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SOLUTION OF SELF-CONSISTENT BEAM DYNAMICS EQUATION IN DIELECTRIC LOADED WAKEFIELD WAVEGUIDES

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Abstract

Self-coordinated transverse dynamics of the high current relativistic electronic bunches used for generation of wake fields in wakefield accelerating structures with dielectric filling is investigated. An analytical approach to solution of self-coordinated beam dynamics is developed. The received solutions are used for a fast assessment of flight range of a bunch in dielectric wakefield structure.

Key words

Beam dynamics, wakefield accelerating structure, dielectric waveguide.

1 Introduction

Wakefield acceleration in dielectric wakefield waveguide structures is one of the most intensively developed direction among new methods of particle acceleration. Linear accelerators are considered also as sources of sequence of electronic bunches for the free electron laser, which is considered now the major candidate for creation of ultra short impulses (an attosecond range) X-ray radiation. Waveguide structures with dielectric filling excited by high current electronic bunches have been investigated intensively for the last years [Andonian et. al., 2012; Jing, Power and Zholents, 2008; Cook et. al., 2006; Sheinman and Kanareykin, 2008; Sheynman, Kanareykin and Sotnikov, 2012]. The main purpose of researches is the prospect of their usage for development of high gradient linear accelerators.

Development of new methods of charged particles acceleration based on principle of wakefield acceleration, inevitably requires a detailed analysis of the self-coordinated beam dynamics taking into account both own and external focusing and deflecting fields.

Wakefield principle is based on a generation by high-current electron bunch in the waveguide structure (Fig. 1) of an electromagnetic wave with a longitudinal component of the electric field up to 100 MV/m. This wake field is used to accelerate a following low-current bunch of high energy in wakefield accelerators or electromagnetic wave is extracted from the waveguide if structure is used as a radiation source (a free electron laser).

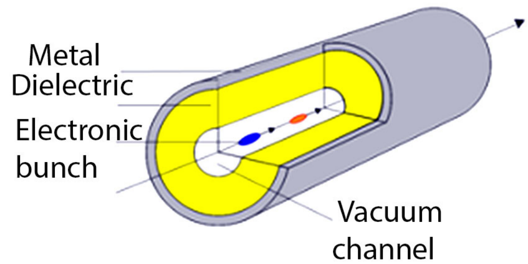

Figure 1. Waveguide structure with dielectric filling excited by a high current electronic bunch

Dielectric wakefield accelerating structures are single or multilayer dielectric cylindrical waveguides with outer metal covering and vacuum channel along the axis. Along with the longitudinal fields there are transverse fields, leading to bunch deflection from the axis of the waveguide and subsidence of particles on its wall. It makes impossible to continue the acceleration.

One of the main problems in realization of the wakefield method is keeping of an intensive electronic bunch in the channel of a wave guide and prevention of subsidence of particles on its wall. In this regard, a key task of the wakefield method of acceleration is modeling of the self-coordinated movement of the relativistic electronic bunch passing through dielectric structure in fields of Vavilov-Cherenkov created by it.

In recent years in tasks of the analysis of self-coordinated dynamics of relativistic electronic bunches in wakefield accelerating structures methods of direct numerical modeling have been developed. These methods are a particle – a particle and a particle – a grid. These methods allow on the set parameters of accelerating structure and an initial condition of the bunch to simulate process of its movement. The results of calculations are determination of the bunch flight range to a contact to them walls of accelerating structure, emittance of the bunch, and also transferred or received by bunch particles energy.

For the analysis of the radial beam dynamics in the accelerating structure, methods of computational experiment are used. To study the self-coordinated beam dynamics in dielectric wakefield accelerating structures we have developed a code "DynPart".

Analysis of the beam dynamics in the developed software is based on the method of macroparticles. In case of high-current relativistic beams electrostatic approximation inapplicable to determine the wakefield. That requires for the beam dynamics calculation using the method of "a particle – a particle". The number of operations in this method increases with the square of the number of macroparticles that increases the calculation time and thus imposes restrictions on their accuracy.

