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QUANTUM CAPACITY FOR A COMPOUND CHANNEL WITH MEMORY

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Abstract
We study a quantum compound Markov chain memory channel, addressing a specific two-state case, combining an amplitude damping channel and a dephasing channel. The sender's knowledge of this channel affects the behavior of its coherent information. We study the quantum capacity of this channel, comparing the case in which transmitter doesn't have any information about the channel in the current use and the case of an informed user. This is done by maximizing coherent information respect to the input of amplitude damping channel and depolarizing channel. We also propose a Hamiltonian for this channel which well describes our model in certain limit conditions, thus substantiating it from the physical point of view. Moreover the Hamiltonian model allows for generalization of the compound channel. This work offers a practical and suitable for applications case-study which demonstrates that uninformed user case represents the "worse case" scenario.

Key words
Nanotechnologies, Quantum information, Stochastic systems

1 Introduction

Quantum mechanics offers new and attractive perspectives for information processing and transmission [Benenti, Casati and Strini 2004] [Benenti, Casati and Strini 2007]. A large scale quantum computer, if constructed, would advance computing power much beyond the capabilities of classical computation, while quantum cryptography allows a provable secure data exchange. However, due to the unavoidable coupling of any quantum system to its environment, decoherence effects appear. This introduces noise, thus disturbing the programmed quantum coherent evolution. The decoherence problem is conveniently formulated in terms of quantum operations. Given the initial state ρ of a quantum system Q and an overall unitary evolution $\mathcal{U}$ of system plus environment, the final state $\rho^{'}$ is obtained after tracing over the environment degrees of freedom: $\rho^{'} = \mathcal{E}(\rho) = Tr_E[\mathcal{U}(\rho \otimes w_0)\mathcal{U}^{\dagger}]$, where w_0 is the initial state of the environment (we assume that initially the system and the environment are not entangled). The map $\mathcal{E}$ is known as a quantum operation or a superoperator. It is interesting to consider $\mathcal{E}$ as a quantum channel. This approach encompasses both noisy propagation in time and in space. In the first case, the map $\mathcal{E}$ describes the evolution from time t_i to time t_f of some piece of quantum hardware, ρ and $\rho^{'}$ being the system states at t_i and t_f , respectively. In the latter, the quantum system Q plays the role of information carrier in a two-party communication scenario: ρ is the quantum state at the entrance of the communication channel and $\rho^{'}$ the output state, corrupted by noise effects described by the quantum operation $\mathcal{E}$. Fundamental quantities characterizing a quantum channel are the *classical* and the *quantum channel capacities*, that are defined as the maximum number of bits/qubits that can be reliably transmitted per channel use. We focus on quantum capacity, which is related to the transmission of quantum information. Usually quantum channels are assumed to be memoryless, that is, the effect of the environment on the input states is always the same for each channel use. In other words, there is no memory in the interaction between carriers and environment: the quantum operation for N channel uses is given by $\mathcal{E}_N(\rho^{(N)}) = \mathcal{E}^{\otimes N}(\rho^{(N)})$, where $\rho^{(N)}$ is the density matrix which describes the quantum source in input to the N channel uses. However, in several physically relevant situations this is not a realistic assumption. Memory effects appear when the characteristic time scales for the environment dynamics are longer than the time between consecutive channel uses. For instance, solid state implementations of quantum hardware, which are the most promising for their scalability and integrability, suffer from low frequency noise. In optical fibers, memory effects may appear due to slow birefringence fluctuations. This introduces correlation among channel uses, i.e., the effect of the environment on one carrier depends on the past interactions between the environment itself and the other carriers. This kind of channels are known as memory channels [Macchiavello and Palma 2002] [Plenio and Virmani 2007] [D'Arrigo, Benenti and Falci 2007].

This paper is organized as follows. We first recall the concept of quantum capacity for a quantum channel.Then we introduce two kind of memoryless channel, namely dephasing channel and amplitude damping channel, discussing about their quantum capacity. We use these results to study a compound channel made up of a dephasing channel plus an amplitude damping channel showing the behavior of its quantum capacity. Finally we give a Hamiltonian model which can describe this specific compound channel, showing the possible physical relevance of this model.

