ON NUMERICAL METHODS OF SOLVING SOME OPTIMAL PATH PROBLEMS ON THE PLANE

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Abstract

T hree numerical methods of solution of some time optimal control problems for a system under phase constraints are described in the paper. Two suggested methods are based on transition to the discrete time model, constructing attainability sets and application of the guide construction. Third method is based on the Deikstra algorithm.

Key words

Optimal control problem, differential inclusions, attainability sets.

1 Introduction

The paper deals with the time optimal control problem connected with studying the dynamic system under phase constraints. The paper continues investigations in [1-6].

Three numerical methods of solution of some time optimal control problems for a system under phase constraints are described in the paper. They are polygons method, grid method and method, based on the Deikstra algorithm.

2 Problem Formulation

Consider the controlled moving object Υ^* in the m-dimensional Euclidean space. Denote by the center O of moving object Υ^* some chosen point inside Υ^*. Orientation of the Υ^* is fixed. Behavior of the center O is described by the equation

$$\dot{x} = f(t, x, u),\ u \in P,\ t \in [t_0, \vartheta],\ t_0 < \vartheta < \infty. \quad (1)$$

Here, x is the m-dimensional phase vector of the system, u is the control, and P is a compact set in the Euclidian space R^r . It is assumed that traditional conditions providing the existing, uniqueness and extendability of solutions of the system (1) at full length of the interval $[t_0, \vartheta]$ are satisfied.

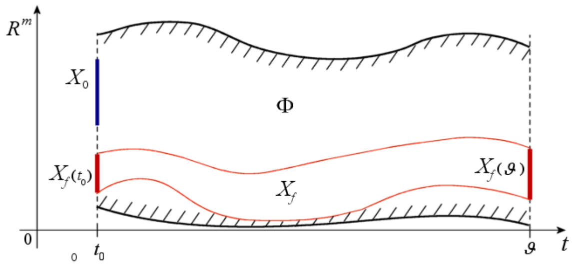

Figure 1.

Along with the system (1), a compacts Φ and X_f from $[t_0, \vartheta] \times R^m$ and compact X_0 from $\Phi(t_0)$ are given. Here the set Φ is a phase constraint for the system (1) and it has nonempty sections $\Phi(t) = \{x \in R^m : (t, x) \in \Phi\}$, $t \in [t_0, \vartheta]$. The set X_f plays a role of a goal set for the control system (1), and the X_0 plays a role of the start set. Let's consider that sections $\Phi(t)$ and $X_f(t)$, $t \in [t_0, \vartheta]$, are changed continuously with a time (see Fig. 1).

By an admissible control $u(t)$, $t \in [t_0, \vartheta]$, we mean any Lebesgue measurable function such, that $u(t) \in P$, $t \in [t_0, \vartheta]$.

Problem 1. *Construct an admissible control $u^*(t)$, $t \in [t_0, \vartheta]$, that steers the phase vector $x[t]$ (trajectory of the center O) of the system (1) from the X_0 into the X_f at minimal time so as $\Upsilon^*(t) \subset \Phi(t)$, $t \in [t_0, \vartheta]$.*

Remark. *Exactly solve formulated problem for a general case is not possible. By this reason we solve this problem approximately. Namely we lead a movement of the center O of the Υ^* from the X_0 to a some chosen neighbourhood of the set X_f. At the same time we construct a control so as the moving object Υ^* is hold in the given neighbourhood of phase constraint Φ.*

3 Scheme of solution

Let's consider the problem of the moving center O instead of the moving object Υ^*. We may do it because

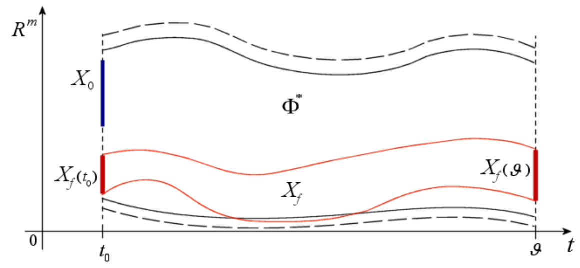

Figure 2.

