Taming Explosive Growth in Complex Networks

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Abstract— In this paper I review results from my research group, published recently in *Scientific Reports (Nature)* 4, 4308, on taming explosive growth in spatially extended systems. Specifically we consider collections of relaxation oscillators, which are relevant to modelling physical, biological and engineering phenomena, under different coupling topologies. We observe that the system witnesses unbounded growth under regular connections on a ring, for sufficiently high coupling strengths. However, when some fraction of the regular connections are rewired to random links, this blow-up is quenched. These results indicate a new direction in controlling blow-ups in complex systems. Lastly, for the case of stochastic switching of links we find scaling relations between the fraction of random links and the link rewiring probability.

I. INTRODUCTION

A prototypical example of self-sustained oscillations in nonlinear systems is the Van der Pol oscillator [1], governed by the second-order differential equation:

$$\ddot{x} + \mu (x^2 - 1)\dot{x} + x = 0 \tag{1}$$

where x is the dynamical variable and μ is a parameter determining the nature of the dynamics. Relaxation oscillations arise in this system for $\mu > 0$, with the limit cycles displaying sudden discontinuous jumps. Such a system is very relevant in modelling phenomena such as heart activity, neurononal spiking [3] and seismology [4]. In this extended abstract we will focus on the dynamics of networks of such oscillators and present a summary of our key observations, illustrated by representative results from the work in my group. The content here is adapted from Refs. [6], [7].

II. TIME VARYING NETWORK OF RELAXATION OSCILLATORS

Consider a generic network constituted of nonlinear dynamical elements at the nodes and a coupling term modeling the interaction between the elements. The local dynamics at each node of the network is given by $\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X})$, where **X** is a *m*-dimensional state vector of the dynamical variables and $\mathbf{F}(\mathbf{X})$ is a typically nonlinear velocity field. So the dynamics of such a system is given by following evolution equation:

$$\dot{\mathbf{X}}_{i} = \mathbf{F}(\mathbf{X}_{i}) + \varepsilon \sum_{j=1}^{N} J_{ij} \mathbf{H}(\mathbf{X}_{i}, \mathbf{X}_{j}), \qquad i = 1, ..., N,$$
(2)

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where J_{ij} are the elements of a connectivity matrix. The coupling strength is given by ε and $\mathbf{H}(\mathbf{X}_i, \mathbf{X}_j)$ is the coupling function determined by the nature of interactions between dynamical elements *i* and *j*.

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We first review our results obtained for the case of coupled Van der Pol oscillators [1], characterized by nonlinear damping, and governed by the second-order differential equation given by Eqn. 1. We take $\mu > 0$ which yields relaxation oscillations characterized by slow asymptotic behavior and sudden discontinuous jumps. Associating $\dot{x} = y$ gives:

$$\dot{x} = f(x, y) = y
\dot{y} = g(x, y) = -\mu(x^2 - 1)y - x$$
(3)

Now we consider a ring of N nonlinearly coupled Vander-Pol oscillators [8]-[10], namely a specific form of Eqn. 2 with m = 2, $\mathbf{X}_i = \{x_i, y_i\}$, $\mathbf{F} = \{f, g\}$ and $\mathbf{H}(\mathbf{X}_i, \mathbf{X}_j) = \{f(x_j, y_j) - f(x_i, y_i), g(x_j, y_j) - g(x_i, y_i)\}$ given as follows:

$$\begin{aligned} \dot{x}_i &= f(x_i, y_i) + \\ & \frac{\varepsilon}{2} [f(x_{i-1}, y_{i-1}) + f(x_{i+1}, y_{i+1}) - 2f(x_i, y_i)] \\ \dot{y}_i &= g(x_i, y_i) + \\ & \frac{\varepsilon}{2} [g(x_{i-1}, y_{i-1}) + g(x_{i+1}, y_{i+1}) - 2g(x_i, y_i)] (4) \end{aligned}$$

Here index i specifies the site/node in the ring, with the nodal on-site dynamics being a Van der Pol relaxation oscillator. Note that this form of coupling has been explored insufficiently in existing literature.

Starting with this regular ring, we consider increasingly random networks formed as follows: we begin with a regular lattice, such as the ring of regular nearest neighbour interactions mentioned above, and then with probability pwe replace ("rewire") the regular links with random connections. So when p is non-zero, random non-local connections exist along-side regular local links, and such networks have widespread relevance [11]. Note that our network preserves degree, namely the number of links for each oscillator remains the same under rewiring.

Additionally we consider the scenario of *time-varying links* or *dynamic links*. Here the set of connections that get rewired to random nodes changes from time to time. So the underlying web of links switches over time [12], [13], [14]. Such time varying connections are widely prevalent, especially where the system responds to environmental influences or internal adaptations [15], [16].

We explored two different methods for changing the connectivity of the network. The first algorithm involves the *periodic switching of links in the network*. Here all the links in the network switch periodically, preserving the qualitative nature of the connectivity matrix. We denote the time-scale of network change by r. So larger r implies infrequent network

changes, with the limit $r \to \infty$ corresponding to the standard static case.

