SHUKOVSKY STABILITY IN DYNAMICAL SYSTEMS. GENERALIZATION OF ANDRONOV–WITT THEOREM

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Andronov-Witt theorem is brilliant achievement of Andronov scientific school [1]. Generalization of this theorem for regular linearization was obtained by Demidovic [2]. Here for investigation of Zhukovskii stability a new research tool – a moving Poincare section – is introduced. With the help of this tool, generalization of theorems of Andronov-Witt and Demidovic for irregular linearization are carried out.

Consider the dynamical systems, generated by the differential equations

$$\frac{dx}{dt} = f(x), \quad t \in \mathbb{R}^1,\ x \in \mathbb{R}^n. \tag{1}$$

We introduce the following denotation

$$L^+(x_0) = \big\{x(t,x_0) \,|\ t \in [0,+\infty)\big\}.$$

Definition 1. *The trajectory $x(t,x_0)$ of dynamical system is said to be Poincare stable (or orbitally stable) if for any number $\varepsilon > 0$ there exists the number $\delta(\varepsilon) > 0$ such that for all y_0, satisfying the inequality $|x_0 - y_0| \le \delta(\varepsilon)$, the relation*

$$\rho\big(L^+(x_0),\, x(t,y_0)\big) \le \varepsilon, \quad \forall\, t \ge 0$$

is satisfied. If, in addition, for the certain number δ_0 and for all y_0, satisfying the inequality $|x_0 - y_0| \le \delta_0$, the relation holds

$$\lim_{t\to+\infty} \rho\big(L^+(x_0),\, x(t,y_0)\big) = 0,$$

then the trajectory $x(t,x_0)$ is said to be asymptotically Poincare stable (or asymptotically orbitally stable).

Here

$$\rho(L,x) = \inf_{y\in\Lambda} |x-y|,$$

$|\cdot|$ is Euclidean norm in R^n.

We introduce now the definition of Zhukovsky stability. For this purpose we need to consider the following set of homeomorphisms

$$Hom = \{\tau(\cdot)\,|\,\tau : [0,+\infty) \to [0,+\infty), \tau(0) = 0\}.$$

The functions $\tau(t)$ from the set Hom play the role of the reparametrization of time for the trajectories of system (1).

Definition 2 [3–7]. *The trajectory $x(t,x_0)$ of system (1.1) is said to be Zhukovsky stable if for any number $\varepsilon > 0$ there exists the number $\delta(\varepsilon) > 0$ such that for any vector y_0, satisfying the inequality $|x_0 - y_0| \le \delta(\varepsilon)$, the function $\tau(\cdot) \in Hom$ can be found such that the following inequality*

$$|x(t,x_0) - x(\tau(t),y_0)| \le \varepsilon, \quad \forall t \ge 0$$

is valid. If, in addition, for the certain number $\delta_0 > 0$ and any y_0 from the ball $\{y|\,|x_0 - y| \le \delta_0\}$ the function $\tau(\cdot) \in Hom$ can be found such that the relation holds

$$\lim_{t\to+\infty} |x(t,x_0) - x(\tau(t),y_0)| = 0,$$

then we shall say that the trajectory $x(t,x_0)$ is asymptotically stable by Zhukovsky.

This means that the stability by Zhukovsky is a stability by Lyapunov for the suitable reparametrization of each of perturbed trajectories.

The following obvious assertions can be formulated.

Proposition 1. *For dynamical system (1) the Lyapunov stability implies the Zhukovsky stability* *and the Zhukovsky stability implies the Poincare stability.*

Definition 3. *The number (or the symbol $+\infty, -\infty$), defined by formula*

$$\lambda = \limsup_{t\to+\infty} \frac{1}{t} \ln |f(t)|,$$

is called a characteristic exponent of the vector-function $f(t)$.

Definition 4. *The characteristic exponent λ of the vector-function $f(t)$ is said to be sharp if there exists the following finite limit*

$$\lambda = \lim_{t\to+\infty} \frac{1}{t} \ln |f(t)|.$$

The value

$$\lambda = \liminf_{t\to+\infty} \frac{1}{t} \ln |f(t)|$$

is called a lower characteristic exponent of the vector-function $f(t)$.