Shortcomings of these methods are considerable duration of calculations for ensuring accuracy of calculations, insistence to volume of random access memory and productivity of computer system. Let us note also that at change of parameters of the bunch and accelerating structure complete recalculation of a problem of a bunch movement is necessary.

For acceleration of calculations an analytical solutions of the equations of relativistic dynamics and an optimization algorithm for wakefield calculating were used.

For design of accelerating structures, solutions of optimization problems in which the structure and the bunch parameters maximizing efficiency of accelerating process are determined are necessary. The solution of such return tasks based on direct numerical modeling of dynamics demands repeated carrying out numerical calculations. Creation of the analytical description of self-coordinated dynamics of the bunch allowing direct parametric research of process in this regard is of interest.

2 Beam Dynamics Equations

The description of self-coordinated movement of an electronic bunch was carried out on the basis of the equations of relativistic dynamics assuming in the assumption of lack of the azimuthal movement of particles. Unlike Kapchirsky-Vladimirsky's model [Kapchinsky, 1982] we will consider the Cherenkov fields created by the bunch in the dielectric waveguide which have the defining impact on transverse dynamics of the bunch [Sheynman, Kanareykin and Sotnikov, 2012]:

$$F_r = d\left(m_e V_r \gamma\right)/dt. \tag{1}$$

Here

$$F_r = F_{fr} - eq \sum_{i,j} \left[\psi_{F_r\, i,\, j} I'{}_i \left(k_{r\, i,j} r(\zeta,t)\right) \times \int_0^{\zeta} f(\zeta_0) \sin\left(k_{z\, i,j}\left(\zeta-\zeta_0\right)\right) I_i(k_{r\, i,j} r(\zeta_0,t)) d\zeta_0 \right], \tag{2}$$

where $r(\zeta, t)$ is a bunch deflection from waveguide axes, $\zeta = z - vt$ is a distance behind the bunch, F_{fr} is a focusing force, e and m_e are charge and mass of electron, q and γ are charge and relativistic factor of the bunch, $k_{z\,i,j}$ and $k_{r\,i,j}$ are longitudinal and radial components of wave vector, $\psi_{F_r\,i,\,j}$ are coefficients of series, depending on geometry, wave guide filling permittivity and initial charge place [Altmark, Sheinman and Kanareykin, 2005], $f(\zeta_0)$ is a function describing longitudinal charge distribution, $I_i(x)$ are modified Bessel function of i-th order.

Both numerical and analytical methods were used to solve obtained equation.

3 Numerical Modeling of the Bunch Dynamics

Numerical solution of equation of the self-coordinated beam dynamics in dielectric wakefield accelerating structures (1) in code DynPart is based on next principles.

Longitudinal, radial and angular fields are obtained by integrating the function describing the radiation field at the point (z, r, θ) from a point charge located at coordinates (z_0, r_0, θ_0), folded with the charge distribution function on the bunch length. Implementing the macroparticle method charge distribution function is realized by generating an array of particles with a given distribution in space, and the integration is replaced by summation over the array. At each time point, the bunch particles coordinates x_0, y_0, z_0 are known, which are used to calculate fields. New particle coordinates at the next time moment and new velocities can be found based on the analytical solution of the relativistic dynamics equations [Sheinman and Kanareykin, 2008; Sheynman, Kanareykin and Sotnikov, 2012].

$$\begin{gathered} z = z_0 + \tfrac{c}{a^3}\left[a a_z\left(\gamma_1-\gamma_0\right) + \right. \\ \left. +\gamma_0\left(\left(a_x^2+a_y^2\right)\beta_{z0} - a_z\left(a_x\beta_{x0}+a_y\beta_{y0}\right)\right)\delta\right] \end{gathered} \tag{3}$$

$$\begin{gathered} x = x_0 + \tfrac{c}{a^3}\left[a a_x\left(\gamma_1-\gamma_0\right) + \right. \\ \left. +\gamma_0\left(\left(a_z^2+a_y^2\right)\beta_{x0} - a_x\left(a_z\beta_{z0}+a_y\beta_{y0}\right)\right)\delta\right] \end{gathered} \tag{4}$$

