Necessary conditions of optimality for measure driven differential inclusions

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In this article, we present and discuss first-order necessary conditions of optimality for a very general class of nonlinear impulsive dynamic control systems whose dynamics are given in the form of measure driven differential inclusions and whose trajectories are subject to both state constraints and endpoint constraints.

$$
\begin{array}{ll}
\text{Minimize} & h(x(0), x(1)) \\
\text{subject to} & dx(t) \in F(t, x(t))dt + \mathbf{G}(t, x(t))d\mu(t) \qquad \forall t \in [0,1] \\
& (x(0), x(1)) \in C \subset I\!R^n \times I\!R^n \\
& l(t, x(t)) \leq 0 \qquad \forall t \in [0,1] \\
& d\mu \in \mathcal{K}
\end{array}
$$

where $h : I\!R^n \times I\!R^n \to I\!R$, and $l : [0,1] \times I\!R^n \to I\!R^q$ are given functions, $F : [0,1] \times I\!R^n \to \mathcal{P}(I\!R^n)$, and $G : [0,1] \times I\!R^n \to \mathcal{P}(I\!R^{n \times q})$ are given set-valued maps, $\mathcal{K} \subset C^*([0,1]; K)$ is the set of control measures supported on $[0,1]$ with range in a given set $K \subset I\!R^q$, and $C \subset I\!R^n \times I\!R^n$ is closed.

This control paradigm can be regarded as an idealization of systems with fast and slow dynamics. This is pertinent to important classes of systems with multi-phase missions or reconfigurable dynamics for which the switching between different "productive" activities represented by slow dynamics are modeled by fast dynamics.

Note that, if we consider $F(t,x) := \{f(t,x,u) : u \in \Omega\}$ and $\mathbf{G}(t,x) := \{G(t,x,u) : u \in \Omega\}$, where the measurable function u plays the role of the conventional control taking values in a given compact set $\Omega \subset I\!R^m$, then it is not difficult to see that this paradigm encompasses impulsive optimal control problems where dynamics are specified by controlled differential equations. Moreover, the dependence of the singular dynamics on the "conventional" control constitutes an interesting challenge with practical implications. This issue is partially addressed in [12]. This optimal control problem has also been considered in [9], but the stated optimality conditions are of different character.

The relationship between hybrid - systems whose evolution is defined by the interaction of time-driven and event-driven discrete dynamics - and impulsive systems has addressed in [4, 5]. The importance of the former is due to the emergence of the so-called networked systems. To better understand the extent to which the measure driven differential paradigm enables the composition of dynamic control systems, property at the crux of hybrid automata, a popular model for hybrid systems, just consider $x = col(y,z)$, a certain index set A, and $\mathcal{Z} = \{z_\alpha : \alpha \in A\}$, and note that the impulsive system

$$
\begin{cases}
\dot{y} = f(y,z,u), & u \in \Omega \\
dz = g(y,z)d\mu
\end{cases}
$$

encompasses a hybrid system specified by a collection of conventional systems

$$\{\dot{y} = f_\alpha(y,u),\ u \in \Omega:\ \alpha \in A\},$$

where $f_\alpha(y,u) := f(y, z_\alpha, u)$, and A is a given discrete or continuum set, being the evolution of the variable α dictated by a transition automaton.

The stated necessary conditions optimality are in the form of both an Hamiltonian inclusion and a maximum principle providing a complete characterization of both optimal and singular - notably that "during" jumps - optimal evolutions of the state trajectory. For this, the solution concept plays an important role. Here, we extend the one discussed in [11]. The basic idea of this solution concept consists in stretching the time variable on the support of the control measure and in filling in the "gaps" of the graph of trajectory of bounded variation by arcs that satisfy the singular (with respect to the Lebesgue measure) dynamics. Thus, the completion of the graph of a trajectory of bounded variation requires a time reparameterization so that the flow of the new time variable reflects, at each moment, the sum of the contributions of the original time and of the control measure variation. In particular, this yields the emergence of nonzero measure intervals whenever there is a discontinuity in the state trajectory, thus enabling the definition of an "equivalent" trajectory solution to an auxiliary conventional differential inclusion.

The presented necessary conditions of optimality are derived in the context of nonsmooth analysis and, in this sense, they extend those derived in [11] in that state constraints are, now, included, and nondegeneracy is ensured through the additional compatibility, regularity and controlability assumptions.

