\hat{T}_o + x_{v,0}T_{v,x}\hat{V}\hat{T}_o \\ & + x_{o,0}T_{o,x}V\hat{V}\hat{T}_o + uT_o[V\hat{V} - I]\hat{T}_o. \end{aligned}$$

If the initial state values of the original and the inverse systems are all zero, the system error e is simplified to

$$e = uT_o[V\hat{V} - I]\hat{T}_o. \tag{19}$$

Thus, if we do not consider the system error generated by the initial state values, the inverse of the outer part, which is a causal realization, will not generate any additional error except acting on the error e_o from the inverse of the inner part and passing it to the output. Because of the finite step computation, $V\hat{V} - I \neq 0$, which generates inversion error within the anti-causal inverse system of the inner factor.

D. Error reduction

In the error analysis of the anti-causal stable inverse of the inner factor part, we have pointed out that for small i, j values, $[V\hat{V}]_{i,j}$ is approaching zero since $l_{A_v} < 1$, and for large i, j values, $[V\hat{V}]_{i,j}$ become non-negligible. This shows that the system error is negligible when k is small and then become larger and larger with k being increased. In order to reduce the non-negligible system error, zeros can be appended to the input sequence. Let the number of zeros appended to the input sequence is K. To show that the inversion error is dependent on the number of zeros appended to the input sequence, we examine the last column of $V\hat{V}$. For $i < (L+1)$,

$$\begin{aligned} [V\hat{V}]_{i,(L+1)} = \; & B_{v,i-1}A_{v,i}\cdots A_{v,L}[C_{v,L+1}\hat{B}_{v,L+1} \\ & + \textstyle\sum_{k=2}^{K} A_{v,L+1}\cdots A_{v,L+k-1}C_{v,L+k} \\ & \hat{B}_{v,L+k}\hat{A}_{v,L+k-1}\cdots\hat{A}_{L+1} - I]\hat{C}_{v,L}, \end{aligned}$$

and

$$\begin{aligned} [V\hat{V}]_{(L+1),(L+1)} = \; & D_{v,L}\hat{D}_{v,L} + B_{v,L}[C_{v,L+1}\hat{B}_{v,L+1} \\ & + \textstyle\sum_{k=2}^{K} A_{v,L+1}\cdots A_{v,L+k-1}C_{v,L+k} \\ & \hat{B}_{v,L+k}\hat{A}_{v,L+k-1}\cdots\hat{A}_{L+1} - I]\hat{C}_{v,L} \end{aligned}$$

For a sufficiently large K, it can be shown that

$$\begin{aligned} & [C_{v,L+1}\hat{B}_{v,L+1} + \textstyle\sum_{k=2}^{K} A_{v,L+1}\cdots A_{v,L+k-1}C_{v,L+k} \\ & \hat{B}_{v,L+k}\hat{A}_{v,L+k-1}\cdots\hat{A}_{L+1}] \to I, \end{aligned}$$

$[V\hat{V}]_{i,(L+1)} \to 0$ and $[V\hat{V}]_{(L+1),(L+1)} \to I$, hence the system error (19) is reduced.

This shows that by appending a sufficiently large number of zeros in the input sequence will reduce the inversion error to some sufficiently small value. Practically, number K can be dependent on the tolerance level of the system error and the total available computational time.

VI. Example

In this section, Example 1 shows the mix-causal inverse system error with zero initial state values. Example 2 is an extension of Example 1, showing the reduced system error by using appended zeros in the input sequence approach.

Example 1. Consider a 2-periodic system with the following state matrices:

for k=0,2,4,...

$$\begin{array}{ll} A_k = \begin{bmatrix} -0.5 & 0.548 \\ 0 & -0.5 \end{bmatrix}, & C_k = \begin{bmatrix} 1 \\ 0.548 \end{bmatrix}, \\ B_k = \begin{bmatrix} 1 & 0.548 \end{bmatrix}, & D_k = 1; \end{array}$$

for k=1,3,5,...

$$\begin{array}{ll} A_k = \begin{bmatrix} 0.5 & 2.333 \\ 0 & 0.333 \end{bmatrix}, & C_k = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ B_k = \begin{bmatrix} 2.5 & 2.333 \end{bmatrix}, & D_k = 1. \end{array}$$

The first step is to obtain the inner and outer factors of the given system using (6). The input signal u is the gaussian random signal with zero mean and variance of 1 with length of 1×500. The output y of the given 2-periodic system is obtained by (10). The output of the anti-causal inverse is given by (16) with y as the input. The reconstructed signal $\hat{u}$ is obtained by (18). All the initial state values are set to be zero. The simulation is done using Matlab program. The input signal, the reconstructed signal and the system error are shown in Fig.1.

