ON THE ASSUMPTIONS AND DECISIONS REQUIRED FOR REDUCED ORDER MODELLING OF ENGINEERING DYNAMICAL SYSTEMS

Matthew P. Cartmell
Department of Mechanical Engineering
University of Glasgow
UK
m.cartmell@mech.gla.ac.uk

David I. M. Forehand
Department of Mechanical Engineering
University of Glasgow
UK
d.forehand@mech.gla.ac.uk

Abstract
This paper considers the principal assumptions and decisions that have to be made when developing reduced order models of problems in engineering dynamics. The work has been motivated by a perception that information loss can be considerable when applying approximate analytical solution methods, despite good solutions *per se* being obtainable with appropriate application. The lost information relates to the evolution of solutions and the consequential inheritance issues. Simplifications, which are normally routinely made during approximate analytical solution, irrespective of the method used, can remove an additional layer of information from the solution. This information is apparently redundant when structural simplification is the principal objective, but if it is retained it can show how the solution has come together, all the principal influences that are acting, and how they interact with each other. This paper takes 10 sample problems in engineering dynamics and highlights the many common and different modelling assumptions that are needed to generate viable and realistic equations of motion, from which inheritance based approximate analytical solutions can be generated, and ultimately visualised.

Key words
Symbolic computational dynamics, approximate analytical solution.

1. Introduction
This paper summarises the main modelling issues that are prevalent when setting up reduced order models in engineering dynamics. Ten different model derivation problems have been investigated, both from the literature and also as derived by the authors, and the modelling assumptions and decisions are summarised. In particular the commonalities between problems are highlighted, as are the major differences. Since the objective in each case is to generate a definitive, but pragmatically solvable analytical model, some comparisons are also offered between the problems studied and the modelling approaches taken.

2. Introduction to the problem set
Ten different problems are chosen and the modelling assumptions are given in the subsections that follow. Further modelling strategy options are also discussed in the next section.

2.1 Forced Nonlinear Cantilever Beam
This first problem is the ubiquitous forced nonlinear cantilever beam with a lumped end mass. Background assumptions are that the beam is mass-less, with linear elasticity on the basis of Euler-Bernoulli modelling, it is definable in modal space, and that this is reducible down to one bending mode, with one associated modal co-ordinate u_0 and the mode is represented by the static deflection curve for a cantilever, normalised at the free end. Linear viscous damping ξ is used and a cubic nonlinear stiffness term intervenes as a result of large deflections. The system is excited by a harmonic force applied directly to the beam, $a_0 \cos\Omega t$. There is one ODE equation of motion in the first bending mode, and this can be nondimensionalised by using convenient length and time scales, in this case the length of the beam l and the reciprocal of the excitation frequency Ω. The problem is attributable to the authors here [Cartmell, 2006]. The physical form of the problem is:

$$\ddot{u}_0 + 2\xi\omega_0\dot{u}_0 + \frac{36}{25l^2}(\ddot{u}_0 u_0^2 + \dot{u}_0^2 u_0) + hu_0^3 + \omega_0^2 u_0 = a_0 \cos\Omega t \tag{1}$$

where $\omega_0^2 = (3EI_z)/(ml^3)$.

2.2 Forced Double Pendulum
This is a different class of problem involving coupled elements which are both assumed to act as rigid bodies, m_1 and m_2, uniform in geometry but of potentially different lengths, l_1 and l_2, and uniform mass distribution. Absolute angular co-ordinates are used for each pendulum, referred to the vertical stable equilibrium, with linear viscous damping, C_1 and C_2, assumed at the joints. The system is planar, orientated vertically, and is excited by horizontal harmonic support motion, $A\cos\Omega t$. Two term McLaurin expansions are used to remove explicit trigonometrical functions and then the equations can be nondimensionalised in terms of mass, length, and time. The nondimensionalisation is based on the length and mass of one pendulum (arbitrarily chosen) and the reciprocal of the excitation frequency Ω. This is a very well known problem frequently used in undergraduate

teaching. The physical model is attributable to the authors in the form used here [Forehand and Cartmell, 2006(a)],

$$\frac{1}{3}(m_1+3m_2)l_1^2\ddot{\theta}_1+\frac{1}{2}m_2l_1l_2\cos(\theta_1-\theta_2)\ddot{\theta}_2+C_1\dot{\theta}_1+$$
$$C_2(\dot{\theta}_1-\dot{\theta}_2)+\frac{1}{2}m_2l_1l_2\sin(\theta_1-\theta_2)\dot{\theta}_2^2+$$
$$\frac{1}{2}(m_1+2m_2)l_2(g\sin\theta_1-A\Omega^2\cos(\Omega t)\cos\theta_1)=0, \quad (2)(3)$$
$$\frac{1}{2}m_2l_1l_2\cos(\theta_1-\theta_2)\ddot{\theta}_1+\frac{1}{3}m_2l_2^2\ddot{\theta}_2+C_2(\dot{\theta}_2-\dot{\theta}_1)-$$
$$\frac{1}{2}m_2l_1l_2\sin(\theta_1-\theta_2)\dot{\theta}_1^2+\frac{1}{2}m_2l_2(g\sin\theta_2-$$
$$A\Omega^2\cos(\Omega t)\cos\theta_2)=0.$$

2.3 Parametrically Excited Nonlinear Pendulum

In this problem we encounter a vertically orientated mass-less pendulum with a lumped end mass m, under vertically orientated harmonic support motion, $q\cos\Omega t$, providing parametric excitation. The system is assumed to be viscously damped, β, and planar, with one absolute angular co-ordinate, referred to the vertical stable equilibrium. The model is nondimensionalised in time and the natural frequency is scaled to unity, using its reciprocal. A two-term McLaurin expansion is used for the trigonometrical term. This is also a very well known problem, treated by many writers. Thomsen's form of the physical problem [Thomsen, 1997] has been used,

$$\ddot{\theta}+2\beta\omega_0\dot{\theta}+(\omega_0^2-q\Omega^2\cos\Omega t)\sin\theta=0, \quad (4)$$

where $\omega_0^2=g/l$ and $\beta=c/(2m\omega_0)$.

