IMPROVING THE REDUCTION PROCESS IN FLEXIBLE MULTIBODY DYNAMICS BY THE USE OF 2^{ND} ORDER POSITION GRAMIAN MATRICES

Jörg Fehr, Peter Eberhard and Michael Lehner
Institute of Engineering and Computational Mechanics
University of Stuttgart
Pfaffenwaldring 9, 70569 Stuttgart, Germany
Tel: ++49(0) 711/685 66388, Fax: -66400, fehr@itm.uni-stuttgart.de

Abstract

An important issue within the field of flexible multibody dynamics is the reduction of the flexible body's degrees of freedom. By using frequency weighted Gramian matrix based reduction techniques, the distribution of the loads is taken into account a priori. Furthermore, an energy interpretation of the reduction procedure and error bounds in the frequency domain are available. This allows a fully automated reduction process. For numerical reasons, the dominant eigen vectors of the Gramian matrix have to be approximated. Within this paper, two different methods are shown. A numerical example demonstrates the approximation capability in the frequency and time domain.

1 Introduction

Flexible multibody dynamics is used to examine the dynamic behavior of gear-boxes, robot-arms, machine-tools, crank-shafts, windmills, impact drills, etc. The concept of flexible multibody systems is described in (Schwertassek and Wallrapp, 1999) and (Shabana, 1998). In this work, the motion of a single elastic body is described within the floating frame of reference formulation. Within this framework, the motion of a flexible body $\boldsymbol{r}$ is separated into an usually nonlinear motion of the reference frame $\boldsymbol{K}_i$ and into a linear elastic deformation with respect to the reference frame $\boldsymbol{u}$, see Fig. 1. A widely used possibility for modelling the linear elastic deformation is the Ritz approach where the elastic deformations are approximated with $\boldsymbol{u} = \boldsymbol{\Phi} \cdot \boldsymbol{q}$, where $\boldsymbol{q}$ is referred to the nodal displacements of a finite element model. Using Jourdain's principles of dynamics, the equation of motion for a single body can be derived, as shown e.g. in (Schwertassek and Wallrapp, 1999). This approach leads to a high number of degrees of freedom. As a consequence, many analysis and design problems cannot be solved within a reasonable computing time. That is why the flexible coordinates $\boldsymbol{q}$ are then approximated by projection on a subspace $\mathrm{span}(\boldsymbol{V})$ by $\boldsymbol{q} = \boldsymbol{V} \cdot \overline{\boldsymbol{q}}$, with $\dim(\bar{\boldsymbol{q}}) \ll \dim(\boldsymbol{q})$, and requiring the residual to be orthogonal on this subspace. This procedure leads to the reduced equations of motion

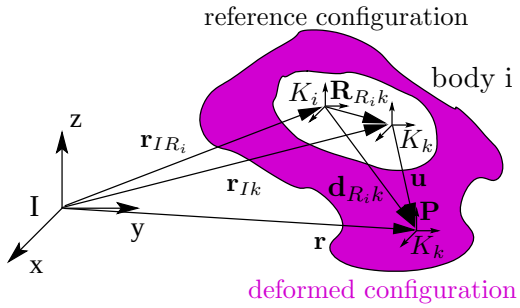

Figure 1. Floating frame of reference formulation

$$\begin{bmatrix} \boldsymbol{M}_r & \boldsymbol{M}_{er}^T \cdot \boldsymbol{V} \\ \boldsymbol{V}^T \cdot \boldsymbol{M}_{er} & \boldsymbol{V}^T \cdot \boldsymbol{M}_e \cdot \boldsymbol{V} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{a} \\ \ddot{\overline{\boldsymbol{q}}} \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{h}}_r \\ \boldsymbol{V}^T \cdot \overline{\boldsymbol{h}}_e \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ -\boldsymbol{V}^T \cdot \boldsymbol{K}_e \cdot \boldsymbol{V} \cdot \overline{\boldsymbol{q}} - \boldsymbol{V}^T \cdot \boldsymbol{D}_e \cdot \boldsymbol{V} \cdot \dot{\overline{\boldsymbol{q}}} \end{bmatrix}, \quad (1)$$