2 Quantum capacity

The *Quantum capacity* $\mathcal{Q}$ [Barnum, Nielsen and Schumacher 1998] refers to the coherent transmission of quantum information; it is the maximum number of qubit that can be reliably transmitted per channel use, in the limit of an infinite number of uses. The value of $\mathcal{Q}$ can be computed, for memoryless channels, as

$$\mathcal{Q} = \lim_{N\to\infty} \frac{\mathcal{Q}_N}{N} \qquad (1)$$
$$\mathcal{Q}_N = \max_\rho I_c(\mathcal{E}_N, \rho^{(N)})$$
$$I_c(\mathcal{E}, \rho) = S(\mathcal{E}(\rho)) - S_e(\rho)$$

Here $S(\rho) = -\mathrm{Tr}\big(\rho \log_2(\rho)\big) = -\sum_i \lambda_i \log_2(\lambda_i)$ is the Von Neumann entropy, $S_e(\rho)$ is the *entropy exchange* [Barnum, Nielsen and Schumacher 1998], which measures the noise introduced by the channel. The quantity $I_c(\mathcal{E}, \rho)$ is called *coherent information* and must be maximized over *all input states* ρ.

The limit $N \to \infty$ in 1 makes the evaluation of $\mathcal{Q}$ difficult. On the other hand this regularization is necessary, since in general I_c is not subadditive. If the channel is degradable [Devetak and Shor 2005], the regularization is not necessary: infact in this case the coherent information is subbaditive and therefore the quantum capacity is given by the 'single letter' formula $\mathcal{Q} = \mathcal{Q}_1 = \max_\rho I_c(\mathcal{E}, \rho)$

3 Amplitude damping channel and dephasing channel

In this section we introduce the behavior of dephasing channel and amplitude damping channel and the expression for their coherent information quantifiers.

3.1 Dephasing channel

This class of quantum channels models those systems in which relaxation times are much longer than dephasing ones. They are characterized by the property that when N qubits are sent through the channel, the states of a preferential orthonormal basis are transmitted without errors. Therefore, dephasing channels are noiseless from the viewpoint of the transmission of classical information, since the states of the preferential basis can be used to encode classical information. Of course superpositions of basis states may decohere, thus corrupting the transmission of quantum information. For instance a dephasing channel can be modelled by the Hamiltonian [D'Arrigo, Benenti and Falci 2007]

$$\mathcal{H}(t) = \mathcal{H}_S + \mathcal{H}_E - \frac{1}{2} X_E F(t) \qquad (2)$$
$$\mathcal{H}_S = \sum_{k=1}^{N} \frac{\Omega}{2} \sigma_z^{(k)}$$
$$F(t) = \lambda \sum_{k=1}^{N} \sigma_z^{(k)} f_k(t)$$

where $\mathcal{H}_S$ describes a series of identical qubits, $\mathcal{H}_E$ is the Hamiltonian of Markovian environment, and X_E is an enviroment operator. The qubits in an orderly way interact with the environment by the functions $f_k(t)$, which switchs on and off the interaction between the k-th qubit and the environment. In this case the coupling between the qubits and the environment is longitudinal so that the Hamiltonian (2) properly describes a dephasing channel.