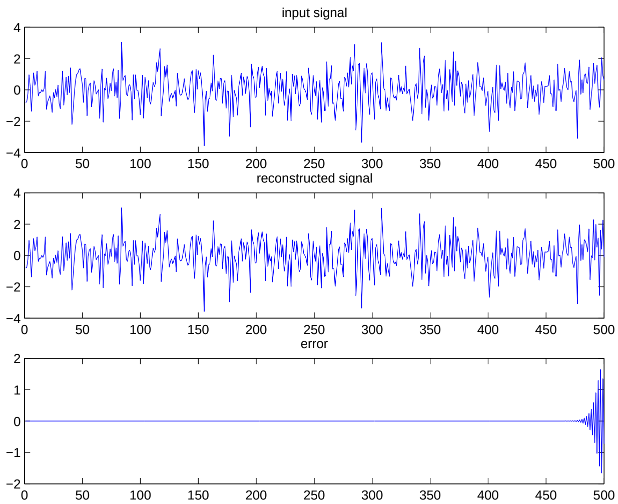

Fig. 1. Input-Output and mix-causal inverse system error

Example 2. This example is to show a result of appending zeros to the input sequence. We append 20 zeros to the last stage of the input sequence of Example 1 and its effect can be seen by comparing the system errors in Fig. 1 and Fig. 2. In Fig.2, it is shown that the error is translated to the extended time period corresponding to the time period of the appended zeros. This practically reduces the inversion error within the time interval of the effective input sequence.

VII. Conclusion

In this paper, we have analyzed the stability of the inverse of the LPTV system with an anti-causal realization. The outer-inner factorization technique is employed to derive the inverse system. The stability of the outer factor and the inverse of the outer factor is analyzed and established using a periodic Lyapunov equation technique. This is further applied to the stability analysis of the inner factor and its inverse, which is realized anti-causally with the state matrices being the Hermitian conjugate of that of the inner factor. Its stability is also shown using the Lyapunov equation technique.

We have also studied the error of the inverse system which contains the anti-causal inverse of the inner factor and the causal inverse of the outer factor. The error of the inverse system is derived in terms of the initial state values and the zero state response. Because of the anti-causal nature of the anti-casual inverse of the inner factor, although $V\hat{V} = I$

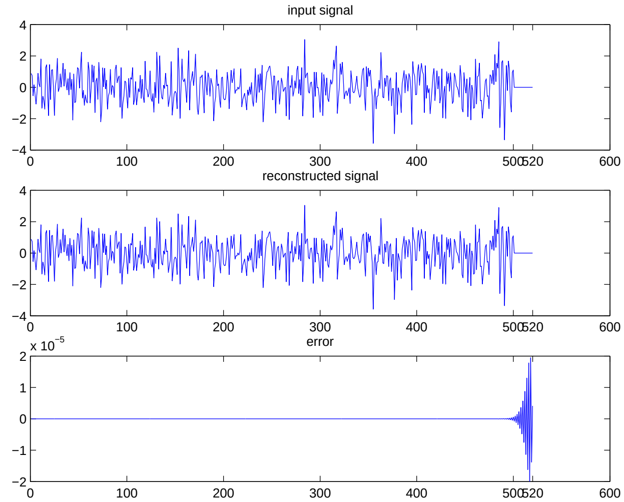

Fig. 2. Input-Output and mix-causal inverse system error with zeros appending

for infinitely long signal computing, $V\hat{V}$ in general is not equal to I for finite length computing of signals. Hence the system inversion error is inevitable by using the anti-causal realization. It is also shown that the inversion error can be reduced by appending zeros to the input sequence.

References

[1] Andras Varga, "Computation of generalized inverse of periodic systems", Proc. of CDC 2004, (Paradise Island, Bahamas), 2004.

[2] Andras Varga, "Computing generalized inverse systems using matrix pencil methods", Int. J. of Applied Mathematics and Computer Science, vol.11, no.8, pp1055-1068, 2001.

[3] C.Zhang and Y.Liao, "A sequentially operated periodic FIR filter for perfect reconstruction", IEEE Trans. on Signal Process, Vol.16, pp.475-786, 1997.

[4] Ching-An Lin and Chwan-Wen King, "Inverting periodic filters", IEEE Transaction on Signal Processing, vol.42, no.1, Jan 1994.

[5] K.Kazlauskas, "Inversion of linear periodic time-varying Digital filters", IEEE Trans.Circuits System II : Analog Digital Signal Process, vol.41, no.2, Feb. 1994.

[6] P.P.Vaidyanathan, Tsuhan Chen, "Role of anticausal inverse in multirate filterbanks part I: system-theoretic fundamentals", IEEE Transactions on Signal Processing, vol.43, no. 5, May 1995.

[7] P.P.Vaidyanathan, Tsuhan Chen, "Role of anticausal inverse in multirate filterbanks part II: the FIR case, factorizations and biorthogonal lappled transforms", IEEE Transactions on Signal Processing, vol.43, no. 5, May 1995.

[8] Jwo-Yuh Wu, Ching-An Lin, Optimal approximate inverse of linear periodic filters, IEEE Transactions on Signal Processing, Vol.52, No.9, Sep. 2004.

[9] P.Dewilde, A.-J. van der Veen, Time-Varying Systems and Computations, Kluwer Academic Publisher, Dordrecht, The Netherlands, 1998.

[10] P.Dewilde, A,-J. van der Veen, Inner-outer factorization and the inversion of locally finite systems of equations, Linear Algebra Application, 313(2000) 53-100.

[11] E.Alijapic, P.Dewilde, Minimal quasi-separable realizations for the inverse of a quasi-separable operator, Linear Algebra Application 414(2006) 445-463.

[12] Huan Zhou, Lihua Xie and Cishen Zhang, A direct approach to H2 optimal deconvolution of periodic digial channels, IEEE Trans. on Signal Processing, Vol.50, No.7, July 2002.

[13] J.J.Hench, A.J.Laub, Numerical solution of the discrete-time periodic Riccati equation, IEEE Trans. on Automatic Control, Vol.39, No.6, June 1994.