where the sub matrix $\boldsymbol{M}_r$ corresponds to the mass matrix known from rigid multibody dynamics, $\boldsymbol{M}_e$, $\boldsymbol{D}_e$ and $\boldsymbol{K}_e$ are the flexible mass, damping and stiffness matrices, whereas $\boldsymbol{M}_{er}$ provides the coupling between the rigid body movement and the elastic deformation. The vector $\boldsymbol{a}$ contains the global accelerations of the floating frame of reference, vectors $\overline{\boldsymbol{h}}_r$ and $\overline{\boldsymbol{h}}_e$ collect generalized inertia forces, gravitational forces and forces acting on the body's surface. In state of the art reduction methods, like modal reduction the projection space $\mathrm{span}(\boldsymbol{V})$ consists of the dominant eigen vectors of the system. Because the spatial distribution of loads is not considered, the convergence of modal reduction can be slow. Modal reduction can be improved

by extending the projection space ($\boldsymbol{V}$) with constraint modes similar to sub-structuring techniques in structural dynamics, see e.g. (Craig, 1968). However, the selection of the dominant eigen modes in combination with the best constraint modes is a slow iterative process which requires experience by the user. By considering all the reaction and applied forces acting on the elastic body as inputs and outputs to the elastic body, the body can be considered as a linear time-invariant second order MIMO-system. Then another approach for model reduction is the approximation of the mapping between specified inputs and output variables of the MIMO-System. One possible reduction technique, widely used in model order reduction of MEMS, is the projection with Krylov-subspaces. By using a Krylov-subspace as projection space $\mathrm{span}(\boldsymbol{V})$, certain moments of the original and reduced transfer matrix of the flexible body match. Overview articles about Krylov-subspace reduction are given e.g. by (Bai, 2002) or (Beattie and Gugercin, 2005). The fact that Krylov-subspace reduction methods are iterative methods and can be applied to large scale models represents their decisive advantage whereas they become inflexible in a system with many inputs if a small model is required which has to be considered as their disadvantage. Another possible reduction technique is the incorporation of Gramian matrices in the reduction process as proposed e.g. by (Sorensen and Antoulas, 2005) or (Stykel, 2004). By using Gramian matrices, we obtain a reduced order model which has very good approximation capabilities in the frequency domain as well as in the time domain. Additionally, methods based on Gramian matrices have a global error bound and moreover, automation of the reduction process is possible.

2 Model Reduction based on Gramian Matrices

In order to establish the proposed model reduction scheme, the linear time-invariant second order system

$$\begin{aligned} \boldsymbol{M}_e \cdot \ddot{\boldsymbol{q}}(t) + \boldsymbol{D}_e \cdot \dot{\boldsymbol{q}}(t) + \boldsymbol{K}_e \cdot \boldsymbol{q}(t) &= \boldsymbol{B}_e \cdot \boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{C}_e^T \cdot \boldsymbol{q}(t) \end{aligned} \tag{2}$$

is considered, with the input matrix $\boldsymbol{B}_e \in \mathbb{R}^{n\times p}$ and the output matrix $\boldsymbol{C}_e^T \in \mathbb{R}^{r\times n}$. Using the Laplace transformation, the $(r \times p)$-transfer matrix of the system

$$\boldsymbol{H}(s) = \boldsymbol{C}_e^T \cdot (s^2\boldsymbol{M}_e + s\boldsymbol{D}_e + \boldsymbol{K}_e)^{-1} \cdot \boldsymbol{B}_e \tag{3}$$

