STABILIZING ℓ_1- SEMIOPTIMAL FRACTIONAL CONTROLLER FOR DISCRETE NON-MINIMUM PHASE SYSTEM UNDER UNKNOWN-BUT-BOUNDED DISTURBANCE

Dmitriy Ivanov
Department of Information Systems Security,
Samara National Research University
Department of Mechatronics,
Samara State University of Transport
Russia
d.ivanov@samgups.ru

Oleg Granichin, Vikentii Pankov
Laboratory "Control of Complex Systems" IPME RAS
Faculty of Mathematics and Mechanics, Science Educational Center,
"Mathematical Robotics and Artificial Intelligence"
St. Petersburg State University,
St. Petersburg, Russia
o.granichin@spbu.ru, st040308@student.spbu.ru

Olga Granichina
Herzen State Pedagogical University of Russia,
St. Petersburg, Russia
olga_granitchina@mail.ru

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Abstract

We consider the problem of optimal stabilizing controller synthesis for a discrete non-minimum phase dynamic plant described by a linear difference equation with an additive unknown-but-bounded disturbance. When considering the 'worst' case of disturbance, solving this optimization problem has combinatorial complexity. However, by choosing an appropriate sufficiently high sampling rate, it becomes possible to achieve an arbitrarily small level of suboptimality using a noncombinatorial algorithm. In this article, we propose using fractional delays to achieve a small level of suboptimality without significantly increasing the sampling rate. We approximate fractional delays by minimizing the ℓ_1-norm of the objective function. The proposed approximation of the fractional delay allows obtaining zero additional error for many non-integer solutions. Furthermore, it is shown that with a non-zero approximation error, the resulting controller may have a smaller additional error than the controller obtained using integer optimization. The theoretical results are illustrated by simulation examples with non-minimum-phase plants of the second and third orders.

Key words

Optimal control, Robust control, Linear systems

1 Introduction

The problem of optimal suppression of external disturbances is one of the main problems in control theory. This problem has primarily been addressed in a stochastic setting, where the solution involves a Linear Quadratic Gaussian (LQG) controller, similar to a linear quadratic controller. Another model of external perturbation is harmonic with an unknown frequency, which can be addressed through H_∞-optimization. More recently, researchers have tackled the problem of arbitrary limited interference. The problem of optimal suppression of arbitrary bounded perturbations for discrete one-dimensional minimum-phase systems was clearly stated in [Yakubovich, 1975] and [Vidyasagar, 1986]; later it was called ℓ_1-optimization. Its solution for the cases of non-minimum-phase systems was obtained in [Barabanov and Granichin, 1984], [Dahleh and Pearson, 1987]. Further development of the theory and methods of ℓ_1-optimization is given in the books [Dahleh and Diaz-Bobillo, 1995; Sánchez-Peña and Sznaier, 1998]. ℓ_1 regularization has also gained extensive application in signal processing, particularly in compressive sensing [Cong et al., 2023; Qaisar et al., 2013], and inverse problems, such as those related to geological models [Khaninezhad, 2013; Pankov and Granichin, 2022].

An important advantage of the ℓ_1-theory turned out to be the obtaining of an explicit formula for the quality index equal to the worst asymptotic error in the class of perturbations. While the ℓ_1-optimization problem can be solved numerically by solving linear programming problem, the order of the optimal controller cannot be estimated in advance. For some examples, the order of the optimal controller can be very large for even simple objects. The synthesis of controllers with a given order for problems with limited noise has been addressed in [Vidyasagar, 1986; Blanchini and Sznaier, 2000; Polyak and Halpern, 2001]. However, this approach is far from optimal. [Granichin, 1990] proposes the optimal synthesis of the controller that results in a controller with sparsity properties. However, the resulting controller is infinite-dimensional and includes fractional delay elements.

In [Chiluka et al., 2021], Virtual Reference Feedback Tuning is used for robust control of non-minimum phase systems. In [Tanaka and Koga, 2019], a linear active disturbance rejection controller based on the linear quadratic regulator is introduced. Paper [Nagarsheth and Sharma, 2020] presents a fractional filter-PID controller for non-minimum phase systems with dead time.