2.4 Flexible Rotor with Snubber Ring

The rotor is driven by an unbalanced mass and is represented by a pair of orthogonal translational co-ordinates originating from the centre of the rotor when in its equilibrium position. In that sense they are absolute co-ordinates and they are nondimensionalised using a radial gap distance γ which is defined as the difference between the radii of the snubber ring and rotor. The time is nondimensionalised using the reciprocal of the natural frequency of the rotor, leading to unity scaled natural frequency. The system is assumed to operate under predominantly linear viscous damping c, and at rotor speed ω. This problem is attributable to [Karpenko, Wiercigroch and Cartmell, 2002],

$$M\ddot{x}+c\dot{x}+k_1x+\begin{Bmatrix}k_2(R-\gamma)\cos\psi, & R\geq\gamma\\ 0, & R<\gamma\end{Bmatrix}=m\rho\omega^2\cos(\varphi_0+\omega t)$$
$$M\ddot{y}+c\dot{y}+k_1y+\begin{Bmatrix}k_2(R-\gamma)\sin\psi, & R\geq\gamma\\ 0, & R<\gamma\end{Bmatrix}=m\rho\omega^2\sin(\varphi_0+\omega t)$$
$$(5)(6)$$

where $R=\sqrt{(x-\varepsilon_x)^2+(y-\varepsilon_y)^2}$, $\cos\psi=(x-\varepsilon_x)/R$, $\sin\psi=(y-\varepsilon_y)/R$, $M=$ total rotor mass plus unbalance and $m=$ unbalance mass.

2.5 Chelomei's Pendulum

A two degree of freedom problem in which an inverted pendulum is fitted with a floating sliding mass. The position of the mass along the pendulum is represented by a translational co-ordinate measured along the length of the pendulum from the based pivot point, and the orientation of the pendulum is given by an absolute angular co-ordinate measured from the vertical equilibrium. The system is parametrically excited by a vertically directed harmonic base motion, $Z(t)=Q\sin(\tilde{\Omega}t)$, and linear viscous damping is assumed for both motions. McLaurin expansions are not used initially to remove the explicit presence of the trigonometrical terms, but appear later during the ordering phase. The equations are nondimensionalised for the translational mass (of mass M) position co-ordinate, $U(t)$, by using the pendulum (of mass m) length, l, and the reciprocal of the natural frequency of pendulum oscillation. The main proponents of this problem are [Chelomei, 1983], [Blekhman & Malakhova, 1986] and [Thomsen, 1997] from which we take inspiration here,

$$(\frac{1}{3}ml^2+MU^2+I)\ddot{\theta}+2c_1\dot{\theta}+2MU\dot{U}\dot{\theta}-(MU+\frac{1}{2}ml)(g-$$
$$Q\tilde{\Omega}^2\sin(\tilde{\Omega}t))\sin\theta=0,$$
$$\ddot{U}+2c_2\dot{U}-\dot{\theta}^2U+(g-Q\tilde{\Omega}^2\sin(\tilde{\Omega}t))\cos\theta=0,\ \ U\in[0;l].$$
$$(7)(8)$$

An order of magnitude analysis by the authors has shown that the solution to equations (7) and (8) is highly unlikely to demonstrate Chelomei's observations, without the involvement of a further, horizontal, excitation.

2.6 Parametrically Excited Cantilever Beam

This beam has a rectangular end mass fitted to the free end. The other end is attached to a rigid support which is excited by a motion in the direction of the stiff plane of the beam, thereby parametrically exciting it. The motion of the beam is defined in modal space, using two bending co-ordinates and one torsion co-ordinate. First and second bending mode shapes are used with normalisation defined by a problem-specific inner product. The torsion mode is similarly treated. The modal translations and the parametric excitation are nondimensionalised by using the beam length and the time is nondimensionalised via the reciprocal of the excitation frequency. Classical linear viscous damping is later assumed. This is a well established problem, and here it is attributable to the authors [Forehand and Cartmell, 2001] in physical, undamped form,

$$(1+(2B_{3,5}u_{1,2}+B_4u_{2,1})^2+B_{1,2}^2\phi_1^2)\ddot{u}_{1,2}+((2B_3u_1+B_4u_2)(B_4u_1$$
$$+2B_5u_2)+B_1B_2\phi_1^2)\ddot{u}_{2,1}+2((2B_{3,5}u_{1,2}+B_4u_{2,1})(B_3\dot{u}_1^2+B_4\dot{u}_1\dot{u}_2$$
$$+B_5\dot{u}_2^2)+B_{1,2}\phi_1\dot{\phi}_1(B_1\dot{u}_1+B_2\dot{u}_2))+2(B_{6,8}/m_0)u_{1,2}$$
$$+(B_7/m_0)u_{2,1}+B_{1,2}\phi_1(\ddot{V}_B+(B_1u_1+B_2u_2)\ddot{\phi}_1)=0,$$
$$B_1\phi_1(B_1u_1+B_2u_2)\ddot{u}_1+B_2\phi_1(B_1u_1+B_2u_2)\ddot{u}_2+((I_0/m_0)+(B_1u_1$$
$$+B_2u_2)^2)\ddot{\phi}_1+2(B_1u_1+B_2u_2)(B_1\dot{u}_1+B_2\dot{u}_2)\dot{\phi}_1+2(B_9/m_0)\phi_1+$$
$$(B_1u_1+B_2u_2)\ddot{V}_B=0,$$
$$(9)(10)(11)$$

where the a,b subscripts represent quantities associated with the two bending mode coordinates respectively, and the $B_{1\text{-}9}$ are mode dependent system constants.