Consider the linear system

$$\frac{dx}{dt} = A(t)x, \quad x \in R^n \tag{2}$$

with the continuous and bounded on $[0, +\infty)$ $n\times n$ matrix $A(t)$.

Let $X(t) = (x_1(t), \ldots, x_n(t))$ be a fundamental matrix of system (2) (i.e. $\det X(0) \neq 0$).

It is well known that under the above conditions the characteristic exponents λ_j of the solutions $x_j(t)$ are numbers.

Definition 5. *Fundamental matrix $X(t)$ is said to be normal if the sum $\sum\limits_{j=1}^{n} \lambda_j$ of the characteristic exponents of the vector-functions $x_j(t)$ is minimal in comparison to other fundamental matrices.*

The following substantial and almost obvious results [8, 9] are well-known.

Theorem 1 (of Lyapunov on a normal fundamental matrix). *For any fundamental matrix $X(t)$ there exists the constant matrix C* ($\det C \neq 0$) *such that the matrix*

$$X(t)C$$

is a normal fundamental matrix of system (2).

Theorem 2. *For all normal fundamental matrices $(x_1(t), \ldots, x_n(t))$ the number of the solutions $x_j(t)$ with the same characteristic exponent is the same.*

By these results the following definitions can be introduced.

Definition 6. *The set of the characteristic exponents $\lambda_1, \ldots, \lambda_n$ of the solutions $x_1(t), \ldots, x_n(t)$ of certain normal fundamental matrices $X(t)$ is called the complete spectrum of linear system (2) and the numbers λ_j are called the characteristic exponents of system (2).*

Thus, any normal fundamental matrix realizes the complete spectrum of system (2).

In the sequel, by $\sum = \sum\limits_{j=1}^{n} \lambda_j$ is denoted the sum of characteristic exponents of system (2).

It is well-known the following Lyapunov inequality [8,9]

$$\sum \geq \limsup_{t\to+\infty} \frac{1}{t} \int\limits_0^t \operatorname{Tr} A(\tau)\, d\tau$$

Here Tr is a spur of the matrix A.

Definition 7. *If the relation*

$$\sum = \limsup_{t\to+\infty} \frac{1}{t} \int\limits_0^t \operatorname{Tr} A(\tau)\, d\tau$$

is satisfied, then system (2) is called regular.

It is well-known [8, 9] that each characteristic exponent of regular system is sharp.

Definition 8. *The number*

$$\Gamma = \Sigma - \liminf_{t\to+\infty} \frac{1}{t} \int_0^t Tr\, A(\tau)\, d\tau$$

is called the coefficient of irregularity (2).

We assume further that $\lambda_1 \geq \ldots \geq \lambda_n$. The number λ_1 is called a higher characteristic exponent.

We give here the stability criteria by the first approximation for the system

$$\frac{dx}{dt} = A(t)x + f(t, x), \quad t \geq 0, \quad x \in \mathbb{R}^n. \tag{3}$$

Here $A(t)$ is a continuous $(n \times n)$-matrix bounded for $t \geq 0$, $f(t, x)$ is a continuous vector-function, satisfying in the certain neighborhood $\Omega(0)$ of the point $x = 0$ the following condition

$$|f(t, x)| \leq \kappa |x|^\nu, \forall t \geq 0, \quad \forall x \in \Omega(0). \tag{4}$$

Here κ and ν are certain positive numbers, $\nu \geq 1$.

Theorem 3 (Chetaev [10], Malkin [11, 12], Massera [13]). *If conditions (4) and the inequality*

$$\lambda_1(1 - \nu) - \Gamma > 0 \tag{5}$$

are satisfied, then the solution $x(t) \equiv 0$ of system (3) is asymptotically Lyapunov stable.

The Zhukovsky stability is none other than the Lyapunov stability of reparametrized trajectories. Therefore for the investigations of Zhukovsky stability it makes possible to apply the arsenal of methods and devices, developed for the study of Lyapunov stability.

The reparametrization of trajectories permits us to introduce interesting and important tool of investigations, namely the *moving Poincare section.*

The classical Poincare section [14–17] is the transversal $(n-1)$-dimensional surface S in the phase space $\mathbb{R}^n$, which possesses a recurring property. The latter means that for the trajectory of dynamical system $x(t, x_0)$ with the initial data $x_0 \in S$ there exists the moment of time $t = T > 0$ such that $x(T, x_0) \in S$.