[Ivanov et al., 2022] presents an approach based on the fractional delay approximation, proposing the use of a first-order filter to approximate the fractional delay. Conditions are obtained to determine cases where the first-order filter is better than the rounding approximation. Nevertheless, within the framework of the proposed approach, the filter for all fractional solutions is suboptimal and it is not possible to answer questions about:

1. What is the optimal order of the fractional lag filter?
2. Which of the many approximations is the best for the ℓ_1 controller implementation?

This article presents a new approach to implement an infinite-dimensional controller. The theorem is formulated that identifies cases where a sparse finite-dimensional controller can be obtained from an infinite-dimensional one. This paper's main contribution consists of the proposed method, which achieves zero additional error for many non-integer solutions through a new approximation of fractional delay. The article shows that by combining the proposed fractional delay approximation and finite upsampling in the controller, zero additional error can be obtained.

The article is organized as follows. Section 2 presents the problem statement and introduction of fractional delay and fractional delay filters for the implementation of fractional delay. Section 3 discusses the approximation errors of the controller. The simulation results are presented in Section 4. Finally, Section 5 concludes this paper.

2 Problem Statement and the Theoretical Framework

The results presented in this section are mostly based on the material stated in [Granichin, 2001].

2.1 Controller for Continuous Non-Minimum Phase System under Unknown-But-Bounded Disturbance

Let us consider a continuous-time control plant with the input–output transfer function

$$G(s) = \frac{g_0(s-\lambda^{(1)})\dots(s-\lambda^{(m)})\dots(s-\lambda^{(n-1)})}{(s-\overline{\lambda}^{(1)})\dots(s-\overline{\lambda}^{(2)})\dots(s-\overline{\lambda}^{(n)})} \quad (1)$$

We assume that the excess of the poles ($\overline{\lambda}$) and zeros (λ) of the system is equal to unity, the first m zeros of the transfer function $G(s)$ are unstable ($Re\lambda(i) > 0, i = 1, \dots, m$) and the remaining zeros are stable ($Re(\lambda(i)) < 0, i = m+1, \dots, n-1$). Here, $Re(\lambda)$ stands for the real part of a complex number λ. Let the poles $\lambda(1), \dots, \lambda(n)$ of the transfer function $G(s)$ not coincide with the first m unstable zeros.

We choose a discretization step $\delta > 0$ and study a family of piecewise constant functions defining the control actions varying at time instants $k\delta, k = 0, 1, 2, \dots$. By considering the discretization of the given continuous-time system in the zero approximation (see [Qiu and Davison, 1993]), we obtain for a sufficiently small value of δ a discrete system with the transfer function

$$H_\delta(z) = \frac{h_\delta z(z-\lambda_\delta^{(1)})\dots(z-\lambda_\delta^{(m)})\dots(z-\lambda_\delta^{(n-1)})}{(z-\overline{\lambda_\delta}^{(1)})\dots(z-\overline{\lambda_\delta}^{(2)})\dots(z-\overline{\lambda_\delta}^{(n)})} \quad (2)$$

with the poles $\overline{\lambda_\delta}^{(1)}, \dots, \overline{\lambda_\delta}^{(m)}, \dots, \overline{\lambda_\delta}^{(n-1)}$ and zeros $\lambda_\delta^{(1)}, \lambda_\delta^{(2)}, \dots, \lambda_\delta^{(n)}$. It is well known that for $\delta \to 0$ (see [Qiu and Davison, 1993]), the poles of the transfer function $H_\delta(z)$ are approximately related to the poles of $G(z)$ by $\overline{\lambda}_\delta^{(i)} = e^{-\delta\overline{\lambda}^{(i)}}, i = 1, \dots, n$; for zeros, these relations are

$$\lambda_\delta^{(i)} \approx e^{-\delta\lambda^{(i)}}, i = 1, \dots, n-1.$$

2.2 Controller for Discrete Non-Minimum Phase Plant under Unknown-But-Bounded Disturbance

Let us consider a discrete dynamic control plant described by the following equation

$$a\left(q^{-1}\right) y_t = b\left(q^{-1}\right) x_t + v_t, \quad (3)$$

where

- y_t , x_t, v_t are the output, input, and disturbance signals at time instant t respectively;
- q^{-1} is the backward shift operator: $q^{-1}y_t = y_{t-1}$;
- $a\left(q^{-1}\right)$ and $b\left(q^{-1}\right)$ are polynomials of q^{-1} ;

$$a(\lambda) = 1 + \sum_{i=1}^{n_a} a_i \lambda^i, b(\lambda) = \sum_{i=1}^{n_b} b_i \lambda^i.$$

It is additionally supposed that $\|v\|_\infty = \max_t |v_t| \leq C_v, C_v > 0$.