2.7 Autoparametrically Coupled Beams

In this system two beams are fitted together into an L shaped structure. The larger horizontal beam is cantilevered out from a substantial support. The smaller vertical beam is attached to the free end of the horizontal beam and orientated so that its most flexible plane is orthogonal to that of the horizontal beam. Two modal co-ordinates are used to represent bending of the beams, once again using normalised mode shape functions. These are nondimensionalised using convenient length scales, and different treatments for this have been offered in the literature, with secondary (vertical) beam length being one convenient solution to this. Classical linear viscous damping is assumed for each beam. Literature examples of this problem, attributable to [Roberts and Cartmell, 1984] and [Bux and Roberts, 1986] exclude time nondimensionalisation, although this could have been introduced, either by means of the reciprocal of a beam natural frequency or the excitation frequency. The excitation frequency is harmonic. The work of [Roberts and Cartmell, 1984] is used in this particular discussion, as shown,

$$\mathbf{M}\ddot{\mathbf{p}} + \mathbf{D}\dot{\mathbf{p}} + \mathbf{K}\mathbf{p} = \mathbf{P}(\mathbf{t}),$$
$$\mathbf{m}\ddot{\mathbf{q}} + \mathbf{d}\dot{\mathbf{q}} + \mathbf{k}\mathbf{q} - \ddot{\mathbf{z}}[\sum_k m_k \psi_{ijk}]\mathbf{q} = \mathbf{0}, \qquad (12)(13)$$

where $\mathbf{p} = [p_1, ..., p_n]^T$ and $\mathbf{q} = [q_1, ..., q_N]^T$ are vectors of n and N generalised coordinates defining the horizontal primary and vertical secondary beams, respectively.

2.8 Motorised Momentum Exchange Tether on a Circular Orbit

This problem is included despite the vastly greater scale of the system it represents. This is a large rigid body operating in space for the purpose of imparting energy to mass payloads through spin about the system's local centre of mass. There is no damping in this system other than negligible internal friction within the excitation device. In this example the tether is excited by a harmonic torque. A mass discretisation is applied to the tether to remove singularities when the numerator and denominator of two potential energy terms tend to zero in the case where the response of the tether is an integer value of π. This discretisation generates a system in which the tether subspans of length L and density ρ are represented by n lumped masses. Expansion of the nonlinear potential energy terms and neglecting terms of order 4 and above in the spin co-ordinate leads to a Duffing type equation. Normalisation of the linear natural frequency of tether libration and time nondimensionalisation using the reciprocal of this leads to an undamped Duffing equation with harmonic forcing. This problem is by the authors [Cartmell, Forehand, D'Arrigo, McKenzie, Wang and Metrikine, 2006], and is stated as follows in its fundamental physical form,

$$\frac{1}{6}(4\rho AL^3 + 3M_m r_m^2 + 6M_p(2L^2 + r_p^2))\ddot{\psi} +$$
$$\frac{\mu M_p r_c L \sin\psi}{(r_c^2 + L^2 - 2r_c L\cos\psi)^{3/2}} - \frac{\mu M_p r_c L \sin\psi}{(r_c^2 + L^2 + 2r_c L\cos\psi)^{3/2}}$$
$$+\mu\rho A \frac{r_c(L + \sqrt{r_c^2 + L^2 + 2r_c L\cos\psi})\sin\psi}{r_c^2 + L^2 + 2r_c L\cos\psi + (r_c\cos\psi + L)\sqrt{r_c^2 + L^2 + 2r_c L\cos\psi}}$$
$$+\mu\rho A \frac{r_c(L - \sqrt{r_c^2 + L^2 - 2r_c L\cos\psi})\sin\psi}{r_c^2 + L^2 - 2r_c L\cos\psi + (r_c\cos\psi - L)\sqrt{r_c^2 + L^2 - 2r_c L\cos\psi}}$$
$$= T_0\cos\tilde{\Omega}t. \qquad (14)$$

2.9 Automotive Disk Brake Vibration

A rotating frictional slider on a circular disk is taken with vertical and horizontal slider stiffness and damping built into the model. Oscillatory motion of the slider as it is driven around the disk at constant angular velocity is defined by an angular co-ordinate, and transverse motion of the slider, which equates to displacement of the disk at the contact point, is represented by a displacement co-ordinate which is transformed into modal space by means of a series form modal expansion, for a finite number of modes. Excitation is provided by the effect of friction F_θ between the slider m and the disk. Some algebraic reorganisation of the problem is necessary to get to the ODE model in modal space. The model is nondimensionalised in time by means of the reciprocal of the critical speed of the disk. Classical linear viscous damping is assumed for the transverse motion at the contact point between slider and disk. The analysis in this particular version of this well researched problem is attributable to [Chan, Mottershead and Cartmell, 1994] and [Ouyang, Mottershead, Cartmell and Friswell, 1998]. In-plane and transverse motions are governed by,

$$\ddot{\varphi} + (1/m)(c_p - \tilde{F}_\theta\alpha/r_0)\dot{\varphi} + \omega_p^2\varphi = -\tilde{F}_\theta(1 - \alpha\tilde{\Omega})/mr_0,$$
$$\rho h\partial^2 w/\partial t^2 + D^*\nabla^4\partial w/\partial t + D\nabla^4 w = -(1/r)\delta(r - r_0)\delta(\theta -$$
$$\varphi - \tilde{\Omega}t)\{m[\ddot{\varphi}\partial w/\partial\theta + (\dot{\varphi} + \tilde{\Omega})^2\partial^2 w/\partial\theta^2 + 2(\dot{\varphi} + \tilde{\Omega})\times$$
$$\partial^2 w/\partial\theta\partial t + \partial^2 w/\partial t^2] + c[(\dot{\varphi} + \tilde{\Omega})\partial w/\partial\theta + \partial w/\partial t] + kw -$$
$$\tilde{F}_\theta[1 - \alpha(\dot{\varphi} + \tilde{\Omega})]\partial w/r\partial\theta\}. \qquad (15)(16)$$

