# Ellipsoidal Techniques in State Estimation Problem for Linear Impulsive Control Systems

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***Abstract*— The paper is devoted to the state estimation problem for impulsive control systems described by linear differential equations containing impulsive terms (or measures). The state estimation algorithms that use the special structure of linear impulsive control problem and are based on techniques of external and internal ellipsoidal approximations of trajectory tubes of uncertain impulsive systems are presented here. The examples of construction of such ellipsoidal estimates of the reachable sets and trajectory tubes of linear impulsive control systems are given.**

## I. Introduction

Number of researches is devoted to the different aspects of the theory of optimization of dynamic systems with generalized (impulse) control. The impulsive control problem of the trajectory tube of the system described by linear differential equations is considered in the paper. This system contains impulsive terms (or measures) and uncertainty on initial date [1]-[4].

There is a long list of publications devoted to impulsive control optimization problems, among them we mention here only the results related to the present investigation [5]-[13]. The question arises how the results of classical control theory established for uncertain dynamical systems can be extended to the case of uncertain impulsive systems. It is impotent that in the considered problem the control influences are limited not only by usual requirement of finiteness of variation but by special restriction of ellipsoidal type. Particularly, the vectors of jumps of generalized controls under this restriction must lie in the given ellipsoid of an appropriate finite-dimension space. Such problems arise when the possibilities of control of impulsive dynamic system are constrained by non-even restrictions for different directions. For example, one can consider the movement of the flying devices near the earth surface or in the narrow gorges. The specific features of an impulse control system result in the necessity of constructing ellipsoidal estimates for a convex hull of the union of a family of ellipsoids (this problem does not arise in the case of estimating states of dynamic systems with controls of classical type).

The aim of the paper is to find the external and internal set-valued estimates of the reachable sets of impulsive control systems with special ellipsoidal constrains on the admissible values of control functions and on the initial state vectors. Basing on the techniques of so-called ellipsoidal calculus [1], [3], [4] a new state estimation approach that uses the special structure of the studied impulsive control problem is presented here. The examples of construction of such ellipsoidal external and internal estimates of reachable sets of linear impulsive control systems are given also.

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## II. Problem Formulation

Consider a dynamic control system described by a differential equation with impulsive control $u(\cdot)$:

$$dx = A(t)xdt + du, \quad x(-0) = x_0, \quad t \in [0, T], \tag{1}$$

or in the integral form [5],

$$x(t){=}x(t; u(\cdot), x_0){=}X(t)x_0 + \int\limits_0^t X(t)X^{-1}(\tau)du(\tau). \tag{2}$$

Here we assume that $A(t)$ is a continuous $n\times n$ - matrix function, $X(t)$ is the fundamental matrix solution $\dot{X} = A(t)X$ ($X(0) = I$), $u(\cdot) \in V_p^n$ where $V_p^n$ means the space of $n$-vector functions $u(\cdot)$ such that $u(t)$ is continuous from the right on $[0, T)$ with $u(-0) = 0$ and

$$V_p[u(\cdot)] = \sup_{\{t_i\}} \sum_{i=1}^{k} \|u(t_i) - u(t_{i-1})\|_p < \infty$$

$$\|u\|_p = \left( \sum_{i=1}^{n} |u_i|^p \right)^{\frac{1}{p}}, \quad (1 \le p < \infty),$$

where $u{=}(u_1, \ldots, u_n)$ and $t_i : 0 = t_0{<}\ldots{<}t_k = T$.

Let $\mathcal{E}_0$ be an ellipsoid in $R^n$:

$$\mathcal{E}_0 = \{l{\in}R^n | \quad l'Q_0 l{\le}1\},$$

where $Q_0$ is a given symmetric positive definite $n\times n$ matrix, we introduce a special restriction $\mathcal{U}$ on the control functions $u(\cdot)$ in the space $V_p^n$ of functions of the bounded variations on $[0, T]$ [12]. Denote $C_q^n$ the space of continuous $n$-vector functions $y(\cdot)$ with the norm

$$\|y(\cdot)\|_{\infty,q} = \max_{0\le t\le T} \|y(t)\|_q.$$

It is well known that the space $V_p^n$ is dual to the space $C_q^n$ ($V_p^n = C_q^{n*}$) where $p = 1$ if $q = \infty$, $p = \infty$ if $q = 1$ and $1 < p < \infty$ if $q = (1 - p^{-1})^{-1}$.