The input state density matrix can be parametrized as:

$$\rho = \begin{pmatrix} 1-p & r \\ r^* & p \end{pmatrix} \qquad (3)$$

where p is the population of the qubit excited state, and r are the qubit coherences. The effect of a dephasing channel results in an output state [Nielsen and Chuang, 2000; Benenti, Casati and Strini 2007]:

$$\rho' = \mathcal{E}_{deph}(\rho) = \begin{pmatrix} 1-p & \beta r^* \\ \beta r & p \end{pmatrix} \qquad (4)$$

where β describes the decay of qubit coherences. This channel is degradable [Devetak and Shor 2005], this implies that in (1) the limit $N \to \infty$ isn't necessary and equation (4) is sufficient to calculate $\mathcal{Q}$. Since it has been shown that coherent information is maximized by mixed input states [Wolf and Garcia 2003], we let $r = 0$ and maximize respect to the population p. The corresponding coherent information is given by:

$$I_c(\mathcal{E}_{deph}(\rho)) = \tag{5}$$
$$= H_2\Big(\frac{1}{2}\big[1 - \sqrt{(1-2p)^2}\big]\Big) +$$
$$-H_2\Big(\frac{1}{2}\big[1 - \sqrt{\beta^2 + (1-\beta^2)(1-2p)^2}\big]\Big)$$

where $H_2(p) = -p\log_2 p - (1-p)\log_2(1-p)$ is the binary entropy. It can be found that (5) reaches its maximum for $p = 1/2$, then:

$$\mathcal{Q}_{deph} = 1 - H_2\Big(\frac{1-\beta}{2}\Big) \tag{6}$$

3.2 Amplitude damping channel

This channel describes the effect of a Markovian environment transversally coupled to the qubit system, in the small temperature limit ($K_B T \ll \Omega$). This situation reproduces the noisy dynamics of solid state qubit at optimal working point, due to high feqrency noise comopenents, as noise from control circuitry [Ithier et al. 2005]. The Hamiltonian model is the same of the one in equantion (2), but in this case the system operator describing the coupling between the k-th qubit and the environment has to be replcaced by $\sigma_x^{(k)}$.
The effect of an amplitude damping channel on the input state (3) is described as follows [Nielsen and Chuang, 2000; Benenti, Casati and Strini 2007]:

$$\mathcal{E}_{AD}(\rho) = \begin{pmatrix} 1-\eta p & \sqrt{\eta}r^* \\ \sqrt{\eta}r & \eta p \end{pmatrix} \tag{7}$$

where η is a decay factor affetting the population of the excited state, and so it describes the channel noise level. Also the amplitude damping channel is degradable [Giovannetti and Fazio 2005], then single letter formula can be used to calculate the quantum capacity. Maximizing the coherent information with respect to the input state, it turns out that maximum is reached for mixed states ($r = 0$); for these states we have [Giovannetti and Fazio 2005]:

$$I_c(\mathcal{E}_{AD}(\rho)) = \tag{8}$$
$$= H_2\Big(\frac{1}{2}\big[1 - \sqrt{(1-2\eta p)^2}\big]\Big)$$
$$-H_2\Big(\frac{1}{2}\big[1 - \sqrt{[1-2(1-\eta)p]^2}\big]\Big)$$

It follows that quantum capacity of an amplitude damping channel is

$$\mathcal{Q}_{AD} = \max_p I_c(\mathcal{E}_{AD}(\rho)) \tag{9}$$

In this case the optmization has to be numerically performed, and the value which optmizes the (9) depends on the channel parameter η.

4 Markov chain memory channel

A Markov chain memory channel is a class of memory channel characterized by the map:

$$\mathcal{E}^{(N)}(\rho^{(N)}) = \tag{10}$$
$$\sum_{i_1,\ldots,i_N} q(i_N|i_{N-1})\ldots q(i_2|i_1)\cdot q(i_1)\cdot$$
$$\cdot\big(\mathcal{E}_{i_1}\otimes\cdots\otimes\mathcal{E}_{i_N}\big)(\rho^{(N)})$$

This mathematical model describes a behaviour of a channel in which each use is correlated with the previous one in a such way that it doesn't depend from all past uses but only from the previous one. Here $j \in \{1,...,N\}$ represent the j-th use of the channel and $q(i_j|i_{j-1})$ is the conditional probability which correlates two successive uses.