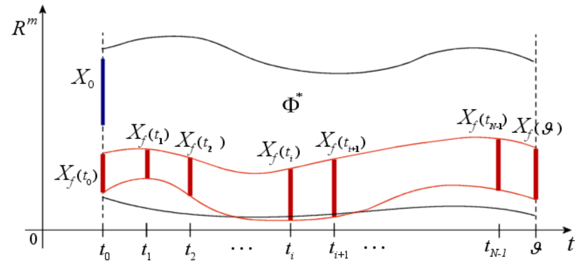

Figure 3.

of fixed orientation of the Υ^*. To perform it the phase constraint is outlined by the Υ^*. In this case a phase constraint Φ for the Υ^* is substituted by a phase constraint Φ^* for the center O (see Fig. 2).

Define the differential inclusion (DI) $F(t,x)$ as following

$$\dot{x} \in F(t,x),\ t \in [t_0, \vartheta], \tag{2}$$

where $F(t,x) = co\{f(t,x,u):\ u \in P\}$.

Divide the interval $[t_0, \vartheta]$; i.e., specify the partition $\Gamma = \{t_0, t_1, \ldots, t_N = \vartheta\}$ of the interval $[t_0, \vartheta]$ such, that the diameter $\Delta = \max\{(t_{i+1} - t_i):\ 0 \le i \le N-1\}$, of the partition Γ is sufficiently small. Further we will consider system (1) only at time moments of partition Γ (see Fig. 3).

Associate a sequence $\{\tilde{X}(t_i)\}$ of sets $\tilde{X}(t_i) \subset R^m$ (attainability sets) with this partition. This sequence is defined recursively as following

$$\tilde{X}(t_0) = X_0,$$
$$\tilde{X}(t_{i+1}) = \Phi^*(t_{i+1}) \cap \tilde{Z}(t_{i+1}; t_i, \tilde{X}(t_i)),$$
$$i = 0,\ 1,\ \ldots,\ N_f - 1.$$

Here, $\tilde{Z}(t^*; t_*, x_*) = x_* + (t^* - t_*)F(t_*, x_*)$, $t_0 \le t_* < t^* \le \vartheta$, $x_* \in R^m$; $\tilde{Z}(t^*; t_*, X_*) = \bigcup\limits_{x_* \in X_*} \tilde{Z}(t^*; t_*, x_*)$. It is assumed that there are instants $t_i \in \Gamma$ in the discrete scheme such, that $\tilde{X}(t_i) \cap X_f(t_i) \ne \emptyset$, and t_{N_f} is the first one (see Fig. 4).

Choose any point $y[t_{N_f}]$ in the $\tilde{X}(t_{N_f}) \cap X_f$ and assign some number $\varepsilon^* > 0$. Formulate the problem 2 whose solution is the approximate solution of the problem 1.

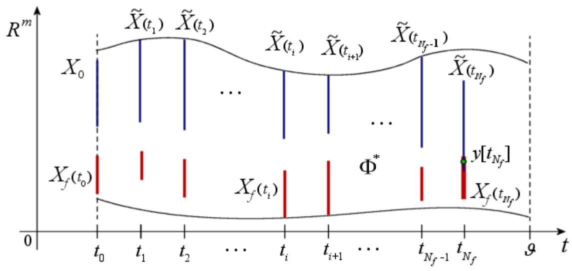

Figure 4.

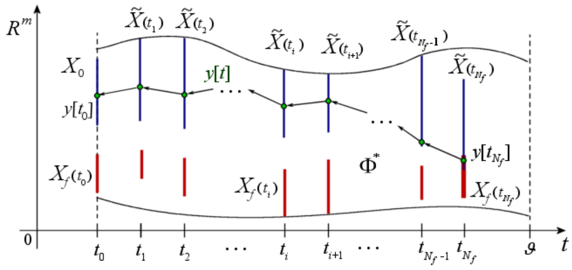

Figure 5.

Problem 2. *It is required to construct an admissible control* $u^*(t)$, $t \in [t_0, t_{N_f}]$, *that leads a phase vector* $x[t]$ *of the system (1) from the* X_0 *into* ε^*-*neighbourhood of the point* $y[t_{N_f}]$ *at the instant* t_{N_f} *so that* $x[t] \in \Phi^*(t)_{\varepsilon^*}$, $t \in [t_0, t_{N_f}]$.

To solve the Problem 2 we also consider the DI with a small parameter $\varepsilon > 0$:

$$\dot{x} \in F(t,x) + \varepsilon Q,\ t \in [t_0, t_{N_f}], \tag{3}$$

where $Q = \{w \in R^m:\ \|w\| \le 1\}$.