2.10 Stephenson-Kapitsa Pendulum

This is a simplification of Chelomei's pendulum (see problem 2.5), or conversely Chelomei's pendulum could be said to be a specialisation of the Stephenson-Kapitsa problem. The difference is that the floating mass is absent in the case of the Stephenson-Kapitsa pendulum [Kapitsa, 1951; Acheson, 1995] and that the system can be shown to be capable of stabilisation in the upright position when the support is oscillating with high frequency and small amplitude, as discussed by [Thomsen, 1997]. The pendulum can be considered as a mass-less link with a lumped mass at the end, with vertical orientation of the pendulum with the pivot at the bottom, and vertically directed parametric excitation. The equation of motion contains a trigonometrical term which is expanded by means of a two-term McLaurin expansion. Linear viscous damping is assumed and a time nondimensionalisation is included, based on the reciprocal of the excitation frequency. This version of the problem is due to [Kapitsa, 1951] and [Acheson, 1995], with some additional notation inserted by the authors,

$$\ddot{\theta} + k\dot{\theta} - (\frac{g}{l} + q\omega_0^2\cos\omega_0 t)\sin\theta = 0. \qquad (17)$$

	1	2	3	4	5	6	7	8	9	10
Physical Assumptions										
massless elem.	y	n	y	n	n	n	n	n	N	y
McLaurin exp.	n	y	y	n	n	n	n	n	N	y
Modal space	y	n	n	n	n	y	y	n	Y	n
Physical space	n	y	y	y	y	n	n	y	N	y
Single DOF	y	n	y	n	n	n	n	y	N	y
MDOF	n	2	n	2	2	3	2	n	2	n
Lin. Vis. Dpg.	y	y	y	y	y	y	y	n	Y	y
Harmonic ex.	y	y	y	y	y	y	y	y	N	y
Trans coord(s)	1	n	n	2	1	2	2	n	1	n
Rot coord(s)	n	2	1	n	1	1	n	1	1	1
Discretisation	n	n	n	n	n	n	n	y	N	n
Nondimens.										
by length	y	y	n	y	y	y	y	n	N	n

by 1/freq	y	y	y	y	y	y	n	y	y	y
by mass	n	y	n	n	n	n	n	n	n	n
Normalis.										
Frequency	n	n	n	y	n	n	n	y	n	n

Table 1. Problem set 1-10 along the top, noting *discretisation* refers to physical discretisation (of mass properties), and underscore, y signifies either yes or no.

It is evident from the problem set that the physical assumptions are entirely problem-specific, but that many of them transcend this, assumptions of linear viscous damping and harmonic excitations being cases in point, and are usually made for reasons of mathematical simplicity rather than engineering desirability. Transformations from physical to modal space are frequently simplifying, as is nondimensionalisation by length and time. Mass nondimensionalisation is generally less common, as is normalisation by natural frequency. Problems 2.3 and 2.10 required relatively little formal consideration to obtain a workable ordering scheme because of their mechanical simplicity. All the other problems involved considerably more effort to get a scheme which properly reflected the working of the mechanical system, the need for mathematical consistency, and reasonable tractability.

3. Equation Ordering for Perturbation Solution

3.1 Forced Nonlinear Cantilever Beam

Continuing from 2.1, we note that the general beam deflection is $u(x,t) = u_0(t) f(x)$, where the static deflection shape $f(l) = 1$. The nondimensionalisation of equation (1) is given by $\mu = u_0 / l$, $\tau = \Omega t$, and $\varpi_0 = \omega_0 / \Omega$, leading to,

$$\mu'' + 2\xi\varpi_0\mu' + \frac{36}{25}\left[\mu''\mu^2 + (\mu')^2\mu\right] + \frac{hl^2}{\Omega^2}\mu^3 + \varpi_0^2\mu = \frac{F_0}{ml\Omega^2}\cos\tau. \tag{18}$$

In practice, $u_0 \ll l$, and so $\mu \ll 1$. If we introduce a small nondimensional parameter ε, and retain the use of this throughout the rest of the paper, such that $\mu = \varepsilon\eta$, and $F_0 = \varepsilon\tilde{F}_0$, where $\eta = O(1)$, then we get the nonlinear terms ordered to $O(\varepsilon^2)$. Alternatively, by using $\mu = \varepsilon^{1/2}\eta$ and $F_0 = \varepsilon^{1/2}\tilde{F}_0$, the nonlinear terms are to $O(\varepsilon)$. This leads to,

$$\eta'' + 2\xi\varpi_0\eta' + \varepsilon^2\frac{36}{25}[\eta''\eta^2 + (\eta')^2\eta] + \varepsilon^2\frac{hl^2}{\Omega^2}\eta^2 + \varpi_0^2\eta = \varepsilon\frac{F_0}{ml\Omega^2}\cos\tau \tag{19}$$

or, alternatively, we get,

$$\eta'' + 2\xi\varpi_0\eta' + \varepsilon\frac{36}{25}[\eta''\eta^2 + (\eta')^2\eta] + \varepsilon\frac{hl^2}{\Omega^2}\eta^2 + \varpi_0^2\eta = \varepsilon\frac{F_0}{ml\Omega^2}\cos\tau \tag{20}$$

noting that the damping can be ordered as required.

