# Linear Algebra techniques in stability problems of systems over rings

A. Sáez-Schwedt (Univ. León) and T. Sánchez Giralda (Univ. Valladolid)

***Abstract*— Systems over rings are a generalization of linear control systems, which are used in the study of evolution processes that can be modelled as differential or difference equations. This has lead to the use of Linear Algebra over rings in Control Theory.**

**This paper deals with stability problems for non-necessarily controllable systems. The systematic use of the residual rank of a system allows us to introduce the classes of PS rings, strong CA rings and strong FC rings, which generalize the known classes of PA, CA and FC rings. If a ring $R$ has the property that all unimodular vectors can be completed to invertible matrices, then we prove that systems over $R$ have good stability conditions.**

## I. Introduction and basic definitions

Many physical systems can be described by state- and input- variables and modelled linearly by:

$$x'(t) = Ax(t) + Bu(t) \text{ or } x(t+1) = Ax(t) + Bu(t),$$

where $t$ denote the continuous or discrete time, the state and input vectors $x, u$ belong to some $n$- and $m$- dimensional spaces, and $A, B$ are matrices of size $n \times n$ and $n \times m$, usually with real or complex coefficients in the continuous case, or integer coefficients in the discrete case. Note that the behavior of such a system is completely determined by the pair of matrices $(A, B)$. For example, all states are reachable at finite time starting from the origin and choosing appropriate inputs if and only if the $n \times mn$ matrix $[B|AB|\cdots|A^{n-1}B]$ has rank $n$.

A major theme in Control Theory is the use of linear (static) state feedback $A \mapsto A + BK$ to achieve the stability of the closed-loop system $(A + BK, B)$, for example by obtaining all eigenvalues $\lambda$ of $A+BK$ with negative real part (in the continuous case) or with $|\lambda| < 1$ (in the discrete case). Moreover, the characteristic polynomial of $A$ is not altered by a change of basis in either the input or state space, which correspond to the operations $(A, B) \to (PAP^{-1}, PB)$ and $(A, B) \to (A, BQ)$, for invertible matrices $P, Q$. Taking together changes of basis and feedback we obtain the feedback group acting on the set of all systems $(A, B)$ of fixed sizes $n, m$. The orbit of a system $(A, B)$ is formed by all systems $(PAP^{-1} + PBK, PBQ)$, with $P, Q$ invertible matrices.

Some classes of systems, including delay differential systems, parameter-depending systems and other examples arising in Coding Theory or finite automata, yield linear dynamical systems with coefficients in commutative rings such as complex or real polynomials, $\mathbb{Z}$, finite fields or rings, rings of functions (continuous,analytical), etc. See [1] for a description of the motivation behind the study of systems over rings.

Let $R$ be a commutative ring with 1. An $m$-input, $n$-dimensional system (or a system of size $(n, m)$) over $R$ will be a pair of matrices $(A, B)$, with $A \in R^{n\times n}$ and $B \in R^{n\times m}$. The residual rank of the system $(A, B)$, denoted by $\mathrm{res.rk}(A, B)$, is defined as the reduced rank of the reachability matrix $A * B = [B|AB|\cdots|A^{n-1}B]$, i. e. $\mathrm{res.rk}(A, B) = \max\{i : \mathcal{U}_i(A * B) = R\}$, where $\mathcal{U}_i(A * B)$ denotes the ideal of $R$ generated by the $i \times i$ minors of the matrix $A * B$, with the convention $\mathcal{U}_0(A * B) = R$. Recall that $\mathcal{U}_i(A * B) \supseteq \mathcal{U}_{i+1}(A * B)$ for all $i$ (see [12] for properties of the ideals of minors). The system $(A, B)$ is reachable or controllable if and only if $\mathrm{res.rk}(A, B) = n$, and $\mathrm{res.rk}(A, B) \geq 1$ if and only if $\mathcal{U}_1(B) = R$, i.e. $B$ has unit content.