A linear stationary stabilizing regulator carries out the control of a plant with known parameters

$$\alpha\left(q^{-1}\right) u_t = \beta\left(q^{-1}\right) y_t \ , \tag{4}$$

where $\alpha(\lambda)$ and $\beta(\lambda)$ are polynomials of q^{-1}:

$$\alpha(\lambda) = 1 + \sum_{i=1}^{n_c} \alpha_i \lambda^i, \beta(\lambda) = \sum_{i=0}^{k_c} \beta_i \lambda^i.$$

If the characteristic polynomial of the closed-loop system described by Equations (3) and (4)

$$\chi(\lambda) = \alpha(\lambda) a(\lambda) - \beta(\lambda) b(\lambda) \tag{5}$$

has no unstable roots (in unit disk), then regulator described by Equation (4) is stabilizing

$$\sup_t \left(|y_t| + |u_t|\right) < \infty, \tag{6}$$

i.e., output y_t and control u_t are bounded.

Denote by $\bar{y} \in l_\infty$ the desired output of the control plant described by Equation (3). The meaningful formulation of the problem is to build the polynomials $\alpha\left(q^{-1}\right)$ and $\beta\left(q^{-1}\right)$, which are guaranteeing Inequality (6)—stabilizing of output and input—and asymptotic bound

$$\overline{\lim_{t\to\infty}} |y_t - \bar{y}_t| \leq J\left(\alpha(\cdot), \beta(\cdot), C_v\right) \tag{7}$$

with performance index

$$J\left(\alpha(\cdot), \beta(\cdot), C_v\right) = \inf_{\alpha(\cdot),\beta(\cdot)} \sup_{\|v\|_\infty \leq C_v} \overline{\lim_{t\to\infty}} |y_t - \bar{y}_t| \,. \tag{8}$$

Let be $\psi_{y/v}(\lambda)$ transfer function of closed-loop system from disturbance v_t to input y_t

$$\psi_{y/v}(\lambda) = \frac{\alpha_{y/v}(\lambda)}{\chi(\lambda)} = \sum_{i=0}^{\infty} \psi_i \lambda^i.$$

So, we have

$$y_t = \sum_{i=0}^{\infty} \psi_i v_{t-i}.$$

The control problem described by Equations (7) and (8) has an infinite dimension.

Due to the arbitrary nature of the disturbance performance index, Equation (8) can be rewritten in the form

$$J\left(\alpha(\cdot), \beta(\cdot), C_v\right) = C_v \inf_{\psi_{y/v}(\cdot)} \sum_{i=0}^{\infty} |\psi_i|. \tag{9}$$

Denote $b_-(\lambda)$ and $b_+(\lambda)$ the unstable and stable part of polynomial $b(\lambda)$. So, we have $b(\lambda) = b_-(\lambda) b_+(\lambda)$.

Assumption 1. *$a(\lambda)$ and $b_-(\lambda)$ are coprime polynomials, and polynomial $b_-(\lambda)$ has s unstable different not zero and not unit roots $\lambda_1, \ldots, \lambda_s$ and does not have unit roots.*

In [Barabanov and Granichin, 1984] it was shown under Assumption 1 that optimal regulator characteristic polynomial $\chi(\lambda)$ is equal to $b_+(\lambda)$ and optimal polynomial $\alpha(\lambda)$ has following structure: $\alpha(\lambda) = f(\lambda) b_+(\lambda)$, where polynomial $f(\lambda)$ has not more s non-zero coefficient (s-sparse structure) and highest possible degree of $f(\lambda)$ is bounded and depended on highest of magnitude on non-stable zeros of polynomial $b(\lambda)$.

The following theorem from [Granichin, 2001] gives the answer to the question about the achievable quality of control. It allows to reformulate the control problem (7)–(8) as a finite dimension problem.