Consider the so-called "ellipsoid" $\mathcal{E}$ in $C_q^n$:

$$\begin{aligned}\mathcal{E} &= \{y(\cdot) \in C_q^n \ | \ \ y'(t)Q_0 y(t) \le 1 \ \ \forall t \in [0, T]\} = \\ &= \{y(\cdot) \in C_q^n \ | \ \ y(t) \in \mathcal{E}_0 \ \ \forall t \in [0, T]\}\end{aligned} \tag{3}$$

and its conjugate $\mathcal{E}^* \subset V_p^m$,

$$\mathcal{U} = \mathcal{E}^* = \{u(\cdot) \in V_p^n | \int_0^T y(t) du(t) \leq 1,$$

$$\forall y(\cdot) \in C_q^n \quad y(t) \in \mathcal{E}_0, \ \forall t \in [0,T]\}.$$

*Definition 1:* A control $u^*$ is admissible if $u(\cdot) \in \mathcal{U}$.

In particular, under such restriction vectors of impulsive jumps of controls $\Delta u = u(t_{i+1}) - u(t_i) \in \mathcal{U}$ have to belong to the ellipsoid

$$\mathcal{E}_0^* = \{z \in R^n | \quad z' Q_0^{-1} z \leq 1\}.$$

We will assume that the initial value $x_0$ for the system (1) is unknown but bounded with a given bound $x_0 \in \mathcal{X}_0$,

$$\mathcal{X}_0 = \{x_0 \in R^n | \quad x_0' R^{-1} x_0 \leq 1\}, \tag{4}$$

where $R$ is a symmetric positively defined $n \times n$ matrix.

Denote

$$\mathcal{X}(t; \mathcal{X}_0) = \bigcup_{x_0 \in \mathcal{X}_0} \bigcup_{u \in \mathcal{U}} x(t; u(\cdot), x_0).$$

The set $\mathcal{X}(t; \mathcal{X}_0)$ is actually the reachable set of the impulsive differential system (1) from the initial set $\mathcal{X}_0$ at the instant $t$ for all possible admissible controls $u(\cdot)$.

So the main problem of the paper is to find the external and internal estimates of ellipsoidal type for the reachable set $\mathcal{X}(T; \mathcal{X}_0)$ basing on the special structure of the data $\mathcal{X}_0$ and $\mathcal{U}$.

## III. Main Results: Ellipsoidal Estimates

In this section we apply the techniques of the ellipsoidal calculus to find the external and internal estimates for the reachable set $\mathcal{X}(T; \mathcal{X}_0)$.

We take $\mathcal{X}_0 = \{0\}$ first and denote $\mathcal{X}(T) = \mathcal{X}(T; \{0\})$ to be a reachable set of (1) and

$$T_* = \{\tau_* \in [0,T] | \ \exists l_* \neq 0, (G(\tau_*, l_*))^{\frac{1}{2}} = \max_{0 \leq \tau \leq T} (G(\tau, l_*))^{\frac{1}{2}}\}, \tag{5}$$

$$G(\tau, l) = l' X(T, \tau) Q_0 X'(T, \tau) l,$$

$$X(T, \tau) = X(T) X^{-1}(\tau).$$

We will consider further, that the following condition is executed

*Assumption P:* The set $T_*$ is finite:

$$T_* = \{\tau_{*1}, \tau_{*2}, \ldots, \tau_{*m}\} \subset [0, T].$$

*Remark 1:* The class of systems for which this assumption holds is not empty, e.g. it is fulfilled in Example.

*Remark 2:* In the general case, the extremal set

$$T_* = \{\tau_* \in [0,T] | \ (G(\tau_*, l))^{\frac{1}{2}} = \max_{0 \leq \tau \leq T} (G(\tau, l))^{\frac{1}{2}}\}$$

should not be finite and may depend on $l$. Therefore the assumption P is essential.