$$y = y_0 + \frac{c}{a^3}\left[a a_y\left(\gamma_1 - \gamma_0\right) + \right.$$
$$\left. +\gamma_0\left(\left(a_z^2 + a_x^2\right)\beta_{y0} - a_y\left(a_z\beta_{z0} + a_x\beta_{x0}\right)\right)\delta\right] \tag{5}$$

$$\beta_z = \xi/\gamma_1, \beta_x = \eta/\gamma_1, \beta_y = \chi/\gamma_1, \tag{6}$$

where

$$\delta = \ln\left|\frac{a^2t + \gamma_0(a_z\beta_{z0} + a_x\beta_{x0} + a_y\beta_{y0}) + a\gamma_1}{\gamma_0(a_z\beta_{z0} + a_x\beta_{x0} + a_y\beta_{y0} + a)}\right|,$$

$$\xi = a_z t + \beta_{z0}\gamma_0, \eta = a_x t + \beta_{x0}\gamma_0, \chi = a_y t + \beta_{y0}\gamma_0,$$

$$a_z = F_z/(m_e c), a_x = F_x/(m_e c), a_y = F_y/(m_e c),$$

$$a = \sqrt{a_z^2 + a_x^2 + a_y^2},$$

$$F_x = F_r\cos\theta - F_\theta\sin\theta + F_{f\,x},$$

$$F_y = F_r\sin\theta + F_\theta\cos\theta + F_{f\,y}.$$

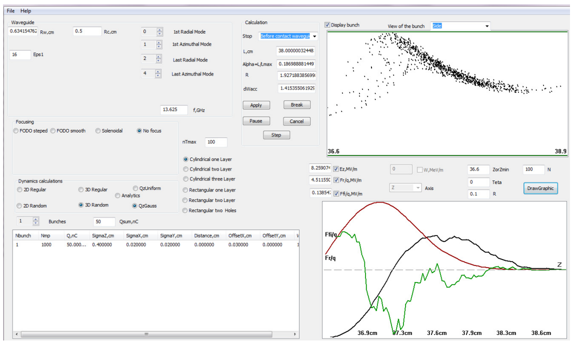

Figure 2. Screen of input data, beam shape and field distribution

Developed software allows you to:

1. Calculate of the single layer cylindrical waveguide parameters.
2. Simulate dynamics for any number of bunches in waveguide.
3. Solve the self-consistent equation of dynamics (1)–(2) in 2D, and 3D models (Fig. 2).
4. Perform parallel computing based on OpenMP for shared memory systems. The result is a substantial increase in performance of about 8 times compared with the linear calculations.
5. Choose parameters of alternating-gradient focusing and weak focusing.
6. Observe the transformation of the beam shape when it moves in the waveguide in the process of calculating (Fig. 2).
7. Display the field distribution in the space inside and outside the beam, and construct a vector diagram fields (Fig. 2).
8. Identify the flight range to prevent beam touching the waveguide wall.
9. Perform optimization of the parameters of the waveguide and the beam focusing system for maximization of flying range and energy extraction from the beam.

comparison of the software "Dynamics of Particles DynPart" with its nearest analogue BBU'3000 code shows a performance gain of 16 [Sheinman and Kirilin, 2014].

4 Analytical Solution for Free Beam

The task of the description of macroparticles movement is self-coordinated: the mutual provision of particles in ensemble influences on the field created by particles which, in turn, leads to change of their position. We consider an analytical method of the solution of the integro-differential equation of self-coordinated dynamics at the following simplifying assumptions:

1. The charge in the bunch is distributed evenly in the longitudinal direction. Thus $f(\zeta_0) = 1/l$, where l is a length of a bunch.