The conditions involve an adjoint equation in the form of a measure driven differential inclusion with a boundary condition reflecting the effect of the cost function as well as endpoint constraints as well as state constraints at the initial and the final times. The solution concept for the adjoint equation is exactly the one adopted for the dynamics of the given control system. Moreover, the definition of the singular component of the Hamiltonian has to be interpreted in a "path-valued" sense on the support of the atomic component of the control measure. The characterization of the optimal control measure involves two additional maximum conditions: one for the atomic and another for the continuous components of the optimal control measure. Moreover, the consideration of state constraints adds some complexity on the representation of the conditions since a measure in the multiplier emerges on the points of the time interval where the state constraints become active.

Nondegenerate first-order necessary conditions of optimality have been derived in [2]. The fact that a free time impulsive control problem with state constraints, besides the control constraints and the nonlinear equality and inequality endpoint state constraints, is considered in this reference brings in not only a lengthier statement of the conditions but also much more complex technical issues in their proof whose methods rely strongly on [1]. A key issue of this result is the nondegeneracy of the obtained conditions. However, the hypotheses assumed on the data of the control problem are smoother than those of (P), and the dynamics are given by a controlled differential equation, for which the vector fields multiplying the control measure depend on the time and state variables only.

A fixed-time optimal control problem without state or control constraints with dynamics similar to those in [2] satisfying even smoother assumptions is considered in [3], where first-order and second-order necessary conditions of optimality are derived. One important point in this work is that second-order information is used in order to select, from all those satisfying the first-order conditions, a subset of multipliers so that nondegeneracy is ensured, thus, dispensing with any a priori normality assumptions.

References

[1] A. Arutyunov. *Optimality Conditions: Abnormal and Degenerate Problems.* Kluwer Academic Publishers, 2000.

[2] A. Arutyunov, D. Karamzin, F. Lobo Pereira. A Nondegenerate Maximum Principle for the Impulsive Control Problem with State Constraints. *SIAM J. of Control & Optim*, **43**, 5, 2005, pp. 1812–1843.

[3] A. Arutyunov, V. Dykhta, F. Lobo Pereira. Necessary Conditions for Impulsive Nonlinear Optimal Control Problems without a priori Normality Assumptions. *J. of Optimization Theory & Applications*, **124**, 1, 2005, pp. 55-77.

[4] J.-P. Aubin. *Impulse differential equations and hybrid systems: A viability approach.* Lecture Notes, University of California at Berkeley, 2000.

[5] M. Branicky, V.S. Borkar, S.K. Mitter. A unified framework for hybrid control: model and optimal control theory. *IEEE Trans. on Automatic Control*, **43**, 1, 1998, pp. 31–45.

[6] A. Bressan and F. Rampazzo. Impulsive control systems without commutativity assumptions. *J. of Optimization Theory & Applications*, **81**, 1994, pp. 435–457.

[7] F.H. Clarke, Yu. Ledyaev, R. Stern, P. Wolenski. *Nonsmooth analysis and control theory.* Graduate Texts in Mathematics, **178**. Springer-Verlag. 1998.

[8] F.H. Clarke. *Optimization and Nonsmooth analysis.* Wiley-Interscience Publications. 1983.

[9] B. Miller, E. Rubinovitch. *Impulsive Control in Continuous and Discrete-Continuous Systems*, Kluwer Academic Publishers, Amsterdam, Holland, 2002.

[10] M. Motta, F. Rampazzo. Space-time trajectories of nonlinear systems driven by ordinary and impulse controls. *Differential and Integral Equations*, 1995.

[11] F.L. Pereira, G.N. Silva. Necessary conditions of optimality for vector-valued impulsive control problems. *Systems & Control Letters*, **40**, 2000, pp. 205–215.

[12] F.L. Pereira, J. Leite. Necessary Conditions of Optimality for Impulsive Control Problems with State Constraints. *FEUP Technical Report*, 2006.

[13] G.N. Silva and R.B. Vinter. Measure differential inclusions. *J. of Mathematical Analysis and Applications*, **202**, 1996, pp. 727–746.

[14] G.N. Silva, R.B. Vinter. Necessary conditions for optimal impulsive control problems. *SIAM J. Control and Optimization*, **35**, 1997, pp. 1829–1846.

[15] S. Zavalischin, A. Sesekin. *Impulsive processes: models and applications.* Nauka, Moscow, 1991.