Using greater possibilities of the DI (3) we construct the Euler polygon for the DI (3), which goes through sets $\tilde{X}(t_i)$ at instants $t_i \in \Gamma^f$ and ends at the instant t_{N_f} in the point $y[t_{N_f}]$. Here $\Gamma^f = \{t_0,\ t_1,\ \ldots,\ t_{N_f}\}$.

This Euler polygon will play a role of a "guide" in construction of the control $u^*(t)$, $t \in [t_0, t_{N_f}]$, solving the Problem 2 for the system (1). We construct the Euler polygon $y[t]$, $t \in [t_0, t_{N_f}]$, starting from the point $y[t_{N_f}]$ and proceeding to the initial instant t_0 (see Fig. 5).

It is possible to construct the Euler polygon

$$y[t] = y[t_i] + (t - t_i)f^*(t_i) + (t - t_i)\varepsilon w^*(t_i),\ t \in [t_i, t_{i+1}],$$

where $f^*(t_i) \in F(t_i, y[t_i])$, $w^*(t_i) \in Q$, $i = 0,\ 1,\ \ldots,\ N_f - 1$. Nodes of this polygon-guide are points $y[t_0] \in \tilde{X}(t_0)$, $y[t_1] \in \tilde{X}(t_1)$, $\ldots$, $y[t_{N_f}] \in \tilde{X}(t_{N_f})$.

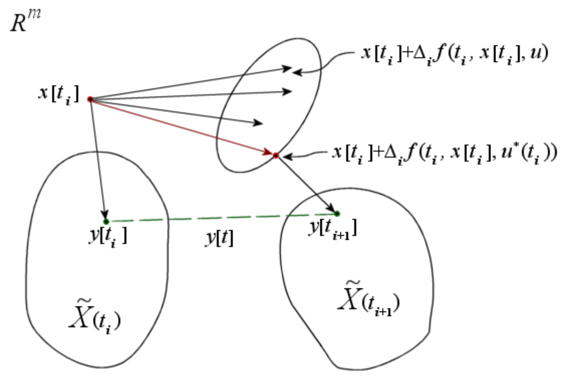

Figure 6.

The following equality is hold:

$$y[t_{N_f}] = y[t_0] + \sum_{i=0}^{N_f-1} \Delta_i f^*(t_i) + \sum_{i=0}^{N_f-1} \Delta_i \varepsilon w^*(t_i),$$

where $f^*(t_i) \in F(t_i, y[t_i])$, $w^*(t_i) \in Q$, $i = 0, 1, \ldots, N_f - 1$.

Now construct an admissible control $u^*(t)$, $t \in [t_0, t_{N_f}]$, that solves the Problem 2.

We construct the control $u^*(t)$, $t \in [t_0, t_{N_f}]$ sequentially at steps $[t_i, t_{i+1})$, $i = 0, 1, \ldots, N_f - 1$, of the partition Γ^f in the form of a piecewise-constant control $u^*(t) \equiv u^*(t_i)$, $[t_i, t_{i+1})$, $i = 0, 1, \ldots, N_f - 1$. Suppose that $u^*(t_0), u^*(t_1), \ldots, u^*(t_{i-1})$, corresponding to intervals $[t_0, t_1)$, $[t_1, t_2)$, $\ldots$, $[t_{i-1}, t_i)$, are constructed and the motion $x[t]$, $t \in [t_0, t_i]$, of the system (1) under the action of the control $u^*(t)$, $t \in [t_0, t_i)$ is realized. The vector function $x[t]$ satisfies the equation $\dot{x}[t] = f(t, x[t], u^*(t))$, $x[t_0] = y[t_0]$, almost everywhere on $[t_0, t_i]$.

Consider the point

$$x[t_i] = y[t_0] + \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} f(t, x[t], u^*(t_k))dt.$$

We choose the vector $u^*(t_i) \in P$, corresponding to the semiopen interval $[t_i, t_{i+1})$, from the condition (see Fig. 6, 7)

$$\|(x(t_i) + \Delta_i f(t_i, x[t_i], u^*(t_i)) - y[t_{i+1}])\| = \quad (4)$$

$$= \min_{u \in P} \|(x(t_i) + \Delta_i f(t_i, x[t_i], u) - y[t_{i+1}])\|.$$

It is possible to achieve that in the case of sufficiently small diameter Δ the following is hold: $x[t] \in \Phi^*(t)_{\varepsilon^*}$, $t \in [t_0, t_{N_f}]$, $x[t_{N_f}] \in O_{\varepsilon^*}(y[t_{N_f}])$.

