The intrinsic hypoelliptic Laplacian and its heat kernel on unimodular Lie groups

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Abstract

We present an invariant definition of the hypoelliptic Laplacian on sub-Riemannian structures with constant growth vector using the Popp's volume form introduced by Montgomery. This definition generalizes the one of the Laplace-Beltrami operator in Riemannian geometry. In the case of left-invariant problems on unimodular Lie groups we prove that it coincides with the usual sum of squares.

We then extend a method (first used by Hulanicki on the Heisenberg group) to compute explicitly the kernel of the hypoelliptic heat equation on any unimodular Lie group of type I. The main tool is the noncommutative Fourier transform. We then study some relevant cases: $SU(2)$, $SO(3)$, $SL(2)$ (with the metrics inherited by the Killing form), and the group $SE(2)$ of rototranslations of the plane.

The study of the properties of the heat kernel in a sub-Riemannian manifold drew an increasing attention since the pioneer work of Hörmander [16].

Since then, many estimates and properties of the kernel in terms of the sub-Riemannian distance have been provided (see [2, 4, 6, 7, 10, 19, 23, 28, 31] and references therein).

In most cases the hypoelliptic Laplacian appearing in the heat equation is the the sum of squares of the vector field forming an orthonormal frame for the sub-Riemannian structure. In other cases it is built as the divergence of the horizontal gradient, where the divergence is defined using any $\mathcal{C}^\infty$ volume form on the manifold (see for instance [30]).

The Laplacians obtained in these ways are not intrinsic in the sense that they do not depend only on the sub-Riemannian distance. Indeed, when the Laplacian is built as the sum of squares, it depends on the choice of the orthonormal frame, while when it is defined as divergence of the horizontal gradient, it depends on the choice of the volume form.

The first question we address in this paper is the definition of an invariant hypoelliptic Laplacian. As far as we know, this question has been raised for the first time in a paper by Brockett [3]. Many details can be found in Montgomery's book [25].

To define the intrinsic hypoelliptic Laplacian, we proceed as in Riemannian geometry. In Riemannian geometry the invariant Laplacian (called the Laplace-Beltrami operator) is defined as the divergence of the gradient where the gradient is obtained via the Riemannian metric and the divergence via the Riemannian volume form.

In sub-Riemannian geometry, we define the invariant hypoelliptic Laplacian as the divergence of the horizontal gradient. The horizontal gradient of a function is the natural generalization of the gradient in Riemannian geometry and it is a vector field belonging to the distribution. The divergence is computed with respect to the sub-Riemannian volume form, that can be defined for every sub-Riemannian structure with constant growth vector. This definition depends only on the sub-Riemannian structure. The sub-Riemannian volume form, called the Popp's measure, was first introduced in Montgomery's book [25], where its relation with the Hausdorff measure is also discussed. The definition of the sub-Riemannian volume form is simple in the 3D contact case, and a bit more delicate in general.

We then prove that for the wide class of unimodular Lie groups (i.e. the groups where the right- and left-Haar measures coincide) the hypoelliptic Laplacian is the sum of squares for any choice of a left-invariant orthonormal base. We recall that all compact and all nilpotent Lie groups are unimodular.

In the second part of the paper, we present a method to compute explicitly the kernel of the hypoelliptic heat equation on a wide class of left-invariant sub-Riemannian structures on Lie groups. We then apply this method to the most important 3D Lie groups: $SU(2)$, $SO(3)$, and $SL(2)$ with the metric defined by the Killing form, the Heisenberg group H_2, and the group of rototranslations of the plane $SE(2)$. These groups are unimodular, hence the hypoelliptic Laplacian is the sum of squares. The interest in studying $SU(2)$, $SO(3)$ and $SL(2)$ comes from some recent results of the authors. Indeed, in [5] the complete description of the cut and conjugate loci for these groups was obtained. These results, together with those presented in this paper, open new perspectives for the clarification of the relation between the presence of the cut locus and the properties of the heat kernel, in line with the result of Neel and Strook [26] in Riemannian geometry. Up to now the only case in which both the cut locus and the heat kernel has been known explicitly was the Heisenberg group [11, 12, 17].[1]

The interest in the hypoelliptic heat kernel on $SE(2)$ comes from a model of human vision. It was recognized in [8, 27] that the visual cortex V1 solves a nonisotropic diffusion problem on the group $SE(2)$ while reconstructing a partially hidden or corrupted image. The study of the cut locus on $SE(2)$ is a work in progress. Preliminary results can be found in [24].