is obtained. The $\mathcal{H}_2$-norm of the system can be written as

$$\|\boldsymbol{H}\|_{\mathcal{H}_2}^2 = \mathrm{trace}(\boldsymbol{C}_e \cdot \boldsymbol{P}_p \cdot \boldsymbol{C}_e^T) \tag{4}$$

with the so-called position controllability Gramian matrix, see (Meyer and Srinivasan, 1996; Sorensen and Antoulas, 2005),

$$\boldsymbol{P}_p = \frac{1}{2\pi}\int_{-\infty}^{\infty} (\boldsymbol{Q}(i\omega) \cdot \boldsymbol{Q}^H(i\omega) d\omega, \tag{5}$$

with

$$\boldsymbol{Q}(i\omega) = (-\omega^2\boldsymbol{M}_e + i\omega\boldsymbol{D}_e + \boldsymbol{K}_e)^{-1} \cdot \boldsymbol{B}_e. \tag{6}$$

Frequently, in mechanical systems a certain range of frequency is of special interest. By applying ideal band pass filters to the input and the output, a certain frequency range $[\omega_{min}, \omega_{max}]$ can be emphasized (Antoulas, 2005). The $\mathcal{H}_2$-norm of the frequency weighted system can then be written as

$$\|\boldsymbol{H}\|_{\mathcal{H}_2}^w = \mathrm{trace}(\boldsymbol{C}_e \cdot \boldsymbol{P}_p^\omega \cdot \boldsymbol{C}_e^T)^{\frac{1}{2}} \tag{7}$$

where $\boldsymbol{P}_p^\omega$ is now the so-called frequency-weighted position controllability Gramian matrix

$$\begin{aligned} \boldsymbol{P}_p^\omega = &\frac{1}{2\pi}\int_{-\omega_{max}}^{-\omega_{min}} (\boldsymbol{Q}(i\omega) \cdot \boldsymbol{Q}^H(i\omega) d\omega \\ &+ \frac{1}{2\pi}\int_{\omega_{min}}^{\omega_{max}} (\boldsymbol{Q}(i\omega) \cdot \boldsymbol{Q}^H(i\omega) d\omega. \end{aligned} \tag{8}$$

Hence, in order to obtain a good approximation of the input-output mapping in a certain frequency range with respect to the $\mathcal{H}_2$-norm of $\boldsymbol{H}(s)$, the frequency-weighted Gramian matrix $\boldsymbol{P}_p^\omega$ should be incorporated in the reduction procedure. For this purpose, the eigen decomposition of

$$\boldsymbol{P}_p^\omega = \begin{bmatrix}\boldsymbol{V}_1 & \boldsymbol{V}_2\end{bmatrix} \cdot \begin{bmatrix}\boldsymbol{diag}(\sigma_i) & 0 \\ 0 & \boldsymbol{diag}(\sigma_j)\end{bmatrix} \cdot \begin{bmatrix}\boldsymbol{V}_1^T \\ \boldsymbol{V}_2^T\end{bmatrix} \tag{9}$$

is considered. The projection matrix $\boldsymbol{V}$ then consists of the dominant eigen vectors $\boldsymbol{V}_1$, which are associated with the largest eigenvalues σ_i of the frequency-weighted position Gramian matrix. The reduced order model is then given by Eq. (1). If only the linear system (2) is accounted for, an a priori error bound in the frequency-weighted $\mathcal{H}_2$-norm can be developed, see (Sorensen and Antoulas, 2005). The error of the reduced system $\boldsymbol{H}_e = \boldsymbol{H} - \bar{\boldsymbol{H}}$ is then bounded in the $\mathcal{H}_2^\omega$-norm by the sum of neglected eigenvalues of the position Gramian matrix, i.e.

$$\|\boldsymbol{H}_e\|_{\mathcal{H}_2}^\omega \leq \kappa(\sum_j \sigma_j)^{\frac{1}{2}}. \tag{10}$$