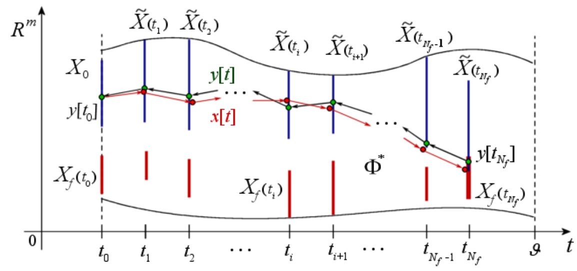

Figure 7.

4 Numerical methods

Two of tree suggested methods use presented scheme. Consider a peculiarity of these methods.

Polygons method.

At this method all sets (the moving polygon Υ^*, start and final sets, attainability sets, the phase constraint) are presented as polygons. Polygons may be non-convex. Each polygon is specified by a set of closed broken lines. One of these broken lines is an external border, others form internal border of polygon (in the common case arbitrary polygon may have number of holes). All operations of constructing attainability sets are based on operations with polygons (union, subtraction and intersection). Because of all polygons are formed by number of closed broken lines it allows to save a lot of memory on personal computer (PC) and in many cases to safe a time of computations in comparison with grid methods. On a contrary, the polygons method has comparatively complicated logic of computations, require a very high calculation accuracy on PC and at the current realization can be applied only for the case on the plane (2-dimensional case).

Example 1. Consider the system

$$\begin{cases} \dot{x}_1 = 2,2x_2 + \sin(0,6t)u_2, \\ \dot{x}_2 = 0,5x_2 - 5\sin(x_1) + 3\cos(0,8t)u_1 + 2,5u_2, \end{cases} \quad (5)$$

where $\|u\| \le 1$, $u \in R^2$, $x \in R^2$, $t \in [0; 10]$, $\Delta = 0,01$. Start conditions are shown on Figure 8

The result of calculation is shown on Figures 9, 10.

Grid method.

Grid method use not only discrete time model, but also use discrete space model. That is the m-dimensional space is broken with the regular grid and all sets are presented as sets of cells of this grid. The advantage of this method is the simple logic of calculations of attainability sets. This future allows to perform calculations on a m-dimensional space. On other side grid method is very time and memory consuming method (especially in the case of high precision of calculations). It leads us to the development of auxiliary methods which decrease a calculation time and decrease an amount of necessary PC memory. One of such algorithms is a border detecting algorithm that allows to exclude internal cells of sets from the calculation process.

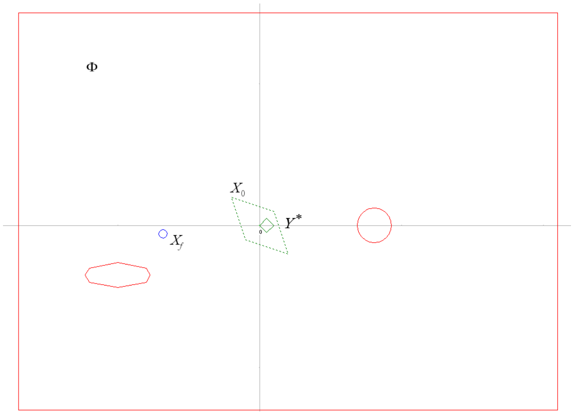

Figure 8. Start conditions.

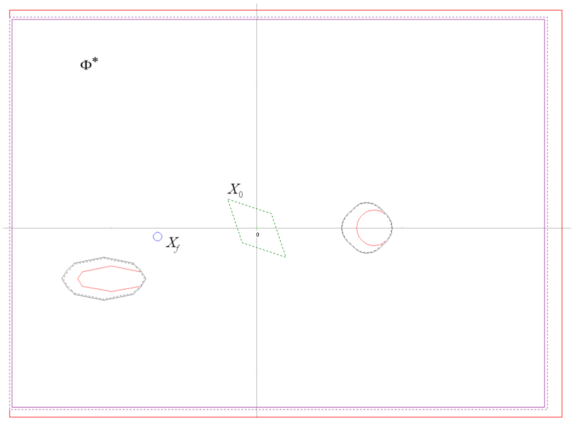

Figure 9. Transition to the moving center O.