The method is based upon the generalized (noncommutative) Fourier transform (GFT, for short), that permits to disintegrate[2] a function from a Lie group G to $\mathbb{R}$ on its components on (the class of) non-equivalent unitary irreducible representations of G. This technique permits to transform the hypoelliptic heat equation into an equation in the dual of the group[3], that is particularly simple since the GFT disintegrate the right-regular representations and the hypoelliptic Laplacian is built with left-invariant vector fields (to which a one parameter group of right-translations is associated).

Unless we are in the abelian case, the dual of a Lie group in general is not a group. In the compact case it is a so called Tannaka category [14, 15] and it is a discrete set. In the nilpotent case it has the structure of $\mathbb{R}^n$ for some n. In the general case it can have a quite complicated structure. However, under certain hypotheses, it is a measure space if endowed with the so called Plancherel measure. Roughly speaking, the GFT is an isometry between $L^2(G,\mathbb{C})$ (the set of complex-valued square integrable functions over G, with respect to the Haar measure) and the set of Hilbert-Schmidt operators with respect to the Plancherel measure.

The difficulties of applying our method in specific cases rely mostly on two points:

i) computing the tools for the GFT, i.e. the non-equivalent irreducible representations of the group and the Plancherel measure. This is a difficult problem in general: however, for certain classes of Lie groups there are suitable techniques (for instance the Kirillov orbit method for nilpotent Lie groups [21], or methods for semidirect products). For the groups disccussed in this paper, the sets of non-equivalent irreducible representations (and hence the GFT) are well known (see for instance [29]);

ii) finding the spectrum of an operator (the GFT of the hypoelliptic Laplacian). Depending on the structure of the group and on its dimension, this problem gives rise to a matrix equation, an ODE or a PDE.

Then one can express the kernel of the hypoelliptic heat equation in terms of eigenfunctions of the GFT of the hypoelliptic Laplacian, or in terms of the kernel of the transformed equation.

For the cases treated in this paper, we have the following (the symbol $\amalg$ means disjoint union):

[1]The Heisenberg group is in a sense a very degenerate example. For instance, in this case the cut locus coincides globally with the conjugate locus (set of points where geodesics lose local optimality) and many properties that one expects to be distinct for more generic situations cannot be distinguished. The application of our method to the Heisenberg group H_2 provides in a few lines the Gaveau-Hulanicki formula [11, 17].

[2]One could also say decompose (possibly continuously).

[3] In this paper, by the dual of the group, we mean the support of the Plancherel measure on the set of non-equivalent unitary irreducible representations of G; we thus ignore the singular representations.

Group	Dual of the group	GFT of the hypoelliptic Laplacian	Eigenfunctions of the GFT of the hypoelliptic Laplacian
H_2	$\mathbb{R}$	$\frac{d^2}{dx^2} - \lambda^2 x^2$ (quantum Harmonic oscillator)	Hermite polynomials
$SU(2)$	$\mathbb{N}$	Linear finite dimensional operator related to the quantum angular momentum	Complex homogeneous polynomials in two variables
$SO(3)$	$\mathbb{N}$	Linear finite dimensional operator related to orbital quantum angular momentum	Spherical harmonics
$SL(2)$	$\mathbb{R}^+ \amalg \mathbb{R}^+ \amalg \mathbb{N} \amalg \mathbb{N}$	Continuous: Linear operator on analytic functions with domain $\{\lvert x\rvert = 1\} \subset \mathbb{C}$ Discrete: Linear operator on analytic functions with domain $\{\lvert x\rvert < 1\} \subset \mathbb{C}$	Complex monomials
$SE(2)$	$\mathbb{R}^+$	$\frac{d^2}{d\theta^2} - \lambda^2 \cos^2(\theta)$ (Mathieu's equation)	2π-periodic Mathieu functions

The idea of using the GFT to compute the hypoelliptic heat kernel is not new: it was already used on the Heisenberg group in [17] at the same time as the Gaveau formula was published in [11], and on all step 2 nilpotent Lie groups in [9, 1]. See also the related work [22].

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