Periodic switching of connections may occur in situations where the links are determined by some global external periodic influence. However a more realistic scenario is a probabilistic model of link switching, such as in [14], [17]. So we introduce a second algorithm where the links *switch* randomly and asynchronously in time. Here each node *i* has probability p_r of changing its links in a certain time interval τ . If the links of a particular node are selected for rewiring, they change such that with probability *p* they connect to random nodes and with probability (1 - p) they connect to nearest neighbors.

In this algorithm, the nodes change their links *independently* and *stochastically*. The probability of the new links being random or regular is determined by p as in the first algorithm (and as in the standard small-world scenario). So, while in the first algorithm link changes occur globally throughout the network, in the second algorithm uncorrelated changes occur at the local level.

We obtained the shapes and sizes of the limit cycles arising in this network, for system sizes ranging from N = 10 to $N = 10^3$, under varying fraction of random links $p \ (0 \le p \le 1)$. We investigated a large range of network switching time periods $r \ (0.01 \le r \le 1)$ for the case of periodically switched networks, and $0 \le p_r \le 1$ for the case of stochastically switched networks (with $\tau = 0.001$). We review below the central results obtained from our extensive numerical simulations as reported in [6], [7].

Spatiotemporal Patterns in a Regular Ring :

First we describe our principal observations for coupling on a ring with two nearest neighbours (i.e. p = 0) under increasing coupling strengths.

For very weak coupling the system shows no regular spatiotemporal pattern. As coupling is increased, regular travelling wave-like behaviour develops. As coupling gets stronger, and approaches a critical value ε_c , the regularity of the pattern breaks up. Figs. 1-3 display representative examples of these spatiotemporal patterns. When the coupling exceeds the critical value ($\varepsilon > \varepsilon_c$), the system experiences a *blow-up*, namely the amplitude of the oscillations grows in an unbounded manner. For instance, the amplitude typically grows from O(1) to $O(10^4)$ in a time interval as short as $\sim 10^{-3}$.

Effect of increasing coupling range:

To further explore the mechanism behind this kind of unbounded growth, we studied long range interactions where each node interacted with k nearest neighbhors (where $1 < k \le N$). We found that long range interactions stabilized the network and the explosive growth was suppressed more efficiently with increasing k. Further it appeared that for low coupling strengths the minimal number of neighbours necessary for preventing blow-ups k_c was independent of system size N, while for high coupling strengths the fraction k_c/N is independent of the network size N.

Spatiotemporal Behaviour of the Oscillators under Random Links:

Interestingly, very different behaviour from that described above, emerges when the links are rewired randomly. Representative results for periodically switched networks is shown in Fig. 4, for coupling strengths greater than the critical value ε_c . For these coupling strengths, under regular coupling, there was a blow-up in the system. However, it is clearly evident that the blow-up has been effectively suppressed for p > 0, and all the limit cycles remain bounded. Note that we obtain qualitatively similar results under stochastic switching of links.

We also explored the minimum fraction of random links necessary to prevent the unbounded growth in the system as a function of the parameter μ . Our observation was as follows: for higher μ , even when the links switch rapidly, the range over which the dynamics remains bounded is quite small. So the nonlinearity of the local dynamics determines how fast the random links need to be switched in order to tame the unbounded growth. Also clearly, when the network changes are fast, a smaller fraction of random links is necessary to enforce boundedness.

Synchronization of the Bounded State

A natural question aries here: when random coupling suppresses blow-ups, does it give rise to a synchronized state? Interestingly, different patterns emerge under different time scales of network change. Networks with rapidly changing connections yield a synchronized state. However, slow network changes gives bounded dynamics that is not synchronized [6]. This was demonstrated quantitatively in [6] through a synchronization order parameter, defined as the mean square deviation of the instantaneous states of the nodes, averaged over time and over different initial conditions.

So this relaxation oscillator network yields three kinds of dynamical states: (i) bounded synchronized motion; (ii) bounded unsynchronized dynamics; and (iii) blow-ups. The dynamical state that will emerge is determined by the interplay of the coupling strength, fraction of random links and frequency of switching links.

Further for the case of stochastic switching of links we found some interesting scaling relations between the fraction of random links p and the link rewiring probability p_r [6]. First we found that the critical coupling strength ε_c beyond which blow-ups occur, scales with the link rewiring probability p_r and fraction of random links p as:

$$\varepsilon_c \sim (p \ p_r)^{\beta}$$
 (5)

where $\beta = 0.119 \pm 0.001$. This scaling relation implies that as the links change more frequently, and there is larger fraction of random connections, the range over which bounded dynamics is obtained becomes larger. Further notice that the quantity that occurs in the scaling relation is the product $p p_r$. So the emergent phenomena is the same if this product remains the same, even though individually p and p_r may differ.