5 Quantum compound channels

This channel class describes a multiple-map channel in which the transmitter and the receiver are uninformed about the behavior of the channel: at the first use, when Alice sends a quantum state through the channel, she doesn't know which map will be applied to the input state; this uncertainty is described by the probability distribution $q_1, q_2, ..., q_N$. For successive uses of the channel, assuming that $\mathcal{E}_i$ is the map applied at the first use, then $\mathcal{E}_i$ will be applied to all successive input states. Quantum compound channels can be considered as a limiting case of Markov chain memory quantum channels, in which there aren't transitions between different states, but only from a state to itself. Their behavior is described by the following map:

$$\mathcal{E}^{(N)}(\rho^{(N)}) = \sum_{i=1}^{N} q_i\mathcal{E}_i^{\otimes N}\big(\rho^{(N)}\big) \tag{11}$$

Several scenarios of utilization of this kind of channel have been studied: *uninformed transmitter, informed transmitter, informed receiver* [Bjelacović, Boche and Nötzel 2008]. We focus on the first one, considering the case in which neither the the transmitter nor the receiver know what channel they actually use to communicate. Indeed, the main assumption commonly made about a communication channel is that both transmitter and receiver know apriori the channel which their information carrier will be sent through. But sometimes this hypothesis cannot be satisfied: transmitter and receiver know only a set of possible map that could be applied to the information carrier, in a non-deterministic way.
In this case the quantum capacity of the channel is

given by the expression [Bjelacović, Boche and Nötzel 2008]:

$$\mathcal{Q}(\mathcal{J}) = \lim_{N\to\infty} \max_{\rho\in\mathcal{S}(\mathcal{H}^{\otimes N})} \left[\min_i I_c(\rho, \mathcal{E}_i^{(\otimes N)}) \right] \quad (12)$$

where $\mathcal{J} = \{\mathcal{E}_1, \mathcal{E}_2, \ldots \mathcal{E}_i, \ldots\}$ is our compound channel, and i is an index that discriminates a channel inside the set of the compound channel.

Alice's lack of knowledge implies worse perfomance in the communication task. Indeed if Alice knew apriori the map that will be applied to the message, she could choose an encoding maximizing the coherent information of this map. This means that maximization is made a priori. Otherwise, in the case of uninformed transmitter scenario, none tells Alice which map will be applied, and she has first to consider the channel that worstly transmit an arbitratry input state: she makes a minimization on the channel; then she can make the best encoding choice: the maximization on the input states.

5.1 A specific model

We propose a simple kind of compund channel, for which is possible to study its quantum capacity utilizing the results of [Bjelacović, Boche and Nötzel 2008]. This consists of a quantum compound channel made up of an amplitude damping channel and a dephasing channel.

Another specific model of quantum compound channel which has been studied [Dorlas and Morgan 2008] in the past consists of an amplitude damping channel and a dephasing channel and the classical capacity was found [Datta and Dorlas 2007]. This offers an example which shows that the classical capacity of a quantum compound channel cannot -in general- be simplified to the expression valid in the informed transmitter scenario.

Here we show that quantum capacity cannot be simplified with the expression for the informed transmitter scenario.

In figure 1 we plot the coherent informations of several amplitude damping channels and dephasing channels, for different values of their respective noise paprameters η and β; we utilized equations (5) and (7), so we have yet partially maximized respect to the input state ($r = 0$), and curves only depends on the qubit excited state population. First of all we can note that both quantities are monotonic respect to the damping parameters of the channels. As a consequence of this, if we consider a quantum compound channel model made only of some amplitude damping channels (or dephasing channels), the minimization in equation (12) is translated into a minimization respect to η (or β). Therefore, we can conclude that this minimization doesn't depend on the input state distribution, hence maximization on the input states and minimization over the set of the maps describing the channel can be inverted.

In general this doesn't hold when different type of channel are considered. Now let's consider a compound channel made up of a dephasing channel and an amplitude damping channel: $\mathcal{J} = \{\mathcal{E}_{deph}, \mathcal{E}_{AD}\}$, choosing specific values of the parameters β and η. In Figure 2 we plot their coherent informations as functions of qubit excited population p. We can clearly observe that curves of coherent informations of the choosen amplitude damping channel and dephasing channel intersect. This results in a non-monotonical behavior of this specific type of compound channel, so that minimization and the maximization operation in equation (12) cannot be inverted. The value of quantum capacity can be found numerically solving the equation (12).