3.2 Forced Double Pendulum

Equations (2) & (3) can be nondimensionalised and ordered as follows,

$$\frac{1}{3}(m+3)l\theta_1'' + \frac{1}{2}\left(1 - \tfrac{1}{2}\varepsilon^2(\theta_1 - \theta_2)^2\right)\theta_2'' + \varepsilon c_1\theta_1' + \varepsilon c_2\left(\theta_1' - \theta_2'\right) + \frac{1}{2}\varepsilon^2(\theta_1 - \theta_2)\theta_2'^2$$
$$+ \frac{1}{2}(m+2)\left(\omega^2\left(\theta_1 - \varepsilon^2\frac{\theta_1^3}{6}\right) - \varepsilon a\left(1 - \varepsilon^2\frac{\theta_1^2}{2}\right)\cos\tau\right) = 0,$$

$$\frac{1}{2}l\left(1 - \frac{1}{2}\varepsilon^2(\theta_1 - \theta_2)^2\right)\theta_1'' + \frac{1}{3}\theta_2'' + \varepsilon l c_2\left(\theta_2' - \theta_1'\right) - \frac{1}{2}\varepsilon^2 l(\theta_1 - \theta_2)\theta_1'^2$$
$$+ \frac{1}{2}\left(\omega^2\left(\theta_2 - \varepsilon^2\frac{\theta_2^3}{6}\right) - \varepsilon a\left(1 - \varepsilon^2\frac{\theta_2^3}{2}\right)\cos\tau\right) = 0,$$

(21)(22)

where the trigonometrical terms have been expanded using two-term McLaurin expansions, ' denotes differentiation w.r.t. nondimensional time $\tau = \Omega t$, and $l = l_1/l_2$, $m = m_1/m_2$, $c_1 = C_1/m_2 l_1 l_2 \Omega$, $c_2 = C_2/m_2 l_1 l_2 \Omega$, $\omega^2 = g/l_2\Omega^2$, $a = A/l_2$. Also, ordering has been achieved by setting $\theta_{1,2} \to \varepsilon\theta_{1,2}$, $c_{1,2} \to \varepsilon c_{1,2}$ and $a \to \varepsilon^2 a$.

3.3 Parametrically Excited Nonlinear Pendulum

In this case we introduce $\tau = \omega_0 t$ and $\omega = \Omega / \omega_0$ to nondimensionalise equation (4) and the ordering is informal, set up so that the damping, the parametric excitation, and the cubic nonlinearity are all $O(\varepsilon)$, *i.e.*,

$$\ddot{\theta} + 2\varepsilon\beta\dot{\theta} + (1 - \varepsilon q\omega^2\cos\omega\tau)\theta + \varepsilon\gamma\theta^3 = 0, \quad \gamma = -\frac{1}{6} \tag{23}$$

We could equally have chosen different ordering schemes.

3.4 Flexible Rotor with Snubber Ring

This is a more difficult problem, in which considerable formal analysis is required to get a consistently ordered pair of equations from equations (5) and (6). We use,

$\tau = \omega_n t$, $\eta = \omega/\omega_n$, $\nu = c/(2\sqrt{k_1 M})$, $\eta_m = m/M$, $\hat{x} = x/\gamma$
$\hat{y} = y/\gamma$, $\hat{\rho} = \rho/\gamma$, $\hat{K} = k_2/k_1$, $\hat{\varepsilon}_x = \varepsilon_x/\gamma$, $\hat{\varepsilon}_y = \varepsilon_y/\gamma$
$\hat{z} = R/\gamma = \sqrt{(\hat{x} - \hat{\varepsilon}_x)^2 + (\hat{y} - \hat{\varepsilon}_y)^2}$,

for nondimensionalisation, leading to,

$$\hat{x}'' + 2\nu\hat{x}' + \hat{x} + \begin{Bmatrix} \hat{K}(\hat{x} - \hat{\varepsilon}_x)(1 - 1/\hat{z}, & \hat{z} \ge 1 \\ 0 & \hat{z} < 1 \end{Bmatrix} = \eta_m\hat{\rho}\eta^2\cos(\varphi_0 + \eta\tau)$$

$$\hat{y}'' + 2\nu\hat{y}' + \hat{y} + \begin{Bmatrix} \hat{K}(\hat{y} - \hat{\varepsilon}_y)(1 - 1/\hat{z}, & \hat{z} \ge 1 \\ 0 & \hat{z} < 1 \end{Bmatrix} = \eta_m\hat{\rho}\eta^2\sin(\varphi_0 + \eta\tau)$$

(24)(25)

Initially, the damping and forcing terms are neglected as they can be ordered in an *ad hoc* manner later. The forcing coefficient can be made arbitrarily small by reducing the o.o.b. mass *m* relative to the rotor *M*. The nonlinear terms are $\hat{K}(\hat{x} - \hat{\varepsilon}_x)(1 - 1/\hat{z}), \hat{z} \ge 1$ and $\hat{K}(\hat{y} - \hat{\varepsilon}_y)(1 - 1/\hat{z}), \hat{z} \ge 1$ in equations (24) and (25). Ordering these terms starts with the assertion that $\hat{K}$ is small, therefore $k_1 \gg k_2$, but $\hat{z} \ge 1$ for the nonlinear terms to take part, so $\hat{z}$ is $O(1)$, which means that $\hat{x}$ and $\hat{y}$ must also be $O(1)$. This challenges the notion of the necessity for a small perturbation about a linear solution. To make progress the nonlinear terms must be expanded in powers of $\hat{x}$ and $\hat{y}$. Analysis of the problem [Forehand and Cartmell, 2006(b)] shows that $(1 - 1/\hat{z})$ can be expanded for $\hat{z} = 1^+$ by a series of even powers of $\hat{z}$, which can be expressed in terms of powers of $\hat{x}$ and $\hat{y}$. Thus we can specifically approximate $g(\hat{z}) = 1 - 1/\hat{z}$ by $p(\hat{z}) = -(1/8)(7 - 10\hat{z}^2 + 3\hat{z}^4)$ for $\hat{z} = 1^+$, with good agreement up to $\hat{z} = 1.14$, or so. The conditions above provide justification for an assumption of small damping, small unbalance mass, and small snubber ring stiffness (relative to the rotor spring stiffness), leading to,

$$\hat{x}'' + 2\varepsilon\nu\hat{x}' + \hat{x} + \varepsilon\hat{K}(\hat{x} - \hat{\varepsilon}_x)(1 - 1/\hat{z}) = \varepsilon\eta_m\hat{\rho}\eta^2\cos(\varphi_0 + \eta\tau)$$
$$\hat{y}'' + 2\varepsilon\nu\hat{y}' + \hat{y} + \varepsilon\hat{K}(\hat{y} - \hat{\varepsilon}_y)(1 - 1/\hat{z}) = \varepsilon\eta_m\hat{\rho}\eta^2\sin(\varphi_0 + \eta\tau)$$
(26)(27)

It should be noted that the quantity $\hat{K}$ in equations (26) and (27) contains Heaviside functions but that there is a precedent for processing such terms within multiple scales perturbation solutions in [Warmiński, Litak, Cartmell, Khanin, Wiercigroch, 2003].