*Theorem 1:* [12] Under the assumption P we have

$$\mathcal{X}(T) = co \bigcup_{\tau \in T_*} \mathcal{E}(0, Q_\tau), \tag{6}$$

$$\mathcal{E}(0, Q_\tau) = \{x \in R^n \mid x' Q_\tau^{-1} x \leq 1\},$$

$$Q_\tau = X(T, \tau) Q_0 X'(T, \tau).$$

*Proof:* The proof of this theorem is based on the structures of the reachable set $\mathcal{X}(T)$ :

$$\mathcal{X}(T) = \bigcup_{u \in \mathcal{U}} \int_0^T X(T, \tau) du(\tau)$$

and the special "ellipsoidal" restriction $\mathcal{U}$ on controls. ■

### A. External Ellipsoidal Estimates

In order to construct the external estimate, we consider the auxiliary problem.

*Problem 1:* Two ellipsoids are given

$$\mathcal{E}_0 = \{x \in R^n \mid x' Q_0^{-1} x \leq 1\},$$

$$\mathcal{E}_1 = \{x \in R^n \mid x' Q_1^{-1} x \leq 1\}. \tag{7}$$

Find an external ellipsoid

$$\mathcal{E}^+ = \{x \in R^n \mid x' (Q^+)^{-1} x \leq 1\}$$

that contains $\mathcal{E}_0 \cup \mathcal{E}_1$ (therefore, the $\mathcal{E}^+$ will contain the convex hull $co(\mathcal{E}_0 \cup \mathcal{E}_1)$). Equivalently, it is required to find the symmetric positive definite matrix $Q^+$ such that for all $l \in R^n$ we have

$$(l' Q_0 l)^{\frac{1}{2}} \leq (l' Q^+ l)^{\frac{1}{2}}, \quad (l' Q_1 l)^{\frac{1}{2}} \leq (l' Q^+ l)^{\frac{1}{2}}, \tag{8}$$

and it is desirable also to find the ellipsoid $\mathcal{E}^+$ that has the minimal possible volume [4].

We need to do three consequent steps to solve the Problem 1.

*Step 1.* Let $\lambda_1, \ldots, \lambda_n$ be the roots of the equation

$$|Q_0 \lambda - Q_1| = 0 \tag{9}$$

(note that $\lambda_i > 0$ $(i = 1 \ldots n)$ are the eigenvalues of matrix $Q_0^{-1} Q_1$).

Denote $B = Q_0^{-\frac{1}{2}} Q_1 Q_0^{-\frac{1}{2}}$. The matrix $B$ is also symmetric and positive definite, and it can easily be seen that $\lambda_i$ are also the eigenvalues of matrix $B$ [14]. There exists an orthogonal matrix $M$ such that [14], $(MM' = M'M = I)$

$$M' B M = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\} = W^2. \tag{10}$$

We transform the coordinates from the vector $x$ to the new variable $s$ that satisfies the equality $x = Q_0^{\frac{1}{2}} M s$. Under this transformation the ellipsoids $\mathcal{E}_0$, $\mathcal{E}_1$ (13) become the ellipsoids

$$\tilde{\mathcal{E}}_0 = \{s \in R^n | s' s \leq 1\}, \tag{11}$$

$$\tilde{\mathcal{E}}_1 = \{s \in R^n | s' (W^2)^{-1} s \leq 1\} \tag{12}$$

where $W^2$ determined in (10).

*Step 2.* We construct the ellipsoid $\tilde{\mathcal{E}}^+ \supseteq \tilde{\mathcal{E}}_0 \cup \tilde{\mathcal{E}}_1$ where

$$\tilde{\mathcal{E}}^+ = \{s \in R^n \mid s'(\tilde{Q}^+)^{-1}s \leq 1\},$$

$$\tilde{Q}^+ = \mathrm{diag}\{\mu_1, \ldots, \mu_n\},$$

$$\mu_i = \max\{1, \lambda_i\}, \quad i = 1, 2, \ldots, n.$$

The following theorem holds.

*Theorem 2:* [12] The ellipsoid $\tilde{\mathcal{E}}^+$ constructed in this way solves the problem 1 for the sets $\tilde{\mathcal{E}}_0$ and $\tilde{\mathcal{E}}_1$ and is minimal with respect to inclusion among all ellipsoids containing $\tilde{\mathcal{E}}_0 \bigcup \tilde{\mathcal{E}}_1$ and having a matrix of diagonal form.