The transversal property means that

$$n(x)^* f(x) \neq 0, \quad \forall x \in S.$$

Here $n(x)$ is a normal vector of the surface S at the point x, $f(x)$ is the right-hand side of the differential equation (1).

We "force" now the Poincare section to move along the trajectory $x(t, x_0)$ (Fig. 1).

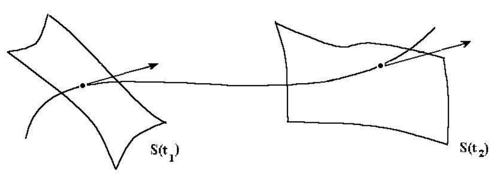

Fig. 1.

We assume further that the vector-function $f(x)$ is twice continuously differentiable and the trajectory $x(t, x_0)$, whose the Zhukovsky stability (or instability) will be considered, is wholly situated in the certain bounded domain $\Omega \subset \mathbb{R}^n$ for $t \geq 0$.

Suppose also that $f(x) \neq 0$, $\forall x \in \overline{\Omega}$. Here $\overline{\Omega}$ is a closure of the domain Ω.

Under these assumptions there exist the positive numbers δ and ε such that the following relation

$$f(y)^* f(x) \geq \delta, \forall y \in S(x, \varepsilon), \quad \forall x \in \overline{\Omega}$$

is satisfied. Here

$$S(x, \varepsilon) = \{y | \, (y - x)^* f(x) = 0, \quad |x - y| < \varepsilon\}.$$

Definition 9. *The set $S(x(t, x_0), \varepsilon)$ is called a moving Poincare section.*

The classical Poincare section allow us to clear up the behavior of trajectories using the information at discrete time of their crossings with the Poincare section. The reparametrization of trajectories makes it possible to organize the motion of trajectories so that at time t all trajectories are situated on the same moving Poincare section $S(x(t, x_0), \varepsilon)$:

$$x(\varphi(t), y_0) \in S(x(t, x_0), \varepsilon). \tag{6}$$

Here $\varphi(t)$ is a reparametrization of the trajectory $x(t, y_0)$, $y_0 \in S(x_0, \varepsilon)$.

This consideration has, of course, a local property and it is only possible for the values t such that

$$|x(\varphi(t), y_0) - x(t, x_0)| < \varepsilon. \tag{7}$$

Now we formulate more precisely the facts stated above.

Lemma 1. (On parametrization). *For any $y_0 \in S(x_0, \varepsilon)$ there exists the differentiable function $\varphi(t) = \varphi(t, y_0)$ such that either relations (6), (7) are valid for all $t \geq 0$ or there exists the value $T > 0$ such that relations (6) and (7) are valid for $t \in [0, T)$ and*

$$|x(\varphi(T), y_0) - x(T, x_0)| = \varepsilon. \tag{8}$$

In this case we have

$$\frac{d\varphi}{dt} = \frac{|f(x(t, x_0))|^2}{f(x(\varphi(t), y_0))^* f(x(t, x_0))} - \\ - \frac{(x(\varphi(t), y_0) - x(t, x_0)) \frac{\partial f}{\partial x}(x(t, x_0)) f(x(t, x_0))}{f(x(\varphi(t), y_0))^* f(x(t, x_0))}. \tag{9}$$

Here we denote by

$$\frac{\partial f}{\partial x}(x(t, x_0))$$

the Jacobian matrix of the vector-function f at the point $x(t, x_0)$.

Proof of Lemma 1. Consider the following function of two variables

$$F(t, \tau) = (x(\tau, y_0) - x(t, x_0))^* f(x(t, x_0)),$$

for which the relation $F(t, \tau) = 0$ is satisfied. From this and from the inclusion $y_0 \in S(t_0, \varepsilon)$ it follows that either $x(\tau, y_0) \in S(x(t, x_0), \varepsilon)$ for all $t \geq 0$, $\tau \geq 0$ or for some $T > 0$ and $\tau_0 > 0$ the relations hold

$$x(\tau, y_0) \in S(x(t, x_0), \varepsilon), \quad \forall \tau \in [0, \tau_0), \quad \forall t \in [0, T),$$