3.5 Chelomei's Pendulum

The system defined by equations (7) and (8) is also very demanding to treat consistently. If the sliding mass (*i.e.* *M*) is removed the problem reduces to the Kapitsa pendulum case, and it should also be noted that Chelomei's observation of an equilibrium position for *M* has not yet been corroborated, theoretically or experimentally. Thomsen and Tcherniak showed that resonant flexural vibrations of the pendulum rod and small symmetry-breaking in the form of off-vertical excitations, were needed to reproduce Chelomei's results, and that constant tuning of the excitation frequency is needed to keep the pendulum upright and the mass floating [Thomsen and Tcherniak, 2001]. Considerable work by the authors of this current paper on a variant of this problem, with added horizontal support motion, has shown that a systematic ordering of equations (7) and (8) leads to,

$$(1 + \gamma + \alpha u^2)\ddot{\theta} + 2\varepsilon\beta_1\dot{\theta} + 2\varepsilon\alpha u\dot{u}\dot{\theta} - (1 + \frac{2}{3}\alpha u)((\varepsilon$$
$$-q_v\Omega^2\sin(\Omega\tau))(\varepsilon\theta - \frac{\varepsilon^3\theta^3}{6}) + q_h\Omega^2\sin(\Omega\tau + \eta)\times \qquad (28)$$
$$(1 - \frac{\varepsilon^2\theta^2}{2})) = 0,$$

$$\ddot{u} + 2\varepsilon\beta_2\dot{u} - \varepsilon\dot{\theta}^2 u + \frac{2}{3}(\varepsilon - q_v\Omega\sin(\Omega\tau))(1 - \frac{\varepsilon^2\theta^2}{2})$$
$$-\frac{2}{3}q_h\Omega^2\sin(\Omega\tau + \eta)(\varepsilon\theta - \frac{\varepsilon^3\theta^3}{6}) = 0. \qquad (29)$$

Note that we have used $\tau = \omega t, \omega^2 = (2/3)g/l,$ together with other definitions, $u = U/l, q = (2/3)Q/l, \Omega = \tilde{\Omega}/\omega,$ $\alpha = 3M/m,$ $\gamma = 1/((1/3)ml^2),$ $\beta_1 = c_1/((1/3)ml^2\omega),$ and, $\beta_2 = c_2/\omega.$ A more compact result is obtained if the nondimensionalisation is based on the excitation frequency $\tilde{\Omega}$ rather than the natural frequency of the pendulum, $\omega = \sqrt{2g/3l}$, noting that $\omega/\tilde{\Omega}$ is likely to be $O(\varepsilon)$. This produces more natural looking equations and removes multiple appearances of $1/\varepsilon$ which would otherwise be obtained. The excitations in equations (28) and (29) are *hard* because, although the amplitudes q_v and q_h are small, the excitation frequency needs to be high enough for the excitation terms to appear in the lowest order perturbation equations in any subsequent analysis.

3.6 Parametrically Excited Cantilever Beam

By introducing the substitutions $u_{1,2} = \varepsilon\bar{u}_{1,2}$ and $\phi_1 = \varepsilon\bar{\phi}_1$, classical linear viscous damping terms to $O(\varepsilon)$, and stating the linear natural frequencies as $\omega_{1,2}^2 = 2(B_{6,8}/m_0)$ and $\omega_T^2 = 2(B_9/m_0)$, leads to three ordered equations of motion, as follows,

$$(1 + \varepsilon^2(2B_{3,5}\bar{u}_{1,2} + B_4\bar{u}_{2,1})^2 + \varepsilon^2 B_{1,2}^2\bar{\phi}_1^2)\ddot{\bar{u}}_{1,2} + \varepsilon^2((2B_3\bar{u}_1 +$$
$$B_4\bar{u}_2)(B_4\bar{u}_1 + 2B_5\bar{u}_2) + B_1B_2\bar{\phi}_1^2)\ddot{\bar{u}}_{2,1} + 2\varepsilon^2((2B_{3,5}\bar{u}_{1,2} +$$
$$B_4\bar{u}_{2,1})(B_3\dot{\bar{u}}_1^2 + B_4\dot{\bar{u}}_1\dot{\bar{u}}_2 + B_5\dot{\bar{u}}_2^2) + B_{1,2}\bar{\phi}_1\dot{\bar{\phi}}_1(B_1\dot{\bar{u}}_1 + B_2\dot{\bar{u}}_2)) +$$
$$2\varepsilon\zeta_{1,2}\omega_{1,2}\dot{\bar{u}}_{1,2} + \omega_{1,2}^2\bar{u}_{1,2} + \varepsilon(B_7/m_0)\bar{u}_{2,1} + B_{1,2}\bar{\phi}_1(\varepsilon\ddot{V}_B +$$
$$\varepsilon^2(B_1\bar{u}_1 + B_2\bar{u}_2)\ddot{\bar{\phi}}_1) = 0,$$
$$\varepsilon^2 B_1\bar{\phi}_1(B_1\bar{u}_1 + B_2\bar{u}_2)\ddot{\bar{u}}_1 + \varepsilon^2 B_2\bar{\phi}_1(B_1\bar{u}_1 + B_2\bar{u}_2)\ddot{\bar{u}}_2 + ((I_0/m_0) +$$
$$\varepsilon^2(B_1\bar{u}_1 + B_2\bar{u}_2)^2)\ddot{\bar{\phi}}_1 + 2\varepsilon^2(B_1\bar{u}_1 + B_2\bar{u}_2)(B_1\dot{\bar{u}}_1 + B_2\dot{\bar{u}}_2)\dot{\bar{\phi}}_1 +$$
$$2\varepsilon\zeta_T\omega_T\dot{\bar{\phi}}_1 + \omega_T^2\dot{\bar{\phi}}_1 + \varepsilon(B_1\bar{u}_1 + B_2\bar{u}_2)\ddot{V}_B = 0.$$
(30)(31)(32)