2.1 Numerical calculation of the 2^{nd} order Gramian matrix

For small asymptotically stable systems, the Gramian matrix $\boldsymbol{P}_p$ can be calculated by evaluating a matrix

logarithm in addition to the solution of a suitable Lyapunov equation, see (Antoulas, 2005). Direct solution of the Lyapunov equation is only possible for small- to medium-scale models because the solution requires $\mathcal{O}(n^3)$ operations and the storage requirement is $\mathcal{O}(n^2)$. For large-scale models, the subspace of dominant eigen vectors of the Gramian matrix has to be generated. Here, we present two approaches, especially suited for the frequency weighted reduction scheme. One approach uses the two-step approach explained in (Lehner and Eberhard, 2007). According to this approach, a medium-scale model is acquired in a first step with the help of Krylov-subspace methods. Subsequently, the Gramian matrix of the medium-scaled model is calculated and then used as an approximation for the large scale Gramian matrix. In the first step applying the Krylov-subspace reduction, it is necessary to choose where the moments of the reduced transfer matrix $\bar{\boldsymbol{H}}$ match with the moments of the original transfer matrix $\boldsymbol{H}$. Up to now, the moment matching properties are chosen by hand. However (Gugercin *et al.*, 2007) proposed recently a method by which the $\mathcal{H}_2$-norm of the error $\boldsymbol{H}_e$ is minimized because the moment matching properties are chosen in an optimal way. The application of this method is under current research.

A second approach for approximating the frequency-weighted Gramian matrix $\boldsymbol{P}_p^\omega$ was introduced in (Lehner, 2007). According to this method the matrix integral $\boldsymbol{P}_p^\omega$ is numerically approximated. By rewriting and resorting the matrix integral (8)

$$\boldsymbol{P}_p^\omega = \frac{1}{\pi}\int_{\omega_{min}}^{\omega_{max}} \tilde{\boldsymbol{Q}}(i\omega)\cdot\tilde{\boldsymbol{Q}}^T(i\omega)d\omega \qquad (11)$$

with the real-valued matrix

$$\tilde{\boldsymbol{Q}}(i\omega) = \left[\mathrm{Re}\left(\boldsymbol{Q}(i\omega)\right)\ \mathrm{Im}\left(\boldsymbol{Q}(i\omega)\right)\right] \qquad (12)$$

and using e.g. the mid-point rule as an approximation rule for the matrix integral, we obtain the approximated Gramian $\tilde{\boldsymbol{P}}_p^\omega$

$$\boldsymbol{P}_p^\omega \approx \tilde{\boldsymbol{P}}_p^\omega = \frac{h_o}{\pi(s-1)}\sum_{l=1}^{s}\tilde{\boldsymbol{Q}}(i\omega_s)\cdot\tilde{\boldsymbol{Q}}^T(i\omega_l), \quad (13)$$

with the interval length $h_0 = \omega_{max} - \omega_{min}$ and the number of sampling points s. The explicit calculation of the eigenvalues of the approximated Gramian matrix $\tilde{\boldsymbol{P}}_p^\omega$ is not necessary, because it is possible to relate the calculation of the dominant eigen vectors of the Gramian matrix to Proper Orthogonal Decomposition (POD) methods. POD based model reduction techniques are for example explained in (Willcox and Peraire, 2002) or in (Volkwein, 2006). By building the data sample matrix

$$\boldsymbol{U} = \left[\tilde{\boldsymbol{Q}}_1\ \tilde{\boldsymbol{Q}}_2\ \ldots\ \tilde{\boldsymbol{Q}}_s\right], \qquad (14)$$

we can rewrite Eq. (13) with

$$\tilde{\boldsymbol{P}}_p^\omega = \frac{h_0}{\pi(s-1)}\underbrace{\boldsymbol{U}\cdot\boldsymbol{U}^T}_{\text{POD Kernel}} \qquad (15)$$

