Reliable Beam Modeling in Control Problems

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The plane controlled motions of a homogeneous rectilinear elastic beam are considered. One end of the beam is free, and the other is clamped on a truck moved in a horizontal direction. In the undeformed state, the beam is fixed in a vertical position. The control action on the beam is the truck acceleration u. Initially, the shape of the beam lateral displacements w and its relative linear momentum density p are given in a reference frame tied to the truck and moved at the velocity v. The truck position is specified by x in an inertial coordinate system, $\dot{x} = v$ and $\dot{v} = u$. It is supposed that the coordinate and velocity of the truck are initially zero.

The equations of beam motions have the form

$$\dot{p} + m'' = -\rho u(t),\ y \in (0,l), \tag{1}$$

$$p = \rho \dot{w}, \quad m = EIw'', \quad t \in (0,T), \tag{2}$$

under the boundary conditions

$$w(t,0) = w'(t,0) = 0, \quad m(t,l) = m'(t,l) = 0; \tag{3}$$

and the initial conditions

$$w(0,y) = f(y), \quad p(0,y) = g(y). \tag{4}$$

Here, m is the bending moment in the beam cross section; l and ρ are the length and linear density of the beam, respectively; EI is its flexural rigidity; and T is the terminal time of the control process. The dotted symbols denote the partial derivatives with respect to t, and the primed symbols stand for the partial derivatives with respect to y.

The problem is to find an optimal control $u(t)$ that moves the truck from its initial to terminal states in the given time T

$$x(T) = x^f, \quad v(T) = v^f, \tag{5}$$

and minimizes a objective function $J[u]$ in the class U of admissible controls:

$$J[u] \to \min_{u \in U}. \tag{6}$$

To solve the initial-boundary value problem (1)–(4), we apply the method of integrodifferential relations (MIDR), described in [1–5]. In this case, it is possible to reduce problem (1)–(4) to a variational problem. If a weak solution p^*, m^*, and w^* exists then the following functional Φ under local constraints (1), (3), (4) reaches its absolute minimum on this solution

$$\Phi(p^*, m^*, w^*) = \min_{p,m,w} \Phi(p, m, w) = 0, \quad \Phi = \int_0^T \int_0^l \varphi(p, m, w) dy dt, \\ \varphi = \frac{(p - \rho\dot{w})^2}{2\rho} + \frac{(m - EIw'')^2}{2EI}. \tag{7}$$

Note that the integrand φ in (7) has the dimension of the energy density and is nonnegative. Hence, the corresponding integral is nonnegative for any arbitrary functions p, m, and w ($\Phi \geq 0$).

To find an approximate solution of the optimization problem defined by Eqs. (1), (3)–(7) a polynomial representation of the unknown functions is applied. The functions p, m, and w are approximated by bivariate polynomials

$$\tilde{p} = \sum_{i+j=0}^{N_p} p_{ij} t^i x^j, \quad \tilde{m} = \sum_{i+j=0}^{N_m} m_{ij} t^i x^j, \quad \tilde{w} = \sum_{i+j=0}^{N_w} w_{ij} t^i x^j; \tag{8}$$

The control u is restricted to a set of time polynomials

$$U = \left\{ u: \quad u = \sum_{i=0}^{N_u} u_i t^i \right\}. \tag{9}$$

Here p_{ij}, m_{ij}, w_{ij}, and u_i are unknown real coefficients.

The basis functions are chosen so that the approximations can exactly satisfy the boundary conditions (3), initial polynomial conditions (4), and the equation of motion (1) by suitably selected integers N_p, N_m, N_w, and N_u.

The resulting finite-dimensional unconstrained minimization problem (7) has a solution $\tilde{p}^*(t, y, u)$, $\tilde{m}^*(t, y, u)$, $\tilde{w}^*(t, y, u)$ for an arbitrary control $u \in U$. The optimal control $u^*(t)$ is determined from condition (6). The total mechanical energy of the beam at the terminal time T is considered as a functional $J[u]$

$$J = \int_0^l \eta(T, y) dy, \quad \eta(t, y) = \frac{\rho\dot{w}^2}{2} + \frac{EI(w'')^2}{2}. \tag{10}$$

The corresponding optimization problem is reduced to a system of linear equations.

An optimal control for the beam motion has been analytically constructed for a number of integers $N_p \leq 20$, $N_m \leq 21$, and $N_w \leq 23$ in (8). In the case when $N_u > 1$, the control u in (9) contains $N_u - 1$ unknown parameters, which have been used for minimizing J. The optimal controls obtained by the MIDR for $N_u \leq 5$ are presented.

The value of the functional Φ is considered as an integral quality criterion for the optimal solution whereas the integrand φ in (7) is a local error characteristic. It is shown that as the number of free parameters of the polynomial control in the optimization problem (1), (3)–(7) increases, the total energy of the beam at the terminal time reduces considerably.

References

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