$$|x(\tau_0, y_0) - x(T, x_0)| = \varepsilon.$$

Since in these cases we have

$$\frac{\partial F}{\partial \tau} = f(x(\tau, y_0))^* f(x(t, y_0)) \geq \delta,$$

then by the implicit function theorem we obtain the existence of $\varphi(t)$ such that relations (6) are satisfied on $[0, T)$. In this case T is a finite number (if (8) is satisfied) or $T = +\infty$ (if (7) occurs for all $t \geq 0$). By the implicit function theorem we have

$$\frac{d\varphi}{dt} = -\frac{\partial F}{\partial t}(t, \varphi(t)) \left[\frac{\partial F}{\partial \tau}(t, \varphi(t))\right]^{-1}.$$

This implies relation (9). $\square$

We can similarly prove the following

Lemma 2. *If the trajectory $x(t, x_0)$ is Zhukovsky stable, then there exists the number $\delta > 0$ such that for any $y_0 \in S(x_0, \delta)$ there exists the differentiable function $\varphi(t) = \varphi(t, y_0)$ such that for all $t \geq 0$ relations (6) and (9) are valid.*

We write now an equation for difference $z(t) = x(\varphi(t), y_0) - x(t, x_0)$, using relations (1), (6) and (9).

By (1)

$$\frac{dz}{dt} = f(x(\varphi(t), y_0))\dot{\varphi}(t) - f(x(t, x_0)), \qquad (10)$$

Rewrite (10), using formula (9), as

$$\frac{dz}{dt} = f(z + x(t, x_0))\frac{|f(x(t, x_0)|^2}{f(z + x(t, x_0))^* f(x(t, x_0))} - \\ -\frac{z^* \dfrac{\partial f}{\partial x}(x(t, x_0))f(x(t, x_0))}{f(z + x(t, x_0))^* f(x(t, x_0))} - \\ -f(x(t, x_0)), \quad z^* f(x(t, x_0)) = 0. \qquad (11)$$

Represent now equation (11) in the form

$$\frac{dz}{dt} = A(x(t, x_0))z + g(t, z), \\ z^* f(x(t, x_0)) = 0\,, \qquad (12)$$

where

$$A(x) = \frac{\partial f}{\partial x}(x) - \frac{f(x)f(x)^*}{|f(x)|^2}\left[\frac{\partial f}{\partial x}(x) + \left(\frac{\partial f}{\partial x}(x)\right)^*\right].$$

We shall show that for system (12) the following relations

$$g(t, z)^* f(x(t, x_0)) = 0 \qquad (13)$$

$$|g(t, z)| = O(|z|^2) \qquad (14)$$

are valid. In fact, from the identity

$$f(x(t, x_0))^* z(t) \equiv 0$$

we have

$$\dot{z}(t)^* f(x(t, x_0)) + z(t)^* \frac{\partial f}{\partial x}(x(t, x_0))f(x(t, x_0)) \equiv 0.$$

Therefore

$$f(x(t, x_0))^*(\dot{z} - A(x(t, x_0))z) = 0.$$

This relation is equivalent to (13).

Estimate (14) results at once from (11) and the definitions of the matrix $A(x)$.

Thus, for system (12) we have the system of the first approximation

$$\frac{dv}{dt} = A(x(t, x_0))v, \quad f(x(t, x_0))^* v = 0. \qquad (15)$$

It differs from the usual system of the first approximation

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}(x(t, x_0))w \qquad (16)$$

in that we introduce here the projector

$$v = \left(I - \frac{f(x(t, x_0)f(x(t, x_0))^*}{|f(x(t, x_0)|^2}\right)w. \qquad (17)$$

It is not hard to prove that the vector-function

$$y(t) = \frac{f(x(t, x_0))}{|f(x(t, x_0))|^2}$$

is the solution of the following system

$$\dot{y} = A(x(t, x_0))y.$$

Therefore we can consider the fundamental matrix of this system

$$Y(t) = \left(\frac{f(x(t, x_0))}{|f(x(t, x_0))|^2}, y_2(t), \ldots, y_n(t)\right),$$

where the solution $y_j(t)$ satisfies the condition

$$f(x(t, x_0))^* y_j(t) \equiv 0, \quad \forall\, t \geq 0,\ \forall\, j = 2, \ldots, n.$$