where $\boldsymbol{U}\cdot\boldsymbol{U}^T$ is the POD Kernel. The POD Kernel is used to solve the POD optimization problem of finding a matrix $\boldsymbol{V}^{pod} \in \mathbb{R}^{n\times k}$ which is the best approximation of reduced order k of the data samples matrix $\boldsymbol{U} \in \mathbb{R}^{n\times s}$. This optimization problem is solved by first solving the eigen problem

$$\frac{1}{s}\boldsymbol{U}\cdot\boldsymbol{U}^T\cdot\boldsymbol{v}_i^{pod} = \sigma_i\boldsymbol{v}_i^{pod} \qquad (16)$$

and then collecting those proper orthogonal modes $\boldsymbol{v}_i^{pod}$ corresponding to big proper orthogonal values σ_i in $\boldsymbol{V}^{pod}$. Thus, a connection between the calculation of the eigen modes of the approximated Gramian matrix $\tilde{\boldsymbol{P}}_p^\omega$ in Eq. (13) and the solution of a POD method in Eq. (16) can be seen. The eigen modes of $\tilde{\boldsymbol{P}}_p^\omega$ can then be calculated by using POD methods where $\boldsymbol{U}$ is builded with Eq. (14). If the number of snapshots s is much smaller than the number of degrees of freedom n, the dominant eigen modes of the POD Kernel can be calculated with the method of snapshots (Sirovich, 1987) which reduces the calculation burden from solving an eigenvalue problem of size $n \times n$ to solving an eigenvalue problem of size $s \times s$.

2.2 Towards an automated reduction process

The sum of neglected eigenvalues $\sum_j \sigma_j$ can be used to determine the size m of the reduced order model because the error is bounded below the sum of the neglected eigenvalues, see Eq. (10). Usually, the eigenvalues of the Gramian matrix σ_i decay rapidly in mechanical systems. This means that the first neglected eigenvalue is the dominant share in the sum of neglected eigenvalues. The ratio of the first neglected eigenvalue and the first eigenvalue σ_{m+1}/σ_1 can then be used as an indicator of an appropriate size of the reduced order model as proposed in (Antoulas, 2005). Characteristic of model reduction is the choice that the ratio σ_{m+1}/σ_1 should be smaller than the square root of the machine precision $\sqrt{\epsilon_{mach}} \approx 1.5\cdot 10^{-8}$. For the automated reduction process, we determine the reduced order size of the system by checking the ratio $\sigma_{m+1}/\sigma_1 < \sqrt{\epsilon_{mach}}$. The work-flow of the automated reduction process can be seen in Fig. 2.

3 Example and Results

The potential of the above explained methods are now shown by reducing a FEM model of a rack as shown in Figure 3 and explained in (Lehner and Eberhard, 2007). We assume, that forces are acting on the lower and the upper plate in all six directions. The lower and the upper plate are assumed to be rigid and are labeled as