Numerical work [Cartmell and Su, 2007] and [Forehand, 2007] has shown that in one installation $B_7/m_0 < \omega_1$ and $\omega_1 < \omega_2$, which in that case vindicates the assertion that $(B_7/m_0)\bar{u}_{2,1} \to O(\varepsilon)$. We do not make any further claims for generality at this stage, but here this step obviates the need for normal mode analysis at zeroth order perturbation, which is useful. The excitation terms emerge at $O(\varepsilon)$, which is also consistent with the specific experimental system used in [Cartmell and Su, 2007].

3.7 Autoparametrically Coupled Beams

A Galerkin representation for each beam, based on selected in-plane and out-of-plane bending modes, together with nondimensionalisation of length, assumption of classical linear viscous damping, and harmonic excitation, leads to,

$$\ddot{X} + 2\varepsilon\zeta_1\omega_1\dot{X} + \omega_1^2 X - \varepsilon\mu(\dot{Y}^2 + Y\ddot{Y}) - \varepsilon\gamma\dot{X}^2 = \varepsilon f_0\cos\Omega t,$$
$$\ddot{Y} + 2\varepsilon\zeta_2\omega_2\dot{Y} + \omega_2^2 Y - \varepsilon\ddot{X}Y = 0,$$
(33),(34)

where X and Y are nondimensional responses at two observation co-ordinates for in and out of plane motion, and a_0 is a reference scale factor. Formal definitions for the excitation force f_0, ε , μ , and γ , and involving the modes shapes of the responses of the two beams, are given in [Roberts and Cartmell, 1984].

3.8 Motorised Momentum Exchange Tether on a Circular Orbit

The numerators and denominators of the last two left hand side terms in equation (14) all tend to zero as $\psi \to \pi$, so L'Hôpital's rule can be used to obtain the finite limits as $\psi \to \pi$. These last two left hand side terms can be discretised so that the tether sub-spans are each approximated by *n* discrete masses. By expanding the left hand side of the discretised form of equation (14), neglecting terms $\geq O(\varepsilon^4)$, choosing a fixed value for *n*, nondimensionalising the time, $\tau = t\omega$, with $\omega^2 = C_2/C_1$ and $\Omega = \tilde{\Omega}/\omega$ [Cartmell, Forehand, D'Arrigo, McKenzie, Wang and Metrikine, 2006], we obtain,

$$\psi'' + \psi + (C_3/C_2)\psi^3 = (T_0/C_2)\cos\Omega\tau \qquad (35)$$

where C_{1-3} are given in full in the reference. The steps needed between equations (14) and (35) are considerable, and involve an algebraically and numerically consistent order of magnitude analysis. It should be noted that the version in the reference cited also contains a parametric excitation term which is absent here. Equation (35) is an undamped Duffing equation with hard excitation.

3.9 Automotive Disk Brake Vibration

We apply a modal discretisation scheme to equations (15) and (16), whereby $w(r,\theta,t)=\sum_{r=0}^{\infty}\sum_{s=-\infty}^{\infty}\psi_{rs}(r,\theta)q_{rs}(t)$, and $\psi_{rs}(r,\theta)=(1/\sqrt{\rho hb^2})R_{rs}(r)\exp(is\theta)$, also, $\tau=\omega_{cr}t$, $\beta_{kl}=\omega_{kl}/\omega_{cr}$, $\beta_p=\omega_p/\omega_{cr}$, $\Omega=\tilde{\Omega}/\omega_{cr}$, $\varepsilon\gamma=m/\rho hb^2$, $\varepsilon\kappa=k/\rho hb^2\psi_{cr}^2$, $\varepsilon\zeta=c/\rho hb^2\omega_{cr}$, $\varepsilon\xi=D^*/D\omega_{cr}$, $\varepsilon f=\tilde{F}_\theta/\rho hb^2 r_0\omega_{cr}^2$, with φ assumed small so that $(\varphi+\tilde{\Omega}t)\approx\tilde{\Omega}t$, noting also that the ordering scheme is informal, resulting in this system of equations:

$$
\begin{aligned}
&\frac{d^2q_{kl}}{d\tau^2}+\varepsilon\xi\omega_{cr}^2\beta_{kl}^2\frac{dq_{kl}}{d\tau}+\beta_{kl}^2q_{kl}=-\sum_{r=0}^{\infty}\sum_{s=-\infty}^{\infty}R_{rs}(r_0)R_{kl}(r_0)\exp[i(s-l)\Omega\tau]\\
&\times\{\varepsilon\gamma\{\frac{d^2q_{rs}}{d\tau^2}+s\Omega[2i+\alpha\varphi_0\omega_p(\exp(-i\beta_p\tau)-\exp(i\beta_p\tau))]\frac{dq_{rs}}{d\tau}-s^2\Omega^2\times\\
&[\frac{i\alpha\varphi_0\omega_p}{2}(\exp(i\beta_p\tau)-\exp(-i\beta_p\tau))+1]^2q_{rs}-\frac{is\alpha\varphi_0\omega_p^2\Omega}{2\omega_{cr}}\times(\exp(-i\beta_p\tau)+\\
&\exp(i\beta_p\tau))q_{rs}\}+\varepsilon\zeta\{\frac{dq_{rs}}{d\tau}+is\Omega[\frac{i\alpha\varphi_0\omega_p}{2}(\exp(i\beta_p\tau)-\exp(-i\beta_p\tau))+1]q_{rs}\}\\
&+\varepsilon\kappa q_{rs}-is\varepsilon f\{1-\alpha\tilde{\Omega}[\frac{i\alpha\varphi_0\omega_p}{2}(\exp(-i\beta_p\tau)-\exp(i\beta_p\tau))+1]q_{rs}\}\}
\end{aligned}
\tag{36}
$$

The k^{th} natural frequency is denoted by ω_{kl}, and ω_{cr} is the first critical speed of the disk. Assumptions in equation (36) are: the friction force does not reverse in direction and the mass slides in one direction without sticking.