The following stability conditions are classically defined for systems: feedback cyclization (FC), also known as Heymann's Lemma; coefficient assignability (CA); and pole assignability (PA). It is well known that FC $\Rightarrow$ CA $\Rightarrow$ PA, and these properties necessarily imply reachability, as proved by Sontag in [15].

In [8] (resp. [14]), the class of PS rings (resp. strong CA rings) is introduced as those rings for which every system is pole assignable (resp. coefficient assignable) in the following sense: if $r = \mathrm{res.rk}(A, B)$ and $f(x) = (x - x_1)\cdots(x - x_r)$, where $x_1, \ldots, x_r$ are arbitrary scalars from $R$ (resp. $f(x)$ is a monic polynomial of degree $r$ over $R$), there exists a matrix $K$ over $R$ such that $\chi(A + BK)$, the characteristic polynomial of $A + BK$, is a multiple of $f(x)$.

Similarly, a strong FC ring is defined in [13] as a ring $R$ such that any system $(A, B)$ over $R$ is feedback cyclizable: there exist a matrix $K$ and a vector $u$ with coefficients in $R$ such that $\mathrm{res.rk}(A + BK, Bu) = \mathrm{res.rk}(A, B)$.

It is immediate that PS rings are PA rings, strong CA rings are CA rings and strong FC rings are FC rings, just take $r = n$ in the previous definitions. Also, strong FC rings are strong CA rings, which are PS rings, generalizing the known result for the reachable case. The following commutative rings are strong FC rings: fields, local and semilocal rings, local-global rings, rings with many units, zero-dimensional rings, in particular von Neumann regular rings and Artinian rings, and Bezout domains with stable range 1, including certain rings of analytical functions.

The paper is organized as follows. In section II, we define the residue rank of a system, which allows us to extend to arbitrary systems all forms of stability usually studied for reachable systems. The main theorem of this section proves that the classical implications FC $\Rightarrow$ CA $\Rightarrow$ PA also hold for the strong classes of rings previously defined.

In section III, we prove that the above implications are strict, and we exhibit a large class of commutative rings for which the classical forms of PA, CA and FC are equivalent to the strong form of the corresponding properties.

Finally, section IV contains some concluding remarks, as well as some comments on the 'dynamic' versions of the pole assignability, coefficient assignability and feedback cyclization properties.

## II. Stability properties and canonical forms

Let $\Sigma = (A, B)$ be a system over a commutative ring $R$, and denote by $Max(R)$ the set of all maximal ideals of $R$. For every $\mathfrak{m} \in Max(R)$, let $\Sigma(\mathfrak{m}) = (A(\mathfrak{m}), B(\mathfrak{m}))$ denote the system over the quotient field $R/\mathfrak{m}$ obtained from $\Sigma$ by reducing all entries modulo $\mathfrak{m}$. The reachability matrix of the system $\Sigma(\mathfrak{m})$ is

$$A(\mathfrak{m}) * B(\mathfrak{m}) = [B(\mathfrak{m})|A(\mathfrak{m})B(\mathfrak{m})|\cdots|A(\mathfrak{m})^{n-1}B(\mathfrak{m})]$$

Since $\mathcal{U}_r(A*B) = R$ if and only if $\mathrm{rank}(A(\mathfrak{m}) * B(\mathfrak{m}) \geq r$ for all maximal ideals $\mathfrak{m}$, we have that

$$\mathrm{res.rk}(A, B) = \min\{\mathrm{rank}(A(\mathfrak{m}) * B(\mathfrak{m}) : \mathfrak{m} \in Max(R)\}$$

**Remark 1:** The residual rank of a linear system is invariant under feedback, a fact that we will use systematically throughout the paper: indeed, if $(A', B')$ is feedback equivalent to $(A, B)$, we have that $(A', B') = (PAP^{-1} + PBK, PBQ)$, for matrices $P, Q, K$ with $P, Q$ invertible. A long but straightforward matricial calculation shows that there is an isomorphism between the $R$-modules $\mathrm{Im}(A'*B')$ and $\mathrm{Im}(A*B)$, then for every maximal ideal $\mathfrak{m}$ of $R$ we have the equality $\mathrm{rank}(A'(\mathfrak{m}) * B'(\mathfrak{m})) = \mathrm{rank}(A(\mathfrak{m}) * B(\mathfrak{m}))$, and hence one has $\mathrm{res.rk}(A', B') = \mathrm{res.rk}(A, B)$.