Theorem 1. *Under Assumption 1, the minimum value of the function $C_v \|F(X)\|_1$ is a lower bound estimate for minimum value of (8), i.e.*

$$\min_{X \in \mathbb{R}_+^s} C_v \|F(X)\|_1 \leq J\left(\alpha(\cdot), \beta(\cdot), C_v\right) \tag{10}$$

where $F(X) = A^{-1}(X) B \in \mathbb{R}^{s+1}$, $X = (x_1, x_2, \ldots, x_s)^T \in \mathbb{R}_+^s$, $x_i \geq 1$, $\|F\|_1 = \sum_{i=0}^{s} |f_i|$,

$$A(X) = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 1 & \lambda_1^{x_1} & \ldots & \lambda_1^{\sum_{j=1}^{s} x_j} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_s^{x_1} & \ldots & \lambda_s^{\sum_{j=1}^{s} x_j} \end{pmatrix},$$

$B = \left(1 \ \frac{1}{a(\lambda_1)} \ \cdots \ \frac{1}{a(\lambda_s)}\right)^{\mathrm{T}}$. *If the minimum of the left side of Equation 10 is achieved in point X_o with integer-value components, then the polynomial of the ℓ_1-optimal stabilizing controller can be obtained by the formulas (see [Granichin, 2001])*

$$\alpha(\lambda) = f(\lambda) b_+(\lambda), \tag{11}$$

$$\beta(\lambda) = \frac{\left(a(\lambda) f(\lambda) - 1\right)}{b_-(\lambda)}, \tag{12}$$

where $f(\lambda) = f_0 + \sum_{j=1}^{m} f_j(X_o) \lambda^{D_j}, D_j = \sum_{i=1}^{j} x_k.$

3 Approximation Error of Controller

In [Ivanov et al., 2022] it is shown that the error caused by the fractional delay approximation does not exceed the following value

$$\|\Delta F\|_1 \leq \|\Delta A\|_1 \left\|A^{-1}\right\|_1 \|F\|_1. \tag{13}$$

It follows from the Inequality 13 that for the minimum error it is required to minimize the ℓ_1 norm of the matrix

$$\|\Delta A\|_1 = \max_{1\leq j\leq s+1} \sum \Delta a_{ij}.$$

The matrix norm depends only on s last rows and columns, since the error in the remaining columns is zero.

Let us consider options for approximating fractional delays to minimize the ℓ_1 norm of the matrix $\|\Delta A\|$. Let us introduce the notation for the integer and fractional parts powers of the matrix A:
$D_j = floor\left(\sum_{i=1}^{j} x_i\right), d_j = \sum_{i=1}^{j} x_i - D_j, j = 1, \cdots, s.$

Representing it in the irreducible fraction of an integer and a fractional part on s last rows and columns

$$\tilde{A} = \begin{pmatrix} \lambda_1^{D_1}\lambda_1^{d_1} & \dots & \lambda_s^{D_s}\lambda_s^{d_s} \\ \vdots & \ddots & \vdots \\ \lambda_s^{D_1}\lambda_s^{d_1} & \dots & \lambda_s^{D_s}\lambda_s^{d_s} \end{pmatrix}$$

The family of nonrecursive filters $C(\lambda)$ minimizing the norm of a matrix $\left\|\Delta\tilde{A}\right\|_1$

$$\max_{1\leq j\leq s} \min_{C(\lambda)} \|\Delta\tilde{A}(C(\lambda))\|_1 = \max_{1\leq j\leq s} \min_{C(\lambda)} \sum_{i=1}^{s} |\Delta\tilde{a}_{ij}(C(\lambda))|$$

is defined as

$$C(\lambda, j) = \sum_{k=0}^{K} c_{kj}\lambda^k$$

In the following theorem, we consider which solutions can be obtained with a filter that completely eliminates additional error.

Theorem 2. *Let a filter family is have the order of the polynomial $K = s-1$, and $s > 1$, then the norm of the matrix $\left\|\Delta\tilde{A}\right\|_1$ is zero and coefficients can be found from solving s systems of equations*

$$\sum_{k=0}^{K} c_{kj}\lambda_i^k = \lambda_i^{d_j(s)}, i = 1, s$$

If the degree of the family of polynomials is $K < s-1$, then the norm of the matrix $\left\|\Delta\tilde{A}\right\|_1 \neq 0$ is determined by the formula

$$\max_{1\leq j\leq s} \min_{c_{kj}} \sum_{i=1}^{s} \lambda_i^{D_j(s)} \left|\sum_{k=0}^{K} c_{kj}\lambda_i^k - \lambda_i^{d_j(s)}\right|.$$

where $c_j = (c_{0,j}, \cdots, c_{K,j})^T$, $D_j(s) = floor(1 + \sum_{i=1}^{j} x_i - s)$, $d_j(s) = 1 + \sum_{i=1}^{j} x_i - D_j(s)$, $j = 1, \cdots, s$.