3.10 Stephenson-Kapitsa Pendulum

We introduce the following into equation (17) in order to develop the ordered version from which we can apply a multiple scales expansion, $\tau=\omega_0 t$, $\omega=\omega_0/\sqrt{g/l}$, $\gamma=k\sqrt{l/g}$, $\theta=\varepsilon\hat{\theta}$, $q=\varepsilon\hat{q}$, $\gamma/\omega=\varepsilon(\hat{\gamma}/\hat{\omega})$, from which the following equation emerges,

$$\ddot{\hat{\theta}}+\varepsilon(\hat{\gamma}/\hat{\omega})\dot{\hat{\theta}}-(\hat{\omega}^{-2}+\varepsilon\hat{q}\cos\tau)(\hat{\theta}-(1/6)\varepsilon^2\hat{\theta}^3)=0 \tag{37}$$

If the only phenomenon of interest is the stability of the upright pendulum the linear version of equation (37) is used; i.e. a damped Mathieu equation [Thomsen, 1997].

4. Conclusions from the Problem Set

The sample set was arbitrarily selected and strongly reflects the authors' interests. Table 1 shows that 30% of the problems contain a mass-less element, 30% involve McLaurin expansion, 40% are expressed in modal space, 40% are SDOF, 90% assume classical linear viscous damping, 90% assume harmonic excitation, 10% involved a necessary discretisation, all were nondimensionalised in some way, 20% were frequency normalised. Nondimensionalisation is universally applicable but frequency normalisation is not. Damping and excitation representations are universally simplifying, but not necessarily accurate. Assumptions of mass-lessness are for ease of modelling, not accuracy. Mass discretisation is only used when mathematically necessary. Consistency throughout each modelling procedure is vital. There is always a lot of highly problem dependent compromise between tractability (solvability) and accuracy.

5. Solution Inheritance and Information Structures

This is a part of a larger programme of research into the information flows during modelling and solution. The paper shows that modelling procedures are highly problem specific, but there are still common themes. The goal is compactness and tractability without loss of information. Tracking of terms and inheritances through the modelling and solution stages reveals considerable hidden information. This can greatly enhance the value of the approximate analytical model and solution approach.

Acknowledgements

EPSRC grant EP/C530446/1 is gratefully acknowledged.

References

Acheson, D. J. (1995). Multiple-nodding oscillations of a driven inverted pendulum. *Proc. R. Soc. Lond*. A, **448**, pp. 89-95.

Blekhman, I. I. and Malakhova, O. Z. (1986). Quasi-equilibrium positions of the Chelomei pendulum. *Sov. Phys. Dokl.* **31**, pp. 229-231.

Bux, S. L. and Roberts, J. W. (1986). Non-linear vibratory interactions in systems of coupled beams. *Journal of Sound and Vibration*. **104**, pp. 497-520.

Cartmell, M. P. (2006). *Unpublished notes*.

Cartmell, M. P., Forehand, D. I. M., D'Arrigo, M. C., McKenzie, D. J., Wang, Y. and Metrikine, A. V. (2006). Approximate analytical solution for a motorised momentum exchange tether on a circular earth orbit. In *Proc. APM*. St. Petersburg, 25 June-1 July.

Cartmell, M.P. and Su, X. (2007). Combination resonances in a vertically orientated and excited cantilever beam with an end mass. In *Proc.2nd Int. Conf. on Nonlin. Dyn*. Kharkov, 25-28 September.

Chan, S. N., Mottershead, J. E. and Cartmell, M. P. (1994). Parametric resonances at subcritical speeds in discs with rotating frictional loads. *Proc. Instn. Mech. Engrs.*. **208**, pp. 417-425

Chelomei, V. N. (1983). Mechanical paradoxes caused by vibrations. *Sov. Phys. Dokl.*, **28**, pp. 387-390.

Forehand, D. I. M. and Cartmell, M. P. (2001). On the derivation of the equations of motion for a parametrically excited cantilever beam. *Journal of Sound and Vibration*, **245**(1), pp. 165-177.

Forehand, D. I. M. and Cartmell, M. P. (2006(a) & (2006(b)). *Unpublished note*.

Forehand, D. I. M. (2007). *Private comm. with X. Su.*

Kapitsa, P. L. (1951). Dynamic stability of a pendulum when the point of suspension is oscillating. *Journ. of Exper. and Theor. Physics*, **21**(5), pp. 588-598.

Karpenko, E. V., Wiercigroch, M. and Cartmell, M. P. (2002). Regular and chaotic dynamics of a discontinuously nonlinear rotor system. *Chaos, Solitons and Fractals*, **13**, pp. 1231-1242.

Ouyang, H., Mottershead, J. E., Cartmell, M. P. and Friswell, M. I. (1998). Friction-induced parametric resonances in discs: effect of a negative friction-velocity relationship. *Journal of Sound and Vibration*. **209**(2), pp. 251-264.

Roberts, J. W. and Cartmell, M. P. (1984). Forced vibration of a beam system with autoparametric coupling effects. *Strain*. **20**, pp. 123-131.

Thomsen, J. J. (1997). *Vibrations and Stability: Order and Chaos*. McGraw-Hill. London.

Thomsen, J. J. and Tcherniak, D. M. (2001). Chelomei's pendulum explained. *Proc. R. Soc. Lond. A*. **457**, pp. 1889-1913.

Warmiński, J., Litak, G., Cartmell, M. P., Khanin, R. and Wiercigroch, M. (2003). Approximate analytical solutions for primary chatter in the non-linear metal cutting model. *Journal of Sound and Vibration*. **259**(4), pp. 917-933.