An important consequence of this fact is that all the stability properties studied are invariant under the feedback action.

**Proposition 2:** Let $(A, B)$ and $(A', B')$ be two equivalent systems over a ring $R$, then $(A, B)$ is pole assignable (resp. coefficient assignable) (resp. feedback cyclizable) iff $(A', B')$ is pole assignable (resp. coefficient assignable) (resp. feedback cyclizable).

**Proof.** Let us prove the coefficient assignability case. Let $r = \mathrm{res.rk}(A, B) = \mathrm{res.rk}(A', B')$, and take matrices $P, K_1, Q$ ($P, Q$ invertible) such that $A' = PAP^{-1} + PBK_1$ and $B' = PBQ$. Suppose that $(A', B')$ is coefficient assignable, and let $f(x)$ be any monic polynomial of degree $r$. Thus, there exists a matrix $K'$ such that $\chi(A' + B'K')$ is divisible by $f(x)$. Substituting and operating, we have that $A' + B'K' = P(A + B(K_1P + QK'P))P^{-1}$, which has the same characteristic polynomial as $A + B(K_1P + QK'P)$, hence we have found a matrix $K = K_1P + QK'P$ such that $\chi(A + BK)$ is divisible by $f(x)$, i.e. $(A, B)$ is coefficient assignable. This also proves the pole assignability case.

Similarly, if $(A', B')$ is feedback cyclizable, there exist a matrix $K'$ and a vector $u'$ such that $\mathrm{res.rk}(A' + B'K', B'u') = r$. Operating as before, $A' + B'K' = P(A + B(K_1P + QK'P))P^{-1}$ and $B'u' = PBQu'$. This system has the same residual rank as the equivalent system $(A + B(K_1P + QK'P), BQu')$, which is of the form $(A + BK, Bu)$, for $K = K_1P + QK'P$ and $u = Qu'$. $\square$

It is a classical result (see [7]) that working with reachable systems one has that FC $\Rightarrow$ CA $\Rightarrow$ PA. As one would expect, the same implications hold for the strong form of these properties, which is the main result of this section. Before proving this fact, we need some auxiliary results.

**Remark 3:** Let $R$ be an Hermite ring in the sense of Lam: stably-free $R$-modules are free, or equivalently, unimodular vectors can be completed to invertible matrices. Consider a single-input $n$-dimensional system $(A, b)$ over an Hermite ring with $\mathrm{res.rk}(A, b) \geq r$. In [14], an almost canonical form is obtained for such systems, by using a block decomposition. We include here an alternative proof, which illustrates most of the algebraic and linear algebraic techniques used in this paper.

First, we claim that $\mathcal{U}_r(N_r) = R$, where $N_r$ is the $n \times r$ matrix $[b|Ab|\cdots|A^{r-1}b]$. This is clear if $r = 1$: if the all the entries of $b$ were in some maximal ideal $\mathfrak{m}$ of $R$, the same would be true for the matrix $N_n = A * B$ and one would have $\mathrm{res.rk}(A, B) = 0$. Now suppose $r > 1$. Since $\mathrm{res.rk}(A, b) \geq r > r - 1$, by induction on $r$ we may assume that $\mathcal{U}_{r-1}(N_{r-1}) = R$. We will derive a contradiction if $\mathcal{U}_r(N_r)$ is contained in some maximal ideal $\mathfrak{m}$. For each $i$, consider the $R/\mathfrak{m}$ vector space $\mathcal{N}_i = \mathrm{im}(N_i(\mathfrak{m}))$, so that $\mathcal{N}_i \subseteq \mathcal{N}_{i+1}$ for all $i$. Since $dim\mathcal{N}_{r-1} \geq r - 1$ and $dim\mathcal{N}_r < r$, we must have $\mathcal{N}_{r-1} = \mathcal{N}_r$, both with dimension $r - 1$. But the maps:

$$\begin{array}{rcl} \varphi_i : \mathcal{N}_i/\mathcal{N}_{i-1} & \longrightarrow & \mathcal{N}_{i+1}/\mathcal{N}_i \\ x + \mathcal{N}_{i-1} & \longmapsto & Ax + \mathcal{N}_i \end{array}$$

are surjective for all $i$ (see [11]), hence $\mathcal{N}_{r-1},\ldots,\mathcal{N}_n$ have all dimension $r-1$, which is impossible because $\mathrm{res.rk}(A, b) \geq r$ means that $N_n(\mathfrak{m}) = A(\mathfrak{m})*B(\mathfrak{m})$ has rank $\geq r$ over $R/\mathfrak{m}$. This proves the claim.

Now, $\mathcal{U}_r(N_r) = R$ means that $N_r$ is left-invertible, so that the exact sequence of $R$-modules

$$0 \to R^r \overset{N_r}{\to} R^n \to R^n/\mathrm{im}(N_r) \to 0$$

is split. If we denote by $\mathcal{M}_r = R^n/\mathrm{im}(N_r)$, we have $R^n \cong \mathrm{im}(N_r) \oplus \mathcal{M}_r$ and $\mathrm{im}(N_r) \cong R^r$. Since stably-free modules over Hermite rings are free, $\mathcal{M}_r$ must be free, and hence there exists an $n \times n$ invertible matrix $P^{-1} = [x_1|\cdots|x_n]$ whose first $r$ column vectors are those of $N_r$, i.e. $Pb = x_1$,$Ax_1 = x_2$, …, $Ax_{r-1} = x_r$, therefore the system $(PAP^{-1}, Pb)$ is of the form

$$\left( \left[\begin{array}{ccccc|ccc} 0 & 0 & \cdots & 0 & * & * & \cdots & * \\ 1 & 0 & \cdots & 0 & * & * & \cdots & * \\ \vdots & 1 & \ddots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & * & * & \cdots & * \\ \hline 0 & 0 & \cdots & 0 & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & * & * & \cdots & * \end{array}\right], \left[\begin{array}{c} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{array}\right] \right),$$

where the first block of $PAP^{-1}$ is $r \times r$. For each $i$ descending from $r$ to 2, we can use the 1 in position $(i, i-1)$ of $PAP^{-1}$ to clean the row $i$ as follows: subtract from columns $i, \ldots, n$ appropriate multiples of column $i-1$, which is done by right multiplication by a suitable matrix $P_i^{-1}$. Then, left multiplication by $P_i$ consits of adding to the row $i-1$ multiples of the rows $i, \ldots, n$, which affects only the positions $i-1, \ldots, n$ (not the 1 in position $i-2$) and has no effect on $Pb$. After this process, we have an equivalent system

$$(P_2 \cdots P_r PAP^{-1} P_r^{-1} \cdots P_2^{-1}, P_2 \cdots P_r Pb).$$

Denote this system by $(\tilde{P}A\tilde{P}^{-1}, \tilde{P}b)$, and note that $(\tilde{P}A\tilde{P}^{-1}$ has no $*$'s in the rows $2, \ldots, r$. As $\tilde{P}b = Pb$ is the first basic vector of $R^n$, there exists a feedback matrix $\tilde{K}$ such that $\tilde{P}A\tilde{P}^{-1} + \tilde{P}b\tilde{K}$ has zeros in the first row. Thus, we have arrived at a system $(\tilde{A}, \tilde{b}) = (\tilde{P}A\tilde{P}^{-1} + \tilde{P}b\tilde{K}, \tilde{P}b)$ in which the upper-left $r \times r$ block of $\tilde{A}$, together with the upper $r \times 1$ block of $\tilde{P}b$, form a reachable system which is in the control canonical form of [7, Lemma 2.4].

Now we can state and prove the implications among the studied stability properties.