2. Changing of a relativistic factor over time is negligible $\gamma(t) = \gamma_0$.

In considered cases $k_{r\ i,j} r(\zeta, t) << 1$, in rejecting field at small deviations of the bunch from an axis the overwhelming contribution is brought by the 1st azimuthal mode $i = 1$. Nonlinear component of the force is negligible. Thus, it is possible to consider that the force acting on charges in the radial direction, depends on r linearly $I_1(kr) \approx kr/2$, $I_1{}'(kr) \approx 1/2$.

$$\frac{\partial^2 r(\zeta,t)}{\partial t^2} - \sum_{j=1}^{N} A_j \int_0^{\zeta} \sin\left(k_{zj}\left(\zeta - \zeta_0\right)\right) r(\zeta_0,\ t) d\zeta_0 =$$
$$= \frac{F_f}{\gamma_0 m_e} \tag{7}$$

Let us consider a one mode regime and $F_f = 0$ first. Then:

$$\frac{\partial^2 r(\zeta,t)}{\partial t^2} - A\int_0^{\zeta} \sin\left(k_z\left(\zeta - \zeta_0\right)\right) r(\zeta_0,\ t) d\zeta_0 = 0 \tag{8}$$

with initial conditions $r(\zeta, 0) = r_0$, $\left.\frac{dr(\zeta,t)}{dt}\right|_{t=0} = 0$.

We apply Laplace transformation on time to reduce of the received integro-differential equation to the integrated equation.

$$r^*(\zeta, p) - \frac{A}{p^2}\int_0^{\zeta} \sin\left(k_z\left(\zeta - \zeta_0\right)\right) r^*(\zeta_0,\ p) d\zeta_0 = \frac{r_0}{p} \tag{9}$$

The received integrated equation has the known solution obtained on the basis of transformation of Laplace on longitudinal coordinate.

$$r^*(\zeta,p) = \frac{r_0}{p} + \frac{r_0 A k_z}{p^3\sqrt{\left|k_z\left(k_z - A/p^2\right)\right|}} \times \int_0^{\zeta} \sinh\left(\sqrt{\left|k_z\left(k_z - A/p^2\right)\right|}\,(\zeta-\zeta_0)\right) d\zeta_0 \tag{10}$$

After integration we use the inverse Laplace transform:

$$r(\zeta,t) = r_0 + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left[\frac{r_0 A}{p^3\left(A/p^2 - k_z\right)} \times \left(\cosh\left(\sqrt{k_z\left(A/p^2 - k_z\right)}\zeta\right) - 1\right)e^{pt}\right]dp \tag{11}$$

The integrating contour is shown on Fig. 3.

Expressing the image of required function, and finding the original Laplace's inverse transformation by image decomposition in a Laurent series, we receive for lack of focusing force $F_f = 0$:

$$r(\zeta,t) = r_0 + v_{r0}t + \sum_{n=0}^{\infty}\left[\frac{(k_z\zeta)^{2n+2}}{(2n+2)!}\sum_{m=0}^{n}\left[\binom{n}{m} \times \frac{(-1)^m\left(t\sqrt{A/k_z}\right)^{2(n-m+1)}}{(2n-2m+2)!}\left(r_0 + \frac{v_{r0}t}{2n-2m+3}\right)\right]\right]. \tag{12}$$

Here we have taken into account initial velocity v_{r_0}.

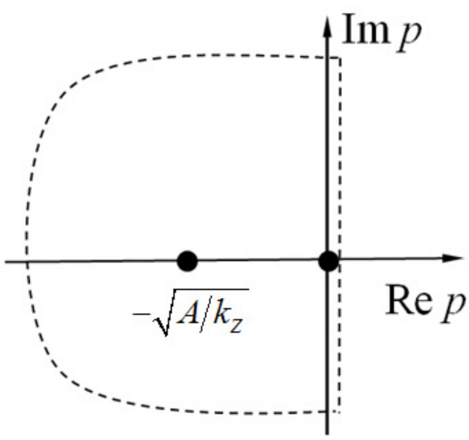

Figure 3. The integrating contour for inverse Fourier transform

If we consider a multi mode excitation the result become more complicated:

$$r(\zeta,t) = r_0 + r_0\sum_{n=0}^{\infty}\frac{\zeta^{2n+2}k_z^{2n+2}}{(2n+2)!}\sum_{m_1=0}^{n}\left[\sum_{m_2=0}^{\min(n-m_1,\, m_1+1)}\left[\frac{n!(-1)^{n-m_1-m_2}b^{m_2}(t/\tau)^{2(1+m_1-m_2)}}{m_1!m_2!(n-m_1-m_2)!(2(1+m_1-m_2))!}\right]\right]. \tag{13}$$