*Theorem 3:* [12] The ellipsoid $\tilde{\mathcal{E}}^+$ is minimal with respect to the volume among all ellipsoids that contain $\tilde{\mathcal{E}}_0 \cup \tilde{\mathcal{E}}_1$.

The proof of these theorems follows from the properties of ellipsoids and the result ([4], corollary 5.1).

*Step 3.* We return to the space of $x$ - coordinates and therefore we get the inclusion

$$\mathcal{E}^+ = \{x \in R^n \mid x'(Q^+)^{-1}x \leq 1\} \supseteq (\mathcal{E}_0 \cup \mathcal{E}_1),$$

$$Q^+ = Q_0^{\frac{1}{2}} M' \tilde{Q}^+ M Q_0^{\frac{1}{2}}.$$

It should be noted that, based on the solution of the auxiliary problem, we can also solve the problem of constructing an external estimate for $\mathcal{X}(T)$. For this purpose, we construct at the first stage an ellipsoid $\mathcal{E}_1^+ = \mathcal{E}^+$ using $\mathcal{E}_{\tau_0}$ and $\mathcal{E}_{\tau_1}$ according to the scheme considered. Then, using $\mathcal{E}_1^+$ and $\mathcal{E}_{\tau_2}$, we construct $\mathcal{E}_2^+$, and so on. The assumption P guarantees the finite number of such steps. The final ellipsoid contains $\mathcal{X}(T)$.

### B. Internal Ellipsoidal Estimates

In order to construct the internal estimate, we consider the auxiliary problem.

*Problem 2:* Two ellipsoids are given

$$\mathcal{E}_0 = \{x \in R^n \mid x'Q_0^{-1}x \leq 1\},$$

$$\mathcal{E}_1 = \{x \in R^n \mid x'Q_1^{-1}x \leq 1\}. \tag{13}$$

Find the internal ellipsoid

$$\mathcal{E}^- = \{x \in R^n \mid x'(Q^-)^{-1}x \leq 1\}$$

that is contained in $co(\mathcal{E}_0 \cup \mathcal{E}_1)$. Equivalently, it is required to find the symmetric positive definite matrix $Q^-$ such that for all $l \in R^n$ we have

$$(l'Q_0 l)^{\frac{1}{2}} \geq (l'Q^- l)^{\frac{1}{2}}, \quad (l'Q_1 l)^{\frac{1}{2}} \geq (l'Q^- l)^{\frac{1}{2}}. \tag{14}$$

We need to do some consequent steps to solve the Problem 2.

*Step 1.* This step is the same as step 1 for external estimates. Therefore, to solve this auxiliary problem 2 we should find at the first step non-singular transformation of matrices $Q_0$, $Q_1$ of ellipsoids $\mathcal{E}_0$, $\mathcal{E}_1$ which leads them simultaneously to diagonal forms. The transformed ellipsoids have forms (11, 12):

$$\tilde{\mathcal{E}}_0 = \{s \in R^n | s's \leq 1\},$$

$$\tilde{\mathcal{E}}_1 = \{s \in R^n | s'(W^2)^{-1}s \leq 1\}.$$

*Step 2.* In order to find the internal ellipsoidal estimation of $co(\mathcal{E}_0 \cup \mathcal{E}_1)$ we use the fact that the intersection and the union for overlapping sets are dual operations [15].

*Lemma 1:* [15] Suppose $\mathcal{M}, \mathcal{M}_1, \mathcal{M}_2$ are convex compacts in $R^n$. Then

1) If $0 \in \mathcal{M}$, then $\mathcal{M}^*$ is convex compacts in $R^n$.
2) If $0 \in \mathcal{M}$, then $(\mathcal{M}^*)^* = \mathcal{M}$.
3) If $\mathcal{M}_1 \subseteq \mathcal{M}_2$ then $\mathcal{M}_1^* \supseteq \mathcal{M}_2^*$.
4) If $A \in R^{n\times n}$ and $\det A \neq 0$ then $(A\mathcal{M})^* = (A')^{-1}\mathcal{M}^*$.
5) If $0 \in \mathcal{M}_1 \cap \mathcal{M}_2$ then $(\mathcal{M}_1 \cap \mathcal{M}_2)^* = co(\mathcal{M}_1^* \cup \mathcal{M}_2^*)$.

