ROBUST OUTPUT ONE AHEAD MODEL PREDICTIVE CONTROL DESIGN

Vojtech Veselý and Danica Rosinová [1]

Slovak University of Technology, Faculty of Electrical Engineering and IT, URPI, Bratislava, Slovak Republic, E-mail : vojtech.vesely@stuba.sk

Abstract: The paper addresses the problem of designing a parameter dependent quadratic stability output/state feedback model predictive control for linear polytopic systems without constraints.

Keywords: Model predictive control, Robust Control, Parameter dependent quadratic stability, Lyapunov function, Polytopic system

1. INTRODUCTION

Model predictive control (MPC) has attracted notable attention in control of dynamic systems. The idea of MPC can be summarized as follows, (Camacho and Bordons, 2004; Maciejovski, 2002, Rositer, 2003):

- Predict the future behaviour of the process state/output over the finite time horizon.
- Compute the future input signals on line at each step by minimizing a cost function under inequality constraints on the manipulated (control) and/or controlled variables.
- Apply on the controlled plant only the first of vector control variable and repeat the previous step with new measured input/state/output variables.

Therefore, the presence of the plant model is a necessary condition for the development of the predictive control. The success of MPC depends on the degree of precision of the plant model. In the most references the principal shortcoming of existing MPC-based control techniques is their inability to explicitly incorporate plant model uncertainty, Kothare et al, 1996. Thus, the present state of robustness problem in MPC can be summarized as follows:

Analysis of robustness properties of MPC.

Zafiriou nad Marchal, 1991 have used the contraction properties of MPC to developed necessary-sufficient conditions for robust stability of MPC with input and output constraints for SISO systems and impulse response model. Polak and Yang, 1993 have analyzed robust stability of MPC using a contraction constraint on the state.

MPC with explicit uncertainty description.

Zheng and Morari, 1993, have presented robust MPC schemes for SISO FIR plants, given uncertainty bounds on the impulse response coefficients. Some MPC consider additive type of uncertainty, de la Pena et al, 2005 or parametric (structured) type uncertainty using CARIMA model and linear matrix inequality, Bouzouita et al, 2007. In Lovaas et al, 2007 for open-loop stable systems having input constraints the unstructured uncertainty is used. The robust stability can be established by choosing the large value for the control input weighting matrix R in the cost function. The authors proposed a new less conservative stability test for determining a sufficiently large control penalty R using bilinear matrix inequality (BMI). The other technique- constrained tightening to design of robust MPC have been proposed in Kuwata et al, 2007. Above approaches are based on idea of increasing the robustness

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of the controller by tightening the constraints of the predicted states. The mixed H_2/H_∞ control approach to design of MPC has been proposed by Orukpe et al, 2007. Robust constrained MPC using linear matrix inequality (LMI) have been proposed by Kothare et al,1996 where the polytopic model or structured feedback uncertainty model have been proposed. The main idea of Kothare et al, 1996 is used of infinite horizon control laws which for state feedback guarantee nominal stability.

In this paper the neccessary and sufficient robust stability conditions for MPC described as the polytopic system with output feedback with one step ahead horizon have been developed for generalized parameter dependent Lyapunov matrix $P(\alpha)$. The proposed robust MPC ensures parameter dependent quadratic stability (PDQS) and guaranteed cost. The developed neccessary and sufficient robust stability conditions for concrete parameter dependent Lyapunov function reduces to sufficient ones and for robust stability analysis of MPC they are in the form of the set of LMIs. For robust MPC design which guarantes PDQS with guaranted cost the developed necessary and sufficient robust stability conditions for concrete parameter dependent Lyapunov function reduces to sufficient ones with bilinear matrix inequality.

The paper is organized as follows: Section 2 present a preliminaries and problem formulation. In Section 3 the main results are given and finally, in Section 4 the simple example using Yalmip BMI solvers shows the effectiveness of the proposed method.

2. PROBLEM FORMULATION AND PRELIMINARIES

We are given a time invariant linear discrete-time system

$$x(t+1) = A(\alpha)x(t) + B(\alpha)u(t) \qquad (1)$$
$$y(t) = Cx(t)$$

where $x(t) \in R^n, u(t) \in R^m, y(t) \in R^l$ are state, control and output variables of the system, respectively; $A(\alpha), B(\alpha)$ belongs to the convex set

$$S = \{A(\alpha) \in R^{n\times n}, B(\alpha) \in R^{n\times m}\} \qquad (2)$$
$$\{A(\alpha) = \sum_{j=1}^{N} A_j\alpha_j \quad B(\alpha) = \sum_{j=1}^{N} B_j\alpha_j, \alpha_j \geq 0\}$$
$$j = 1, 2...N, \sum_{j=1}^{N} \alpha_j = 1$$