**Theorem 4:** Strong FC rings are strong CA rings, and strong CA rings are PS rings.

**Proof.** It is immediate that strong CA rings are PS rings. Now, let $R$ be a strong FC ring and let $(A, B)$ be a system over $R$ with $\mathrm{res.rk}(A, B) = r$, and let $f(x) = x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0$ be a monic polynomial of degree $r$ over $R$. By the strong FC property, there exist $K_1, u$ with $\mathrm{res.rk}(A + BK_1, Bu) = r$. Note that strong FC rings are Hermite: they are FC rings, and hence GCU rings [2], a property which by [3] implies $R$ is an Hermite ring. By Proposition 2, we can suppose that the single-input system $(A + BK_1, Bu)$ is in the reduced form given in Remark 3. Therefore, it is immediate to find a matrix $K_2$ such that $A + BK_1 + BuK_2$ is of the form

$$\left[\begin{array}{ccccc|c} -a_{r-1} & \cdots & \cdots & -a_1 & -a_0 & \\ 1 & 0 & \cdots & 0 & 0 & \\ 0 & 1 & \ddots & \vdots & \vdots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 & \\ 0 & 0 & \cdots & 1 & 0 & \\ \hline \multicolumn{5}{c|}{A_2} & A_3 \end{array}\right],$$

with characteristic polynomial $f(x)\chi(A_3)$. Thus, we have found a matrix $K = K_1 + uK_2$ with $\chi(A + BK)$ divisible by $f(x)$, hence $R$ is a strong CA ring. $\square$

**Remark 5:** As in Remark 3, we will derive a normal form for a multi-input system $(A, B)$ with $\mathrm{res.rk}(A, B) \geq r$ over a strong FC ring $R$ (cf. [13]). By the strong FC property, there exist $K, u$ such that $\mathrm{res.rk}(A + BK, Bu) \geq r$. Since strong FC rings are Hermite, there exist matrices $\tilde{P}, \tilde{K}$ such that the single-input system

$$(\tilde{P}(A + BK)\tilde{P}^{-1} + \tilde{P}Bu\tilde{K}, \tilde{P}Bu)$$

is in the reduced form of Remark 3. We can now complete $u$ to an invertible matrix $Q = [u|*]$, such that the first column of $\tilde{P}BQ$ is the first basic column vector of $R^n$. By elementary column operations (right multiplication by an invertible matrix), we may assume that $\tilde{P}BQ$ has its first row $(1, 0 \cdots 0)$. Finally, $(A, B)$ is equivalent to the system

$$(\tilde{P}(A + BK)\tilde{P}^{-1} + \tilde{P}Bu\tilde{K}, \tilde{P}BQ)$$

in the following normalized form with $r$ 1's:

$$\left( \left[\begin{array}{ccccc|ccc} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & 1 & \ddots & \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & 0 & \cdots & 0 & * & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & * & * & \cdots & * \end{array}\right], \left[\begin{array}{c|c} 1 & 0 \cdots 0 \\ 0 & * \cdots * \\ \vdots & \vdots \\ \vdots & \vdots \\ 0 & * \cdots * \\ \hline 0 & * \cdots * \\ \vdots & \vdots \\ 0 & * \cdots * \end{array}\right] \right).$$

## III. Examples

Before describing the content of this section we need to recall some known definitions. A matrix $B$ over $R$ is good if there exists $A$ with $(A, B)$ reachable. $R$ is a GCU ring if given a good matrix $B$ there exists a vector $u$ such that $Bu$ is unimodular (its entries generate $R$). GCU means "Good Contains Unimodular". If there exists $u$ with $Bu$ unimodular for any $B$ with unit content, then $R$ is called a UCU ring, where UCU stands for "Unit-content Contains Unimodular". This property, which implies the GCU property, was introduced in [7] and was named UCU in [2] and $BCU$ in [5]. Examples of UCU rings include principal ideal domains.

It can be seen that UCU rings belong to the class of rings for which the usual forms of the PA, CA and FC properties are equivalent to the strong forms, as we will see in the next proposition, whose part (iii) was conjectured at the end of [13].

**Proposition 6:** Let $R$ be a UCU ring, then the following conditions hold:

(i) $R$ is a PA ring iff $R$ is a PS ring.