From Lemma 1 it follows that for transformed ellipsoids $\tilde{\mathcal{E}}_0$ and $\tilde{\mathcal{E}}_1$ we have

$$\tilde{\mathcal{E}}^- \subseteq co(\tilde{\mathcal{E}}_0 \cup \tilde{\mathcal{E}}_1) \Leftrightarrow (\tilde{\mathcal{E}}^-)^* \supseteq \tilde{\mathcal{E}}_0^* \cap \tilde{\mathcal{E}}_1^*.$$

Notice that $\tilde{\mathcal{E}}_0^* = \tilde{\mathcal{E}}_0$,

$$\tilde{\mathcal{E}}_1^* = \{s \in R^n | s'W^2 s \leq 1\},$$

$$W^2 = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\},$$

$\lambda_i > 0$ $(i = 1 \ldots n)$ are the eigenvalues of matrix $Q_0^{-1}Q_1$, defined in (9).

*Step 3.* Now we need to construct the upper ellipsoidal bound for $\tilde{\mathcal{E}}_0^* \cap \tilde{\mathcal{E}}_1^*$.

There are different approaches to the ellipsoidal estimation of intersection of two ellipsoids, e.g. [1]-[4], [11], but unfortunately non of them provided the minimal volume upper estimate.

The approach given below allows to estimate the reachable set enough precisely and it can be easy calculated.

At first we construct the ellipsoid $\tilde{\mathcal{E}}_2^*$ that is contained in the intersection $\tilde{\mathcal{E}}_0^* \cap \tilde{\mathcal{E}}_1^*$:

$$\tilde{\mathcal{E}}_2^* = \{s \in R^n | s'\tilde{Q}_2^{-1}s \leq 1\}, \tilde{Q}_2 = \mathrm{diag}\{\eta_1, \ldots, \eta_n\},$$

$$\eta_i = \min\{1, 1/\lambda_i\}, \quad i = 1, 2, \ldots, n.$$

Further we find the ellipsoid similar to $\tilde{\mathcal{E}}_2^*$ with minimal possible volume among all ellipsoids containing the intersection $\tilde{\mathcal{E}}_0^* \cap \tilde{\mathcal{E}}_1^*$. It is not difficult to calculate similarity coefficient:

$$k = \frac{\eta_2(1-\lambda_2) + \eta_1(\lambda_1 - 1)}{\eta_1\eta_2(\lambda_1 - \lambda_2)}.$$

Therefore we get the ellipsoid

$$(\tilde{\mathcal{E}}^-)^* = \{s \in R^n | s'(\tilde{Q}^-)^{-1}s \leq 1\},$$

where

$$\tilde{Q}^- = k\tilde{Q}_2 = \mathrm{diag}\{k\eta_1, \ldots, k\eta_n\}, \tag{15}$$

$\eta_i$ is defined above.

*Step 4.* The conjugate ellipsoid to the $(\tilde{\mathcal{E}}^-)^*$ is the internal estimate of the union of transformed ellipsoids $\tilde{\mathcal{E}}_0$ and $\tilde{\mathcal{E}}_1$,

$$\tilde{\mathcal{E}}^- = \{s \in R^n | s'\tilde{Q}^- s \leq 1\},$$

where $\tilde{Q}^-$ is in (15).

*Theorem 4:* The ellipsoid $\tilde{\mathcal{E}}^-$ constructed in this way solves the problem 2 for the sets $\tilde{\mathcal{E}}_0$ and $\tilde{\mathcal{E}}_1$.