Now we apply to the system of solutions, making up the matrix $Y(t)$, the procedure of orthogonalization

$$v_1(t) = \frac{f(x(t), x_0))}{|f(x(t), x_0)|^2}$$

$$v_2(t) = y_2(t) - v_1(t)^* y_2(t)\frac{v_1(t)}{|v_1(t)|^2}$$

$$\cdots\cdots\cdots$$

$$v_n(t) = y_n(t) - v_1(t)^* y_n(t)\frac{v_1(t)}{|v_1(t)|^2} - \cdots - \\ - v_{n-1}(t)^* y_n(t)\frac{v_{n-1}(t)}{|v_{n-1}(t)|^2}.$$

Further, from the orthogonal vector-functions $v_j(t)$ $(j = 1, \ldots, n)$ we make up the unitary matrix

$$U(t) = \left(\frac{v_1(t)}{|v_1(t)|}, \cdots, \frac{v_n(t)}{|v_n(t)|}\right).$$

It is obvious that in the case considered we have

$$\frac{v_1(t)}{|v_1(t)|} = \frac{f(x(t, x_0))}{|f(x(t, x_0))|}. \qquad (18)$$

It is clear that in this case the identity holds

$$f(x(t, x_0))^* U(t) \equiv (|f(x(t, x_0))|, 0 \ldots 0). \qquad (19)$$

Therefore by the unitary transformation

$$z = U(t)u$$

system (12) can be reduced to the form

$$\dot{u} = B(t)u + U(t)^* g(t, U(t)u), \qquad (20)$$

where

$$B(t) = U(t)^* A(x(t, x_0))U(t) - U(t)^* \dot{U}(t).$$

It is well-known that if $A(t)$ is bounded for $t \geq 0$ then $B(t), U(t), \dot{U}(t)$ are also bounded for $t \geq 0$. [9]

Relations (19) implies the equivalence of the following identities

$$f(x(t, x_0))^* z(t) \equiv 0$$

$$u_1(t) \equiv 0. \tag{21}$$

Here $z(t)$ is the solution of system (12) and $u_1(t)$ is the first component of the vector-function $u(t)$ being the solution of system (20).

Thus, system (12) can be reduced to system (20) of order $n-1$, where relation (21) is satisfied. The latter makes it possible to apply Theorem Chataev–Malkin–Massera about Lyapunov asymptotic stability to the study of the Zhukovsky stability. For this purpose we give some simple propositions.

Proposition 2. *If the zero solution of system (12) is Lyapunov stable, then the trajectory $x(t,x_0)$ is Zhukovsky stable. If the zero solution of system (12) is asymptotically Lyapunov stable then the trajectory $x(t,x_0)$ is asymptotically Zhukovsky stable.*

This assertion follows at once from Lemma 1 and transformations (10)–(12). Here as $\tau(t)$ (see Definition 2) we choose the reparametrization $\varphi(t)\,:\,\tau(t)=\varphi(t)$. □

Proposition 3. *If the zero solution of system* (12) *is Lyapunov unstable, then the trajectory $x(t,x_0)$ is Zhukovsky unstable.*

Proof. If the trajectory $x(t,x_0)$ is Zhukovsky stable, then by Lemma 2 there exists reparametrization $\varphi(t)$ for which relations (12) are valid and from the condition $|z(0)|\leq\delta$ the inequality $|z(t)|\leq\varepsilon,\ \forall t\geq 0$ follows. This means that the zero solution of system (12) is Lyapunov stable. The latter is in the contrast to condition of Proposition 3. This contradiction concludes the proof of Proposition 3. □

Proposition 4. *The Lyapunov stability, the asymptotic Lyapunov stability, and the Lyapunov instability for the zero solutions of systems (12) and (20)–(21) are equivalent.*

This proposition results from the unitary transformation $U(t)$, which reduces system (12) to system (20)–(21).

Propositions 2, 4, and Theorem 3 imply the following

Theorem 4. *If for system (15) the inequality*

$$\Lambda+\Gamma<0,$$

is satisfied, then the trajectory $x(t,x_0)$ is asymptotically Zhukovsky stable.

Here Λ is the higher characteristic exponent of system (15), Γ is a coefficient of irregularity.