(ii) $R$ is a CA ring iff $R$ is a strong CA ring.

(iii) $R$ is an FC ring iff $R$ is a strong FC ring.

**Proof.** It is proved in [14]. $\square$

With the previous result and recalling some known examples, we are able to show that the implications given in Theorem 4 are strict.

In [7] it is proved that $\mathbb{Z}$ is a PA ring but not a CA ring. Being a principal ideal domain, $\mathbb{Z}$ is a UCU ring and hence by the above proposition it is a PS ring and not a strong CA ring.

On the other side, in [6] it is shown how to construct a polynomial ring $R = k[y]$ which is a CA ring but not an FC ring: take $k$ algebraically closed with characteristic different from zero. Since $R$ is a UCU ring, $R$ is a strong CA ring but not a strong FC ring, by the above proposition.

At this point, one can give examples of rings which satisfy all the stability properties. We recall that a ring $R$ is a local-global ring if every polinomial admitting unit values locally, admits unit values.

**Proposition 7:** The following commutative rings are strong FC rings and hence also strong CA rings and PS rings:

*(i)* Local-global rings,
*(ii)* Fields, local rings and semi-local rings,
*(iii)* Rings with many units,
*(iv)* Zero-dimensional rings, in particular von Neumann regular rings and artinian rings.
*(v)* Bezout domains with stable range 1, in particular the ring $H(\Omega)$ of holomorphic functions on a noncompact Riemann surface.

**Proof.** (i) In [2, Prop. 3], it is shown that local-global rings are FC rings. Also, local-global rings are UCU rings (see [5, p.68], where UCU is called BCU). By Proposition 6, local-global rings are strong FC rings.

(ii),(iii) and (iv) All these rings are local-global: it is immediate that fields, local rings and semi-local rings are local-global. For rings with many units and zero-dimensional rings, see [12]. More examples of local-global rings are given in [2, p.282].

(v) A Bezout domain $R$ with stable range 1 is an elementary divisor domain, therefore it is a UCU ring by [7] and an FC ring by [3]. Now, by Proposition 6, $R$ is a strong FC ring. The validity of the example $H(\Omega)$ is given in [10]. $\square$

## IV. Conclusions

The results contained in this paper are a new and up to date contribution to the problems of fixing the existence of canonical forms and stating the validity of stability conditions such as Heymann's lemma for systems over commutative rings, which are open questions.

We believe that the systematic use of the residual rank of a system is a very useful tool which can be exploited to further study in the above open problems and to generalize known results from reachable systems to arbitrary systems, by using induction techniques.

Finally, we want to remark that the stability conditions that we have studied are 'static', in the sense that there exists also a 'dynamic' approach to the same properties. See [1] for a discussion of the dynamic pole assignment, the dynamic coefficient assignment and the dynamic feedback cyclization problems for systems over rings.

Let $(A, B)$ be a system of size $(n, m)$ over a ring $R$ with $\text{res.rk}(A, B) = r$, and suppose that one can in some sense detect or isolate a reachable part $(A_1, B_1)$ as in section II:

$$A = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

where $A_1 \in R^{r\times r}$, $A_2 \in R^{n-r\times r}$, $A_3 \in R^{n-r\times n-r}$, $B_1 \in R^{r\times m}$ and $B_2 \in R^{n-r\times m}$, with $(A_1, B_1)$ reachable.

It is easy to see that the following system, augmented from $(A, B)$:

$$\tilde{A} = \begin{bmatrix} 0_s & 0 & 0 \\ 0 & A_1 & 0 \\ 0 & A_2 & A_3 \end{bmatrix}, \tilde{B} = \begin{bmatrix} I_s & 0 \\ 0 & B_1 \\ 0 & B_2 \end{bmatrix}$$

has residual rank $r + s$ and is feedback cyclizable, where $s$ is the size of the augmentation necessary for the reachable system $(A_1, B_1)$ to be cyclizable (see [1]). It is an open problem to determine the minimum size of augmentation of a reachable system needed to obtain a feedback cyclizable system.

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