*Proof:* The proof follows from the algorithm of constructing the diagonal matrix $(\tilde{Q}^-)^{-1}$ of ellipsoid $\tilde{\mathcal{E}}^-$, properties of ellipsoids and Lemma 1. ■

*Step 5.* Then one can return to the original coordinates. The final calculation gives the ellipsoid $\mathcal{E}^-$ (the internal estimate)

$$\mathcal{E}^- = \{x \in R^n \mid x'(Q^-)^{-1}x \leq 1\} \subseteq co(\mathcal{E}_0 \cup \mathcal{E}_1),$$

$$Q^- = Q_0^{\frac{1}{2}} M'(\tilde{Q}^-)^{-1} M Q_0^{\frac{1}{2}}.$$

Assume now that the ellipsoid

$$\mathcal{X}_0 = \{x_0 \in R^n | \ \ x_0' R^{-1} x_0 \leq 1\}$$

is done instead of the above assumption $\mathcal{X}_0 = \{0\}$. In this case we have

$$\mathcal{X}(T, \mathcal{X}_0) = \mathcal{E}(0, X(T,0)RX'(T,0)) + \mathcal{X}(T),$$

and we need only to apply the estimate procedure for sum of the ellipsoid $\mathcal{E}(0, X(T,0)RX'(T,0))$ and the ellipsoid which is contained in $\mathcal{X}(T)$ [3], [4].

Based on the solution of the auxiliary problem 2, we can also solve the problem of constructing an internal estimate for reachable set $\mathcal{X}(T)$. For this purpose we apply this procedure consequently. assumption The assumption P guarantees the finite number of such steps. The final ellipsoid is contained in $\mathcal{X}(T)$.

### C. General Case

It was assumed before that the assumption P is valid. Next we omit the assumption P and consider the general case. The following theorem is true.

*Theorem 5:* For any $\varepsilon > 0$ there exist $\delta > 0$ and a finite set $T_\delta \subset [0, T]$ such that for all $l \in R^n$ the inequalities hold

$$\max_{\tau\in T_\delta}(G(\tau,l))^{\frac{1}{2}} \leq \max_{0\leq\tau\leq T}(G(\tau,l))^{\frac{1}{2}} \leq \max_{\tau\in T_\delta}(G(\tau,l))^{\frac{1}{2}} + \varepsilon\|l\|. \tag{16}$$

*Proof:* Introduce the notation

$$f(\tau, l) = \rho(l|\mathcal{X}(T, \mathcal{X}_0)) = (l'X(T,\tau)Q_0X'(T,\tau)l)^{\frac{1}{2}}.$$

It is obvious that the function $f(\tau, l)$ is continuous in $[0,T] \times S$, where $S = \{l \in R^n | \ \|l\|_2 \leq 1\}$. Therefore, for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$|f(\tau, l) - f(\tilde{\tau}, l)| \leq \varepsilon \tag{17}$$

for any $l \in S$ and any $\tau, \tilde{\tau} \in [0,T]$ such that $|\tau - \tilde{\tau}| < \delta$. Let set

$$T_\delta = \{0 = \tau_0 < \tau_1 < \ldots < \tau_k = T\}, \ \max_{1\leq i\leq k}|\tau_i - \tau_{i-1}| < \delta.$$

Then, we obtain from (17)

$$\max_{\tau\in T_\delta} f(\tau,l) \leq \max_{0\leq\tau\leq T} f(\tau,l) \leq \max_{\tau\in T_\delta} f(\tau,l) + \varepsilon.$$

Using that this and the fact that the function $f(\tau, l)$ is positively homogeneous, we obtain inequality (16). ■

*Corollary 1:* If the conditions of the theorem 5 are valid then the inclusions are true

$$co(\bigcup_{\tau\in T_\delta} \mathcal{E}_\tau) \subset co(\bigcup_{\tau\in[0,T]} \mathcal{E}_\tau) \subset co(\bigcup_{\tau\in T_\delta} \mathcal{E}_\tau) + \varepsilon S.$$

*Proof:* The proof follows from the formula

$$\rho(l|co\bigcup_{\tau\in T_\delta} \mathcal{E}_\tau) = \max_{\tau\in T_\delta} f(\tau, l)$$

and estimate (16) of Theorem 5. ■

Applying the procedure described above for the case of a finite $T_* = T_\delta$, the internal ellipsoidal estimate of the set $\mathcal{X}(T; \mathcal{X}_0)$ and the external ellipsoidal approximations of the set $\mathcal{X}(T; \mathcal{X}_0) + \varepsilon S$ (for any $\varepsilon > 0$) are found.