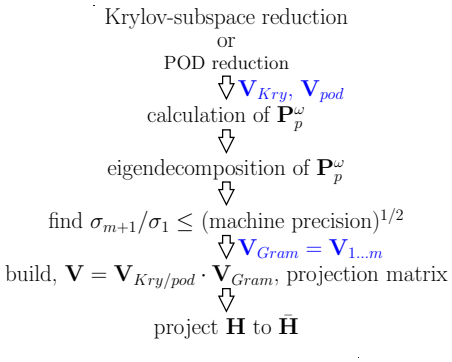

Figure 2. Workflow of the automated reduction process

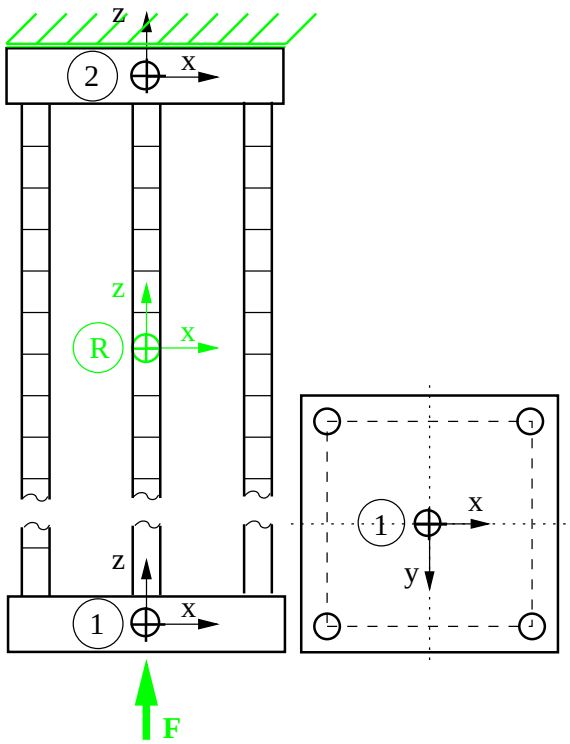

Figure 3. Model of the rack. In green boundary conditions for the dynamic simulation

node 1 and node 2. Accordingly, the rack is considered as a MIMO System with 12 inputs and the inputs coincide with the outputs, $\boldsymbol{B} = \boldsymbol{C}$. The interesting frequency range is from 10 to 500 Hz. The automated reduction process is applied to a Krylov-reduced medium-scaled system of order 63 and a POD based model reduction where the number of snapshots s is equal to 30 is then applied to the elastic body. This makes it possible to calculate the two error indicators introduced in Section 2.2 by calculating the eigenvalues of the Gramian matrix $\boldsymbol{P}_p^\omega$. In Fig. 4 the two error indicators are plotted over the dimension m of the reduced order model and it can be seen that they indicate about the same cut-of dimension. The automated reduction process generates reduced order models of size

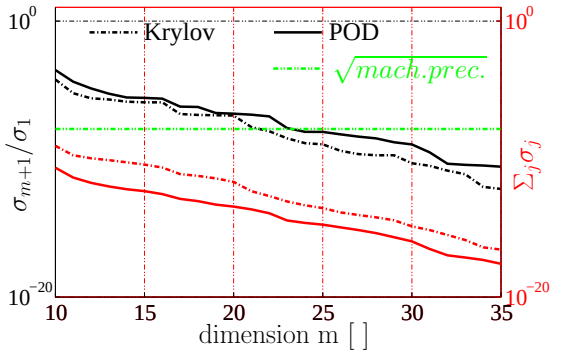

Figure 4. Error indicator used for the reduction process

$m = 21$ if the two-step approach is used whereas it generates a reduced order model of size $m = 23$ by using POD based model order reduction techniques. In Fig. 5, the relative error

$$\epsilon(\omega) = \frac{\|\boldsymbol{H}(\omega) - \bar{\boldsymbol{H}}(\omega)\|_F}{\|\boldsymbol{H}(\omega)\|_F} \tag{17}$$

in the Frobenius norm is plotted. A reduced system of order 63 generated by a Krylov based approach is the most accurate model. Furthermore, the system of order 21 respectively order 23, obtained by a Krylov + Gramian matrix based approach or a POD based approach, are much more accurate then the model of size 23 obtained by traditional modal reduction. Because of the high error of the modal model we cannot expect useful simulation results there.