where $\tau = \frac{\sqrt{\sum_{j=1}^{n}\left(A_j k_{zj}^3\right)}}{\sum_{j=1}^{n}\left(A_j k_{zj}\right)}$,

$$b = \frac{\sum_{j=1}^{n-1}\sum_{l=j+1}^{n}\left(A_j A_l k_j k_l\left(k_l^2 - k_j^2\right)^2\right)}{\left(\sum_{j=1}^{n}\left(A_j k_j^3\right)\right)^2},$$

$$k_z^2 = \frac{\sum_{j=1}^{n}\left(A_j k_{zj}^3\right)}{\sum_{j=1}^{n}\left(A_j k_{zj}\right)}.$$

In case the contribution of one of modes is overwhelming, and the others can be neglected, the solution (13) coincides with (12).

5 Comparison of the Analytical Expression with Numerical Modeling

Comparison of the received analytical expression was carried out with numerical modeling of the bunch dynamics by the method of macroparticles based on the DynPart program. Comparative calculations were made at the following parameters of the waveguide and the bunch: $R_c = 0.5$ cm, $R_w = 0.634$ cm, $\varepsilon_1 = 16$, $W = 16$ MeV, $f = 13.625$ GHz, $Q = 100$ nC, $l = 6\sigma = 1.2$ cm, $r_0 = 0.01$ cm, $v_0 = 0$ m/s.

The DynPart program realizes modeling of uniform and Gaussian distribution of the bunch charge [3]. The bunch with the Gaussian profile of charge distribution exponential suppresses excitement of high modes of the waveguide that allows its comparison to analytical calculation of dynamics of the homogeneous bunch taking into account only one main mode. Program was terminated after the bunch had contacted with the vacuum channel wall.

Results of comparison are presented on Fig. 4, 5.

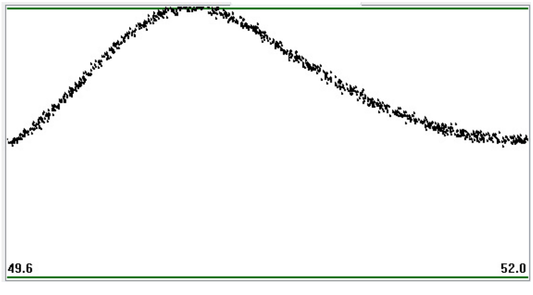

Figure 4. Charge distribution according to numerical calculation by macroparticles method

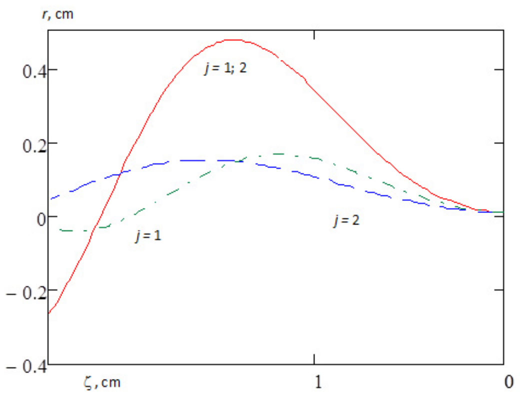

Figure 5. Charge distribution according to analytical solution

Flight range of the bunch to the contact of the waveguide wall according to analytical solution taking into account of two modes (Fig. 4) is $L = 53$ cm. Flight range has been found using numerical calculations (Fig. 5) is $L = 51$ cm.

It means that flight ranges are practically coincided. It proves a correctness of the assumptions chosen for analytical model.