Proof. The value of the objective function is equal to 0 because the choice of parameters is equivalent to solving a determined system of linear algebraic equations.

The application of Theorem 2 makes it possible to implement a controller with zero additional error, in contrast to the implementation of the controller proposed in [Ivanov et al., 2022] .

Corollary 1. *The combination of the proposed fractional delay approximation and finite upsampling in the controller provides zero additional error.*

Corollary 2. *For $s = 2$, a linear filter will give an error equal to zero in all s matrix columns. Thus, for $s \leq 2$ the linear filter allows reaching the zero level of additional error.*

Filter coefficients for each column are determined from the minimum condition

$$\min_{c_{0j}c_{1j}} \sum_{i=1}^{s} \lambda_i^{D_j} \left|c_{0j} + c_{1j}\lambda_i - \lambda_i^{d_j}\right|. \tag{14}$$

4 SIMULATION RESULTS

4.1 Example 1 [Ivanov et al., 2022]. Number of unstable zeros are $s = 2$

The class of non-minimum phase plants is described by the equation

$$\begin{gathered} y_t - 1.91y_{t-1} + 5.2y_{t-2} = \\ \lambda_1\lambda_2 u_{t-1} - (\lambda_1 + \lambda_2) u_{t-2} + u_{t-3} + \upsilon_t, \end{gathered} \tag{15}$$

The [Ivanov et al., 2022] article provides figures for additional error for various values for implementing a controller using fractional rounding and a fractional delay filter.

It follows from the results of Theorem 1 that the additional approximation error in approximating the fractional delay by the objective function (14) is equal to 0.

Table 1 shows the mean and standard deviation of additional errors for all λ_1, λ_2.

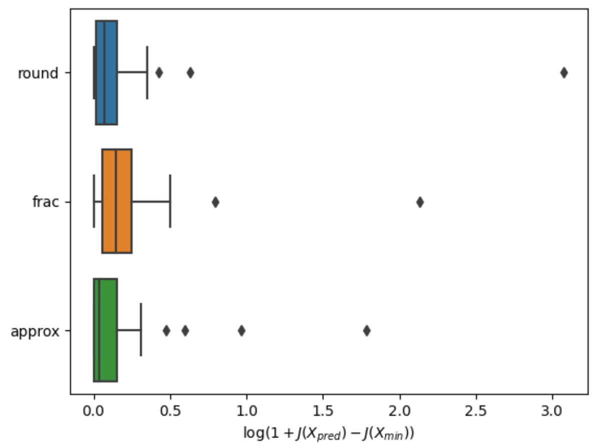

Figure 1. Boxplot of additional errors caused by rounding (round), fractional lag filter (frac), and approximation 14 (approx)

Table 1. Mean and standard deviation of additional errors

Method	Accuracy characteristics	
	Mean	**Std**
Rounding	0.0591	0.2050
Fractional filter	0.0468	0.1018
Approximation (14)	0	0

4.2 Example 2. Number of unstable zeros are $s = 3$

Consider designing a controller using a first-order filter with non-zero error. The class of non-minimum phase plants is described by the equation

$$y_i - 1.91y_{i-1} + 5.2y_{i-2} = -0.5\lambda_1\lambda_2 u_{i-1} + \\ +(0.5(\lambda_1+\lambda_2) + \lambda_1\lambda_2)u_{i-2} - \\ -(0.5 + \lambda_1 + \lambda_2)u_{i-3} + u_{i-4} + v_i, \quad (16)$$

where the corresponding matrices are:

$$A(X) = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0.5^{x_1} & 0.5^{x_1+x_2} & 0.5^{x_1+x_2+x_3} \\ 1 & \lambda_1^{x_1} & \lambda_1^{x_1+x_2} & \lambda_1^{x_1+x_2+x_3} \\ 1 & \lambda_2^{x_1} & \lambda_2^{x_1+x_2} & \lambda_2^{x_1+x_2+x_3} \end{Vmatrix},$$

$$B = \begin{pmatrix} 1 \\ 1/a(0.5) \\ 1/a(\lambda_1) \\ 1/a(\lambda_2) \end{pmatrix},$$

and $C_v = 1$.