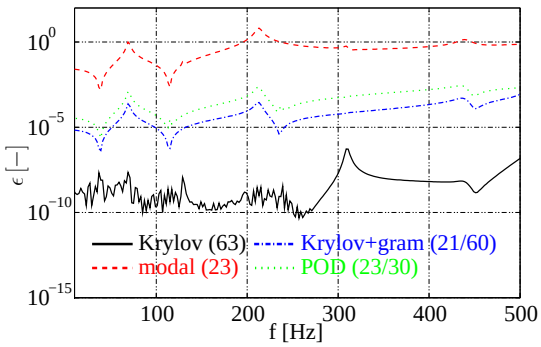

Figure 5. Relative reduction error using different approaches

3.1 Emphasizing certain frequency ranges

One of the advantages of model reduction with frequency-weighted Gramian matrix $\boldsymbol{P}_p^\omega$ is the possibility to emphasize certain frequency ranges. By changing the frequency range $[\omega_{min}, \omega_{max}]$ for which $\boldsymbol{P}_p^\omega$ is calculated with the two-step approach, we obtain reduced order models of different size, depending on the frequency range of interest, which have good approximation behavior in the preferred frequency range,

see Fig. 6. For a smaller frequency range we only need reduced order models of smaller size. However, the reduced order models of smaller size only have a good approximation behavior in the emphasized frequency range.

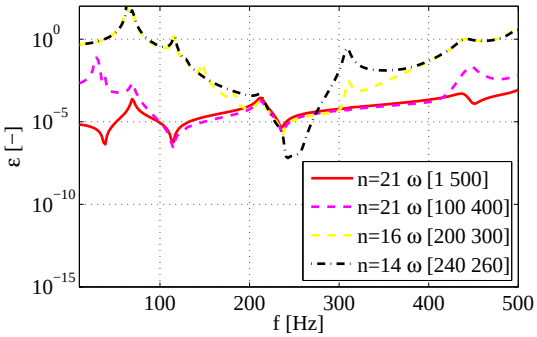

Figure 6. Relative reduction error using different frequency intervals

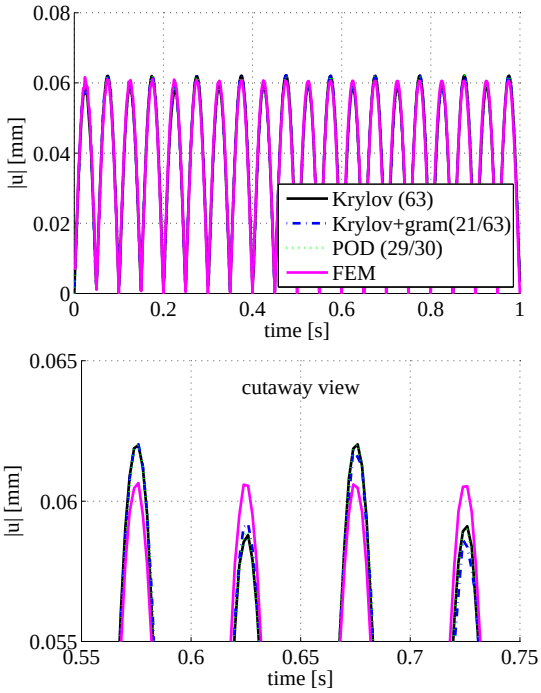

Figure 7. Results of the dynamic simulation

3.2 Comparisons in the time domain

The time domain is of great importance in flexible multibody dynamics. For this purpose, also a dynamic simulation is considered and the different approaches are compared with respect to accuracy and computation time. The body is clamped to the surrounding at node 2 and the body is actuated at node 1 with a time-dependent force

$$\boldsymbol{F} = \begin{bmatrix} 0 \\ 0 \\ -10000\sin(2 \cdot 10\pi t) \end{bmatrix} N.$$