6 Beam Dynamics with Focusing

Significant amplitude of own rejecting fields generated by high current bunches affecting his tail, emphasize focusing system. To keep the high current beam is appropriate to use a rigid focusing system based on FODO focusing [Wangler, 2008; Pavlov, 2008]:

$$F_f = -k(z)r = -ecr\partial B(z)/\partial r. \tag{14}$$

The period of the focusing system radial force can be approximated by the harmonic dependence for taking part a potential "sagging" between quadruple lenses:

$$F_f = -ecrB_0 \cos(2\pi z/L_s)/(2R_w). \tag{15}$$

Denoting $g_1 = ecB_0/(2m_e\gamma_0 R_w)$ we obtain:

$$F_f = -g_1 r m_e \gamma_0 \cos\kappa(\zeta + vt). \tag{16}$$

To simplify the beam dynamics equation let us consider that restoring force is linear increasing with deflection from waveguide axis without alternating sign component: $F_f = -g_0 r m_e \gamma_0$, where $g_0 = kg_1$, $k \approx 1/4$. Solving such integro-differential equation by Laplace transform method, we receive:

$$\begin{aligned} r(\zeta,t) &= r_0\cos\left(\sqrt{g_0}t\right) + \\ &+\frac{r_0}{a^2}\sum_{n=1}^{\infty}\left\{\frac{(k_z\zeta)^{2n+2}}{(2n+2)!}\times\right. \\ &\left.\times\sum_{j=0}^{n-1}\left[\frac{j+1}{a^j}\sum_{m=0}^{(n-j-1)}\binom{n}{m}\frac{(-1)^{m+j}\tau^{2(n-m-j-1)}}{(2(n-m-j-1))!}\right]\right\}+ \\ &+r_0\cos\left(t\sqrt{g_0}\right)\left(\frac{\cos(\lambda\zeta)-1}{(a+1)^2}+\frac{k_z\zeta\sin(\lambda\zeta)}{2\sqrt{a}(a+1)^{3/2}}\right)- \\ &-\frac{r_0t\sqrt{g_0}}{2(a+1)}\sin\left(t\sqrt{g_0}\right)\left(\cos\left(\lambda\zeta\right)-1\right), \end{aligned} \tag{17}$$

where $\tau = t\sqrt{A/k_z}$, $a = g_0k_z/A$, $\lambda = k_z\sqrt{(a+1)/a}$.

The received solutions can be used for fast assessment of flight range of the bunch in the dielectric wakefield structure, Fig. 6, 7.

Increasing of g_0 leads to increasing the flight range. The most significant increasing the flight range can be achieved by decreasing the beam offset from waveguide axes. This method increasing the flight range is limited by technical possibilities of the beam alinement before its inserting to dielectric filled accelerating structure.

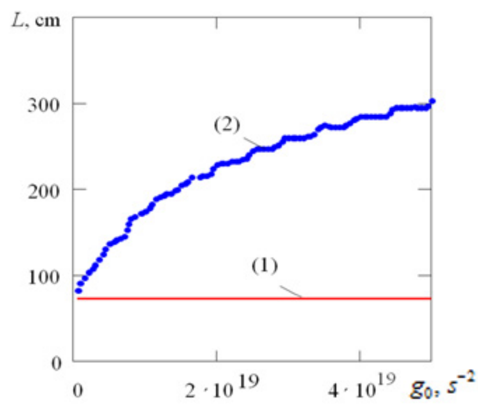

Figure 6. Flying range vs of force value

7 Conclusion

The methods of analytical calculation of self-coordinated bunch dynamics in wakefield accelerating structures can be used for development of new optimization tasks of accelerators for physics of high energy and perspective sources of radiation in THz frequency range.

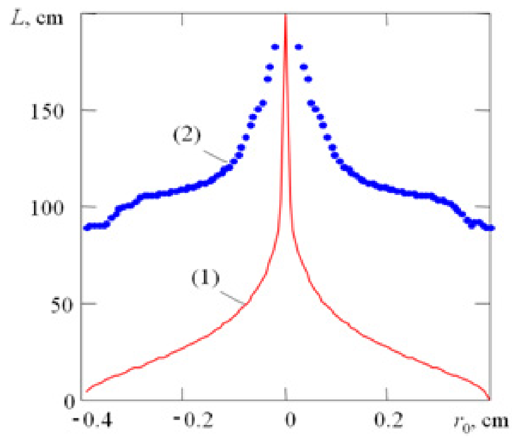

Figure 7. Flying range vs of initial offset

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