Furthermore the minimum fraction of random links p_c necessary to suppress blow-ups, at a particular coupling strength, varies with link switching probabilities p_r as

$$p_c \sim (p_r)^{\delta}$$
 (6)

Representative values of the exponent are: $\delta = -1.04 \pm 0.008$ for $\varepsilon = 0.7$ and $\delta = -0.98 \pm 0.008$ for $\varepsilon = 0.9$ [7].

Lastly, we checked the generality and scope of our results by studying the behaviour of a network of Stuart-Landau oscillators given by:

$$\dot{z} = (1 + i\omega - |z|^2)z$$
 (7)

where ω is the frequency, and z(t) = x(t) + iy(t). Fig. ?? shows representative results. It is clear that networks that change rapidly supress blow-ups over a larger range of p. This implies that even when the number of random links is small, blow-ups are prevented in rapidly varying networks, while slow varying networks need a larger number of random links in order to suppress unbounded growth. Further, timevarying links yield more synchronized states.

We also studied nonlinearly coupled networks of FitzHugh-Nagumo oscillators modeling neuronal populations, where the dynamics of the membrane potential x and the recovery variable x is described by [19], [20]:

$$\dot{x} = x - x^3/3 - y + I$$

$$\dot{y} = \Phi(x + a - by)$$
(8)

with *I* being the magnitude of stimulus current. We also explored the *heterogeneous* systems, such as the case where the nonlinearity parameter μ was distributed over a range of positive values for the local Van der Pol oscillators. Our extensive numerical simulations showed qualitatively similar behavior for all of the above systems [6]. This strongly indicates that the prevention of blow-ups through time-varying random connections is quite general.

Conclusions

In summary, we have reviewed our recent results on the dynamics of a collection of relaxation oscillators under varying coupling topologies, ranging from a regular ring to a random network. Our central result is the following: the coupled system experiences unbounded growth under regular ring topology, for sufficiently strong coupling strength. However when some fraction of the links are dynamically rewired to random connections, this blow-up is suppressed and the system remains bounded. So our results suggest an underlying mechanism by which complex systems can avoid a catastrophic blow-up. Further from the stand point of potential applications, our observations indicate a method to control and prevent blow-ups in coupled oscillators that are commonplace in a variety of engineered systems.

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Fig. 1. (Top to Bottom) Phase-space plot of all the van der Pol oscillators coupled to nearest neighbours on a ring with $\mu = 1.5$ (i = 1, 2, ... 100), with the x-axis representing position, and the y-axis representing velocity, and the different colors representing the different oscillators; Plot of the time evolution of position $x^i(t)$ of all the oscillators, with different colors representing the different oscillators; Density plots of the spatiotemporal evolution, with the site indices on the x-axis and time along the y axis, and the magnitude of the variables $x_i(t)$ represented by the colour scale. Here coupling strength is 0.007. For coupling strengths greater than the critical value $\varepsilon_c \sim 0.5$, the dynamics of all the oscillators is unbounded. This figure is adapted from [7].

Fig. 2. (Top to Bottom) Phase-space plot of all the van der Pol oscillators coupled to nearest neighbours on a ring with $\mu = 1.5$ (i = 1, 2, ... 100), with the *x*-axis representing position, and the *y*-axis representing velocity, and the different colors representing the different oscillators; Plot of the time evolution of position $x^i(t)$ of all the oscillators, with different colors representing the different oscillators; Density plots of the spatiotemporal evolution, with the site indices on the *x*-axis and time along the *y* axis, and the magnitude of the variables $x_i(t)$ represented by the colour scale. Here coupling strength is 0.3. For coupling strengths greater than the critical value $\varepsilon_c \sim 0.5$, the dynamics of all the oscillators is unbounded. This figure is adapted from [7].



Fig. 3. (Top to Bottom) Phase-space plot of all the van der Pol oscillators coupled to nearest neighbours on a ring with $\mu = 1.5$ (i = 1, 2, ... 100), with the x-axis representing position, and the y-axis representing velocity, and the different colors representing the different oscillators; Plot of the time evolution of position $x^i(t)$ of all the oscillators, with different colors representing the different oscillators; Density plots of the spatiotemporal evolution, with the site indices on the x-axis and time along the y axis, and the magnitude of the variables $x_i(t)$ represented by the colour scale. Here coupling strength is 0.5. For coupling strengths greater than the critical value $\varepsilon_c \sim 0.5$, the dynamics of all the oscillators is unbounded. This figure is adapted from [7].



Fig. 4. Spatiotemporal evolution of the oscillators under random links, with the site indices on the x-axis and time along the y axis, and the magnitude of the variables $x_i(t)$ (i = 1, ..., N), represented by the colour scale, for network switching time period (left) r = 0.1 (right) r = 0.01. Here N = 100, $\mu = 1.5$, fraction of random links p = 0.6 and coupling strength $\varepsilon = 0.6 > \varepsilon_c$. Note that regular coupling (p = 0) yields unbounded dynamics at this coupling strength [7].