As a reference frame for the dynamic simulation a Buckens-frame is chosen, such that the origin of the reference frame coincides with the center of gravity, compare Fig. 3 green. The reduced order models are simulated with the multibody dynamics simulation tool SIMPACK for one second. The resulting system of ODEs is solved numerically with the standard solver of SIMPACK. In Fig. 7 the magnitude $|\boldsymbol{u}|$ of the translation deflection of the actuated lower plate is shown for models of different size obtained with different reduction methods. In addition we compare those results with the translation of the nonlinear finite element model. The accuracy of the reduced order model is good. It pretty much shows the same response as the nonlinear FEM model with 5982 degrees of freedoms. The reduced order model of size 21 obtained with the two step frequency-weighted Gramian matrix approach has the same accuracy as a model of size 29 obtained by a POD based model reduction. The accuracy of those two models based on frequency-weighted Gramian matrix reduction techniques is comparable to the accuracy of the model of size 63 obtained with a Krylov-subspace based approach. However with the model of size 21 we gain a speedup of the calculation time as shown in Table 1. As could be already expected from the error in the frequency response shown in Fig. 5, the results from the modal model of order 24 are worse than the other results and that is why they are not depicted here.

Table 1. Normalized computation times for several reduced order models

model	computation time
Krylov+gram (21/63)	1.0
POD (23/30)	1.57
Krylov (63)	8.05
FEM	161.3

4 Conclusion

By using automated reduction techniques based on frequency weighted Gramian matrices, the spatial distribution of loads is considered during the reduction process and the stability properties of the original system are preserved. With these methods we obtained excellent reduction results in the frequency domain as well as in the time domain. Furthermore, the necessary order of the reduced model is chosen automatically which simplifies the reduction process for the user.

References

Antoulas, A.C. (2005). *Approximation of Large-Scale Dynamical Systems*. SIAM. Philadelpia.

Bai, Z. (2002). Krylov subspace techniques for reduced-order modeling of large-scale dynamical systems. *Applied Numerical Mathematics* **43**, 9–44.

Beattie, C.A. and S. Gugercin (2005). Krylov-based model reduction of second-order systems with proportional damping. In: *44th IEEE Conference on Decision and Control, 2005 European Control Conference*. pp. 2278–2283.

Craig, R.R. (1968). Coupling of substructures for dynamic analysis. *AIAA Journal* **6(7)**, 1313–1319.

Gugercin, S., A.C. Antoulas and C.A. Beattie (2007). $\mathcal{H}_2$ model reduction for large-scale linear dynamical systems. *accepted, SIAM Journal on Matrix Analysis and Applications*.

Lehner, M. (2007). *Modellreduktion in elastischen Mehrkörpersystemen*. PhD thesis (in German). Schriften aus dem Institut für Technische und Numerische Mechanik der Universität Stuttgart, Band 9. Shaker Verlag. Aachen.

Lehner, M. and P. Eberhard (2007). A two-step approach for model reduction in flexible multibody dynamics. *Multibody System Dynamics* **17**(2-3), 157–176.

Meyer, D.G. and S. Srinivasan (1996). Balancing and model reduction for second-order form linear systems. *IEEE Transactions on Automatic Control* **41**(11), 1632–1644.

Schwertassek, R. and O. Wallrapp (1999). *Dynamik flexibler Mehrkörpersysteme (in German)*. Vieweg. Braunschweig.

Shabana, A.A. (1998). *Dynamics of Multibody Systems*. Cambridge University Press. Cambridge.

Sirovich, L. (1987). Turbulence and the dynamics of coherent structures. *Quart. Appl. Math.* **45**, 561–590.

Sorensen, D.C. and A.C. Antoulas (2005). On model reduction of structured systems. In: *Dimension Reduction of Large-Scale Systems* (P. Benner, G. Golub, V. Mehrmann and D.C. Sorensen, Eds.), pp. 83–115. Springer-Verlag, Berlin/Heidelberg.

Stykel, T. (2004). Gramian-based model reduction for descriptor systems. *Math. Control Signals Systems* **16**, 297–319.

Volkwein, S. (2006). Model reduction using proper orthogonal decomposition. Script, accessed at http://www.uni-graz.at/imawww/volkwein/POD.pdf on 8.11.2007.

Willcox, K. and J. Peraire (2002). Balanced model reduction via the proper orthogonal decomposition. *AIAA Journal* **40**, 2323–2330.