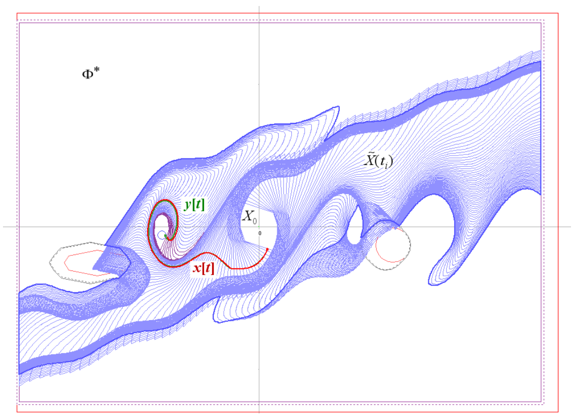

Figure 10. Attainability sets $\tilde{X}(t_i)$, polygon-guide $y[t]$ and trajectory $x[t]$.

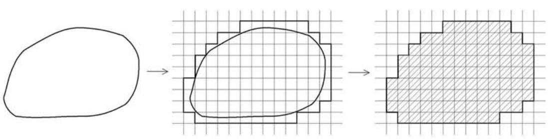

Figure 11. Grid presentation of a set on the plane.

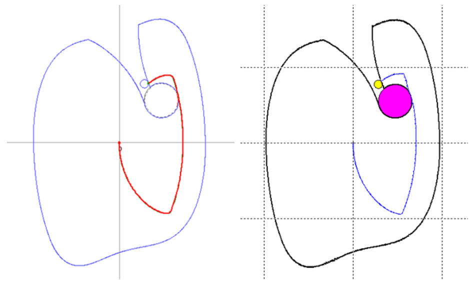

Figure 12.

Example 2. Consider the system

$$\begin{cases} \dot{x}_1 = -x_2, \\ \dot{x}_2 = x_1 + 2 \cdot x_1^3 - u, \end{cases} \tag{6}$$

where $u \in [-1;1]$, $x \in R^2$, $t \in [0;3]$, start set is a point $(0;0)$, $\Delta = 0,02$, size of grid sell is equal $0,001$.

The result of calculation is shown on the Figure 12. Left figure was calculated with help of polygon method (calculation time 12,5 minutes) and right one was calculated by grid method (more then 2 hours).

Method, based on Deikstra algorithm.

For a case of the stationary system (1) where the phase constraint $\Phi(t)$ is fixed in a time we apply some kind of the grid method – method based on the Deikstra algorithm. Here we also replace the problem of moving object Υ^* by the problem of the moving center O, but we don't apply a three stage method here.

At this method m-dimensional space is broken with the regular grid and all sets are presented as sets of cells of this grid. Instead of previous approaches we consider all cells as vertexes of some weighed graph. The weigh of each rib of graph is the time that needed for moving along this rib. Ribs of graph and their weights are calculated during the calculation process and depend on the system (1) and the form of the set P. Advantages of this method are comparatively small calculation time and possibility to consider a changeable orientation of the moving object. In the case of changeable orientation we have additional dimensions. The shortcoming of this method is that we can apply this method only for the case of the fixed $\Phi(t)$.

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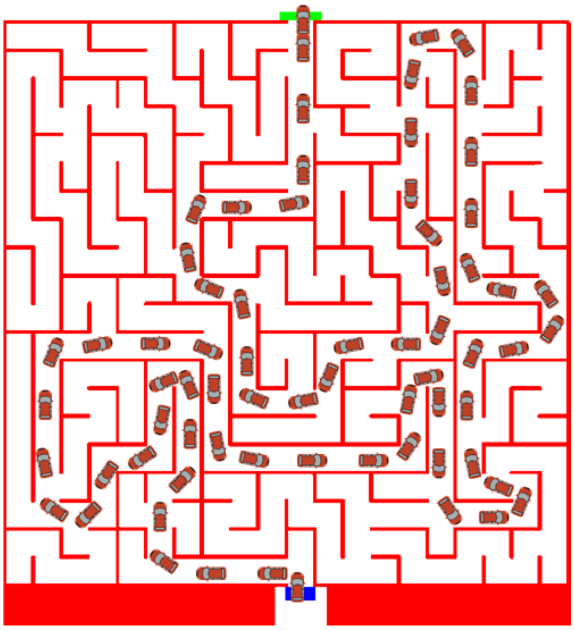

Figure 13.

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