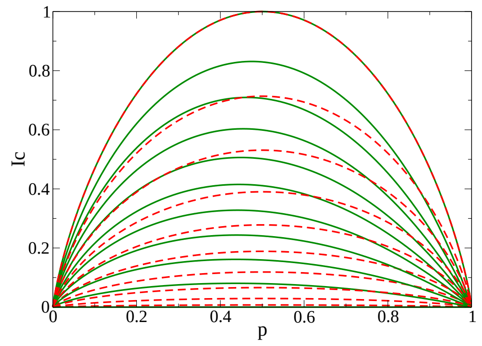

Figure 1. Coherent information vs population of input state, of some amplitude damping channels (green straight line) and dephasing channels (red dashed line), for $\eta \in \{0.5, 0.55, ..., 1\}$ and $\beta \in \{0, 0.1, ..., 1\}$ respectively.

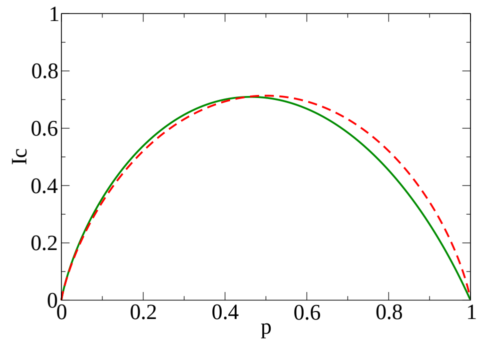

Figure 2. Coherent information of an amplitude damping channel (green straight line, $\eta = 0.9$) and a dephasing channel (red dashed line, $\beta = 0.9$).

6 Hamiltonian model

We propose a Hamiltonian model for the compound channel introduced in the previous section. This model may describe a semiconductor-based memory cell. The cell is coupled to an impurity, which has bistable behavior with a characteristic time $1/\gamma$. The compound channel $\mathcal{J} = \{\mathcal{E}_{deph}, \mathcal{E}_{AD}\}$ can be described by the following Hamiltonian:

$$\begin{aligned}\mathcal{H}_{compound} &= \mathcal{H} + \delta\mathcal{H} \\ \delta\mathcal{H} &= -\frac{v}{4}\sigma_x(1-\tau_z) + \gamma^*\tau_+ + \gamma\tau_- \end{aligned} \quad (13)$$

in which $\mathcal{H}$ is the Hamiltonian in (2). We assume that $v \gg \Omega$ and $t_{use} \ll 1/\gamma$, this last condition ensuring that during the time t_{use} in which we use the channel we can neglect the dynamics of the impurity, i.e. the impurity doesn't change its state. When the expectation value of the impurity operator τ_z is 1, the coupling between each qubit and the environment is longitudinal so that the Hamiltonian (13) describes a dephasing channel. Otherwise, when this expectation value is -1, due to the fact that $v \gg \Omega$, the impurity drastically changes the qubit working point so that the coupling between each qubit and the environment turns to be transverse: in this case the Hamiltonian (13) can effectively describe an amplitude damping channel.

7 Conclusions

The dominant mechanism for quantum capacity of our compound channel model depends on initial encoding: the transmitter has a given knowledge of the system which lays him to make a proper choice on the way to encode his message; if he knows a priori the map that will be applied to the system he can make an apriori maximization over the input states. Otherwise he has to firstly consider the worse case for all the input states, and then choose the maximizing source.

We speculate that the model we study is interesting for applications and propose a Hamiltonian model which relates to a physical system already studied [Paladino et al. 2002; Falci et al. 2005]. Notice that removing the limit condition $t_{use} \ll 1/\gamma$ in the Hamiltonian allows to make calculation in different regimes, corresponding to more general error models.

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