Matrix C is known matrix of corresponding dimension. Consider the region of the complex plain defined by

$$D = \{z \in C : R_{11} + R_{12}z + R_{12}^T z^* +\} \qquad (3)$$
$$\{+R_{22}zz^* < 0\}$$

where $R_{11} = R_{11}^T \in R^{d\times d}$ and $R_{22} = R_{22}^T \in R^{d\times d}$ are submatrices of matrix R_0 such that

$$R_0 = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix} \qquad (4)$$

and d is called the order of region. It is assumed that R_{22} is positive semidefinite (definite) matrix, for detail see Peaucelle et al, 2000. Typical region in stability analysis of discrete-time system is the unitary disk centered at the origin by the following choice of R_0

$$R_0 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (5)$$

Lemma 1

Closed-loop system matrix of discrete-time system (1) is robustly D-stable if and only if there exists a symmetric positive definite parameter dependent Lyapunov matrix $P(\alpha)$ such that

$$R_{11}P(\alpha) + R_{22}A_m(\alpha)^T P(\alpha)A_m(\alpha) \leq 0 \qquad (6)$$

where $A_m(\alpha)$ is the closed-loop system matrix for system (1).

The problem studied in this paper is to derive a parameter dependent quadratic stability conditions for output one step ahead robust model predictive controller when control algorithm is given as folows

$$u(t) = F_1(y(t) - w(t)) + \qquad (7)$$
$$F_2(y(t+1) - w(t+1))$$

A cost function to be minimized is given as follows

$$J = \sum_{t=0}^{\infty} J(t) \qquad (8)$$

where

$$J(t) = (y(t) - w(t))^T Q_1(y(t) - w(t)) +$$
$$(y(t+1) - w(t+1))^T S_1(y(t+1) - w(t+1)) +$$
$$u(t)^T Ru(t)$$

and Q_1, S_1, R are positive definite matrices of corresponding dimensions.

Definition 1

Consider the system (1). If there exists a control algorithm $u(t)^*$ and a positive scalar J^* such that the closed-loop system for (1) and (7) is stable and closed-loop value cost function (8) satisfies $J \leq J^*$, then J^* is said to be guaranteed cost and $u(t)^*$ is said to be guaranteed cost control law for

the system (1).
Substituting control algorithm (7) to (1) one obtains

$$x(t+1) = A_m(\alpha)x(t) - \quad (9)$$
$$B_m(\alpha)(F_1 w(t) + F_2 w(t+1))$$

vhere

$$A_m(\alpha) = (I - B(\alpha)F_2 C)^{-1} A_c(\alpha)$$

,

$$B_m(\alpha) = (I - B(\alpha)F_2 C)^{-1} B(\alpha)$$

and $A_c(\alpha) = A(\alpha) + B(\alpha)F_1 C$. Because the vectors $w(t), w(t+1)$ are independent from vector$x(t)$ and if vectors $w(t), w(t+1)$ belong to the class of L_2 stability and robustness properties of closed-loop system (9) are determined by the closed-loop system matrix of $A_m(\alpha)$. The origin of the state vector $x(t)$ has to be recalculated to a new steady state given by the setpoint vectors $w(t), w(t+1)$. Due to Lyapunov function approach and new recalculated state vector origin below we assume that $w(t) = w(t+1) = 0$.
Lemma 2
Consider the system (1) with control algorithm (7). The control algorithm (7) is the guaranteed cost control law for the closed-loop system if and only if the following condition holds

$$B_e = A_m(\alpha)^T P(\alpha) A_m(\alpha) - P(\alpha) + Q + \quad (10)$$
$$A_m(\alpha)^T S A_m(\alpha) +$$
$$(F_1 C + F_2 C A_m(\alpha))^T R (F_1 C + F_2 C A_m(\alpha)) \leq 0$$

where $Q = C^T Q_1 C, S = C^T S_1 C$.

3. MAIN RESULTS

Main results of this paper can be summarized in the following theorem.
Theorem 1.
The closed -loop system (9) is parameter dependent quadratically stable with parameter dependent Lyapunov function $V(t) = x(t)^T P(\alpha) x(t)$ if and only if there exits matrices N_1, N_2, F_1, F_2 such that the following bilinear matrix inequality holds.

$$B_e = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \leq 0 \quad (11)$$

where

$$M_{11} = N_1^T A_c(\alpha) + A_c(\alpha)^T N_1 - P(\alpha) +$$
$$Q + C^T F_1^T R F_1 C$$
$$M_{12} = N_1^T M_c(\alpha) - A_c(\alpha)^T N_2 + C^T F_1^T R F_2 C$$
$$M_{22} = -N_2^T M_c(\alpha) - M_c(\alpha)^T N_2 + S +$$
$$C^T F_2^T R F_2 C$$
$$M_c(\alpha) = B(\alpha) F_2 C - I$$