The Global Optimization Toolbox for Matlab was used to find the minimum of the objective function. The best solution obtained using the genetic and particle swarm algorithms was chosen.

Figure 1 shows boxplot representing the distributions of additional errors for the compared methods. They illustrate that errors caused by using the approximation in (14) have a lower mean, variance, and maximum values compared to the rounding and fractional lag methods.

Table 2 shows the mean and standard deviation of additional errors for all λ_1, λ_2.

Table 2. Mean and standard deviation of additional errors

Method	Accuracy characteristics	
	Mean	**Std**
Rounding	0.2423	1.6610
Fractional filter	0.2342	0.6031
Approximation	0.1339	0.4266

Let us consider one of the plant in more detail. The non-minimum phase plants is described by the equation

$$y_i - 1.91y_{i-1} + 5.2y_{i-2} = 0.3u_{i-1} + \\ 1.375u_{i-2} - 2.05u_{i-3} + u_{i-4} + v_i, \quad (17)$$

with the following matrices A and B:

$$A(X) = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 1 & 0.5^{x_1} & 0.5^{x_1+x_2} & 0.5^{x_1+x_2+x_3} \\ 1 & 0.75^{x_1} & 0.75^{x_1+x_2} & 0.75^{x_1+x_2+x_3} \\ 1 & 0.8^{x_1} & 0.8^{x_1+x_2} & 0.8^{x_1+x_2+x_3} \end{Vmatrix},$$

$$B = \begin{pmatrix} 1 \\ 1/a(0.5) \\ 1/a(0.75) \\ 1/a(0.8) \end{pmatrix}.$$

The minimum of the functional (9) is at the point $X_{min} = (1.4055, 1, 6.8605) J_{min} = 3.6185$.

Polynom $f_{min}(q^{-1})$ is

$$f_{\min}\left(q^{-1}\right) = 1 - 1.3542q^{-1.4055} + \\ 4.4284 \cdot 10^{-5} q^{-2.4055} + 1.2662q^{-9.2928}. \quad (18)$$

To optimize the objective function with the integer condition, the minimum is at the point $X_{int} = (2, 1, 9), J(X_{int}) = 3.7729$. Polynom $f_{int}(q^{-1})$ is

$$f_{\text{int}}\left(q^{-1}\right) = 1 - 0.6260q^{-2} \\ +0.8207q^{-3} + 1.3262q^{-9}. \quad (19)$$

There is an additional error $J(X_{int}) - J(X_{min}) = 0.1544$.

When rounding the resulting delays [Granichin, 2001] to the nearest integers , the minimum point is $X_{round} = (2, 1, 7)$, $J(X_{round}) = 9.4012$. Polynom $f_{round}(q^{-1})$ is

$$f_{\text{round}}\left(q^{-1}\right) = 1 - 0.998820q^{-1} \\ -2.9035q^{-2} + 4.4995q^{-9}. \tag{20}$$

There is an additional error $J(X_{int}) - J(X_{min}) = 5.7827$.

When using the following first order fractional delay filters [Ivanov et al., 2022]:

$$C_1\left(q^{-1}\right) = 0.5945 + 0.4055q^{-1},\\ C_2\left(q^{-1}\right) = 0.5945 + 0.4055q^{-1},\\ C_3\left(q^{-1}\right) = 0.7340 + 0.2660q^{-1},$$

Polynom $f_{fd}(q^{-1})$ is

$$f_{\text{fd}}\left(q^{-1}\right) = 1 + 0.05816q^{-1} - 0.8483q^{-2} \\ -0.6056q^{-3} + 0.9448q^{-9} + 0.3424q^{-10}. \tag{21}$$

There is an additional error

$$J_{fd} = 3.8786, J_{fd} - J(X_{min}) = 0.2601$$

When using the approximation (14), the minimizing norm of the matrix $\|\Delta A\|_1$, the following family of first-order filters was obtained:

$$C_1\left(q^{-1}\right) = 0.4908 + 0.5284q^{-1},\\ C_2\left(q^{-1}\right) = 0.4908 + 0.5284q^{-1},\\ C_3\left(q^{-1}\right) = 0.6858 + 0.3208q^{-1},$$

Polynom $f_{approx}(q^{-1})$ is

$$f_{approx}(q^{-1}) = 1 + 0.0104q^{-1} + 0.7041q^{-2} + \\ 0.7460q^{-3} + 0.9112q^{-9} + 0.4260q^{-10}. \tag{22}$$

There is an additional error $J_{approx} = 3.7660$, $J_{approx} - J(X_{min}) = 0.1475$.