Remark that for system (16) the vector–function $f(x(t,x_0))$ is the solution of it and the following relation

$$\lim_{t\to+\infty}\frac{1}{t}\ln|f(x(t,x_0))|=0 \tag{22}$$

is valid.

Hence

$$\liminf_{t\to+\infty}\frac{1}{t}\int\limits_0^t \operatorname{Tr}A(x(s,x_0))\,ds=$$

$$\lim_{t\to+\infty}\inf\left[\frac{1}{t}\int\limits_0^t\left(\operatorname{Tr}\,(\frac{\partial f}{\partial x}(x(s,x_0))\right)-\right.$$

$$-\operatorname{Tr}\left(\frac{f(x(s,x_0))f(x(s,x_0))^*}{|f(x,s,x_0)|^2}\left(\frac{\partial f}{\partial x}(x(s,x_0))\ +\right.\right.$$

$$\left.\left.\left.\left(\frac{\partial f}{\partial x}(x(s,x_0))\right)^*\right)\right)\right)\ ds\right]=$$

$$=\lim_{t\to+\infty}\inf\left[\frac{1}{t}\int\limits_0^t\left(\operatorname{Tr}\,(\frac{\partial f}{\partial x}(x(s,x_0))\right)-\right.$$

$$-\operatorname{Tr}\left(\frac{1}{|f(x(s,x_0)|^2}(f(x(s,x_0))^\bullet f(x(s,x_0))^*+\right.$$

$$\left.\left.+f(x(s,x_0))(f(x(s,x_0))^*)^\bullet\right)\right)\ ds]=$$

$$=\liminf_{t\to+\infty}\left[\frac{1}{t}\int\limits_0^t\left(\operatorname{Tr}\left(\frac{\partial f}{\partial x}(x(s,x_0))\right)-\right.\right.$$

$$\left.\left.-\frac{(|f(x(s,x_0))|^2)^\bullet}{|f(x(s,x_0))|^2}\right)\ ds\right]=$$

$$=\liminf_{t\to+\infty}\left[\frac{1}{t}\left(\int\limits_0^t\left(\operatorname{Tr}\left(\frac{\partial f}{\partial x}(x(s,x_0))\right)\right)-\ ds\right.\right.$$

$$\left.\left.-\ln|f(x(t,x_0))|^2\right)\right]=$$

$$=\lim_{t\to+\infty}\inf\frac{1}{t}\int\limits_0^t\operatorname{Tr}\left(\frac{\partial f}{\partial x}(x(s,x_0))\right)\ ds \tag{23}$$

and system (16) has the one null characteristic exponent λ_1. Denote by $\lambda_2\geq\cdots\geq\lambda_n$ the rest of characteristic exponents of system (16).

¿From relations (17) it follows that the characteristic exponents of system (15) are not greater than the corresponding characteristic exponents of system (16). From this and from (23) we have the inequality

$$\gamma\geq\Gamma. \tag{24}$$

Here γ is the coefficient of irregularity of system (16).

Besides we have

$$\lambda_2\geq\Lambda. \tag{25}$$

Theorem 4 and inequalities (24) and (25) give the following

Theorem 5. *If for system (16) the following inequality*

$$\lambda_2+\gamma<0 \tag{26}$$

is satisfied, then the trajectory $x(t, x_0)$ is asymptotically Zhukovsky stable.

This result is the generalization of the well-known Andronov– Witt theorem.

Theorem 6 (Andronov, Witt [1,9]). *If the trajectory $x(t, x_0)$ is periodic, differs from equilibria and for system (16) the inequality*

$$\lambda_2 < 0$$

is satisfied, then the trajectory $x(t, x_0)$ is asymptotically orbitally stable (asymptotically Poincare stable).

Theorem 6 is a corollary of Theorem 5 since system (16) with the periodic matrix

$$\frac{\partial f}{\partial x}(x(t, x_0))$$

is regular [9].

Recall that for the periodic trajectories the asymptotic stability by Zhukovsky and by Poincare are equivalent.

The theorem of Demidovich is also a corollary of Theorem 5.

Theorem 7 (Demidovich [2]). *If system (16) is regular (i.e. $\gamma = 0$) and the inequality $\lambda_2 < 0$ is satisfied, then the trajectory $x(t, x_0)$ is asymptotically orbitally stable.*

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