Note that (11) is affine with respected to α. Substituting (2) and $P(\alpha) = \sum_{i=1}^N \alpha_i P_i$ to (11) for the polytopic system the following BMI is obtained

$$B_e = \begin{bmatrix} M_{11i} & M_{12i} \\ M_{12i}^T & M_{22i} \end{bmatrix} \leq 0 \quad i = 1, 2, ...N \quad (12)$$

where

$$M_{11i} = N_1^T A_{ci} + A_{ci}^T N_1 - P_i +$$
$$Q + C^T F_1^T R F_1 C$$
$$M_{12i} = N_1^T M_{ci} - A_{ci}^T N_2 + C^T F_1^T R F_2 C$$
$$M_{22i} = -N_2^T M_{ci} - M_{ci}^T N_2 + S + C^T F_2^T R F_2 C$$
$$M_{ci} = B_i F_2 C - I \quad A_{ci} = A_i + B_i F_1 C$$

If the solution of (12) is feasible with respect to symmetric matrices $P_i = P_i^T > 0, i = 1, 2...N$,and matrices N_1, N_2, the gain matrices F_1, F_2 guarantee for one ahead predictive control closed-loop system (9) the guaranteed cost and parameter dependent quadratic stability within the convex set defined by (2).
Note that:

- BMI robust stability conditions "if and only if" in (11) for concrete matrix $P(\alpha) = \sum_{i=1}^N \alpha_i P_i$ reduces in (12) to BMI conditions " if".
- If in (12) $P_i = P_j = P, i \neq j = 1, 2...N$ the feasible solution of (12) with respect to matrices N_1, N_2, and symmetric positive definite matrix P the gain matrices F_1, F_2 guarantee the guaranteed cost and quadratic stability within the convex set defined by (2) for one ahead predictive control closed-loop system (9).

4. EXAMPLE

Due to experiments for the linear affine type discrete-time uncertain system one obtaines model as follows

$$A = \bar{A}_0 + \sum_{i=1}^{p} \theta_i \bar{A}_i \quad (13)$$

$$B = \bar{B}_0 + \sum_{i=1}^{p} \theta_i \bar{B}_i$$

where $\bar{A}_0, \bar{A}_1 ... \bar{B}_0, \bar{B}_1 ...$ are constant matrices of appropriate dimensions, $\theta = [\theta_1 ... \theta_p] \in R^p$ is a vector of uncertain and possibly time varying real parameters with $\theta_i \in <\underline{\theta_i}, \overline{\theta_i}>, i = 1, 2, ...p$, p is the number of uncertainties. When one substitutes to (13) for $i = 1, 2...p$ lower or upper bound of uncertainties, the polytopic system (2) is obtained, where $N = 2^p$.

Let we are given the following matrices

$$\bar{A_0} = \begin{bmatrix} .1 & .4 \\ .2 & .5 \end{bmatrix} \bar{A_1} = \begin{bmatrix} .003 & .01 \\ .003 & .001 \end{bmatrix}$$

$$\bar{B_0} = \begin{bmatrix} .1 & 1 \\ .9 & .1 \end{bmatrix} \bar{B_1} = \begin{bmatrix} .001 & .01 \\ .1 & .0001 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In this example two vertices are calculated. Using YALMIP, Lőfberg, 2004 with GLPK solver, Kvasnica et al, 2004, for the input data $R = I, Q = 10I, S = 10I, ro = 500(\lambda_{max}(Ljapunovmatrix) < ro)$, the following results for quadratic stability and guaranteed cost are obtained.

Gain matrices F_1 and F_2

$$F_1 = \begin{bmatrix} 2.3477 & .4834 \\ 60.41969 & 6.2783 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} 3.7655 & 14.6986 \\ 136.7213 & 240.3946 \end{bmatrix}$$

Maximal eigenvalue of two vertex matrices of closed-loop MPC is given as follows: $Maxeig = 0.2311$, guaranted cost is determined by $\lambda_{max}(P) = 489.047$ where P is the Lyapunov matrix. The existence of $P = P^T > 0$ guarantees the closed-loop quadratic stability and guaranteed cost.

For the above input data and parameter dependent quadratic stability (two Lyapunov matrices) and guaranteed cost the following results are obtained.

Gain matrices F_1 and F_2

$$F_1 = \begin{bmatrix} 7.604 & 1.0654 \\ 1.8944 & 3.1749 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -.4357 & 167.5213 \\ 123.7151 & 72.5922 \end{bmatrix}$$

$Maxeig = 0.03581$, $\lambda_{max}(P_1, P_2) = 487.4061$. The feasible solution or existence of two symmetric and positive definite matrices P_1, P_2 guarantees the closed-loop parameter dependent quadratic stability and guaranteed cost. The feasible solution of bilinear matrix inequality conditions have been obtained by YALMIP with GLPK solver.

5. CONCLUSION

The paper addresses the problem of designing a parameter dependent quadratic stability static output/state feedback one step ahead model predictive control for linear polytopic systems without constraints. The new robust stability conditions for one step ahead model predictive control are given in *Theorem 1.* The feasible solution of BMI has been obtained by Yalmip with GLPK solver.

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