It follows from Theorem 2 that the use of a family of second-order filters will make it possible to obtain a controller implementation with zero additional error.

When using the approximation (14), the minimizing norm of the matrix $\|\Delta A\|_1$, the following family of second-order filters was obtained:

$$C_1\left(q^{-1}\right) = -0.0677 + 0.7110q^{-1} + 0.3590q^{-2},\\ C_2\left(q^{-1}\right) = -0.0677 + 0.7110q^{-1} + 0.3590q^{-2},\\ C_2\left(q^{-1}\right) = -0.0580 + 0.8357q^{-1} + 0.2240q^{-2},$$

Polynom $f_{\text{approx 2}}\left(q^{-1}\right)$ is

$$f_{\text{approx 2}}\left(q^{-1}\right) = 0.6104 - 3\cdot 10^{-6}q^{-1} \\ -0.4862q^{-2} + 1.6\cdot 10^{-5}q^{-3} \\ -0.0734q^{-8} + 1.0582q^{-9} + 0.2837q^{-10} \tag{23}$$

4.3 Example 3. Flexible-link manipulator

The following transfer function is obtained by identification of a flexible-link manipulator [Ho and Tu, 2005; Merrikh-Bayat and Bayat, 2013]

$$G(s) = \frac{\sum_{i=1}^{6} b_i s^i + b_0}{\sum_{i=1}^{9} a_i s^i + a_0},$$

where the values of the parameters of the identified model are given in Table 3.

Table 3. Parameters of the identified model in Example 3

i	a_i	b_i
9	1	
8	1486.7	
7	69317.7	
6	1.616×10^7	-14340.4953
5	1.062×10^9	4.446×10^6
4	6.167×10^{10}	5.697×10^8
3	2.624×10^{12}	1.908×10^{10}
2	3.595×10^{13}	9.354×10^{11}
1	1.42×10^{14}	6.919×10^{12}
0	0	2.839×10^{14}

This system has three non-minimum phase zeros located at $z_1 = 400.0282, z_2 = 45.0015$, and $z_3 = 19.9982$.

The poles and zeros of the discrete transfer function with sampling time $T_d = 0.003$ are given in the Table 4.

The minimum of the functional is at the point $X_{\min} = \left(1\ 1.7396\ 14.6249\right), J\left(X_{\min}\right) = 9.3265$.

To optimize the objective function with the integer condition, the minimum is at the point

$$X_{\text{int}} = \left(1\ 2\ 14\right), J\left(X_{\text{int}}\right) = 9.3826.$$

When rounding the resulting delays [Granichin, 2001] to the nearest integers, the minimum point is

$$X_{\text{round}} = \left(1\ 3\ 13\right), J\left(X_{\text{int}}\right) = 9.4012$$

When using the following first order fractional delay filters [Ivanov et al., 2022], $J_{fd}(X) = 9.5630$.

When using the approximation (14), the minimizing norm of the matrix $\|\Delta A\|1$, $J_{\text{aprox}}\left(X\right) = 9.4658$.

It follows from Theorem 2 that the use of a family of second-order filters will make it possible to obtain a controller implementation with zero additional error.

Table 4. Poles and zeros of transfer function in Example 3

Poles	Zeros
1.0000	**0.3011**
3.3198	1.3909
$0.8956 + 0.5001i$	**0.8737**
$0.8956 - 0.5001i$	**0.9417**
$0.9939 + 0.1714i$	1.0778
$0.9939 - 0.1714i$	1.0618
1.1446	
1.0366	
1.021	

5 Conclusion

The article proposes a controller implementation based on a new fractional delay approximation. This makes it possible to obtain zero additional error for many non-integer solutions. It is also shown that with a non-zero approximation error, the resulting controller may have a lower level of error than the controller obtained using integer optimization.

The implementation of a controller with an additional error equal to zero has less computational complexity than the implementation of a controller with a non-zero additional error. This is explained by the fact that the optimization problem is reduced to solving the determined system of equations.

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