## IV. Example

Consider the following control system $(0 \leq t \leq T)$:

$$\begin{cases} dx_1(t) &= x_2(t)dt + du_1(t), \\ dx_2(t) &= du_2(t). \end{cases} \tag{18}$$

Find the ellipsoidal estimations of the reachable set. Here we take $\mathcal{X}_0 = \{0\}$ and the set $\mathcal{U}$ generated by the ellipsoid

$$\mathcal{E}_0 = \left\{l \in R^2 \ | \ \ l'Q_0^{-1}l \leq 1\right\}, \ \ Q_0 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix},$$

where $a, b > 0, \ \ a, b \in R$.

In these example the reachable set is the convex hull of union two ellipsoids $\mathcal{E}_0 = \mathcal{E}(0, Q_0)$ and $\mathcal{E}_1 = \mathcal{E}(0, Q_1)$

$$\mathcal{X}(T, \{0\}) = co(\mathcal{E}_0 \cup \mathcal{E}_1), \tag{19}$$

$$\mathcal{E}_0 = \left\{x \in R^2 \mid x'Q_0^{-1}x \leq 1\right\}, \ Q_0 = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix},$$

$$\mathcal{E}_1 = \left\{x \in R^2 \mid x'Q_1^{-1}x \leq 1\right\}, \ Q_1 = \begin{pmatrix} a^2 + b^2T^2 & b^2T \\ b^2T & b^2 \end{pmatrix}.$$

The formula (19) was calculated using the maximum principle for impulse system (18) with given ellipsoidal restriction [13].

Algorithm of constructing the external estimate of the reachable set may be illustrated in this example.

Figure 1 shows the transformed ellipsoids $\tilde{\mathcal{E}}_0$ (it is marked by number **1** at Fig. 1) and $\tilde{\mathcal{E}}_1$ (it is marked by number **2** at Fig. 1). The ellipsoid $\tilde{\mathcal{E}}^+$ (**3**, Fig. 1) is minimal with respect to inclusion among all ellipsoids containing $\tilde{\mathcal{E}}_0 \bigcup \tilde{\mathcal{E}}_1$ and having a matrix of diagonal form.

When we return to the space of $x$ - coordinates we get the ellipsoids $\mathcal{E}_0$ (**1**, Fig. 2) and $\mathcal{E}_1$ (**2**, Fig. 2). The ellipsoid $\mathcal{E}^+$ (**3**, Fig. 2) is the solution of auxiliary problem 1.

The external ellipsoidal estimates and exact reachable set are presented at Fig. 3 for some values of $T$.

The tube of trajectories of the system (18) and the dynamics of external estimates of $\mathcal{X}(T)$ are indicated at Fig. 4.

Fig. 5–8 illustrate the internal estimation algorithm. The transformed conjugate ellipsoids $\tilde{\mathcal{E}}_0^*$ and $\tilde{\mathcal{E}}_1^*$ are shown at Fig. 5 (**1** and **2** respectively). The ellipsoid $\tilde{\mathcal{E}}_2^*$ (**3**, Fig. 5) is the maximal ellipsoid with respect to inclusion among all ellipsoids with diagonal matrices containing in the intersection $\tilde{\mathcal{E}}_0 \bigcap \tilde{\mathcal{E}}_1$. Denote by symbol $(\tilde{\mathcal{E}}^-)^*$ the upper estimate ellipsoid (**4**, Fig. 5) for the intersection $\tilde{\mathcal{E}}_0^* \cap \tilde{\mathcal{E}}_1^*$. This ellipsoid is similar to the ellipsoid $\tilde{\mathcal{E}}_2^*$ (see step 3 of the above algorithm of internal estimation).

We construct after that the conjugate ellipsoid $\tilde{\mathcal{E}}^-$ and return to the space of $x$ - coordinates. The exact reachable set $\mathcal{X}(T)$ is given at Fig. 6 (**4**). The set $\mathcal{X}(T)$ is the convex hull of two ellipsoids $\mathcal{E}_0$ (**1**, Fig. 6) and $\mathcal{E}_1$ (**2**, Fig. 6).

The internal ellipsoidal estimate $\mathcal{E}^-$ (it is marked as **3**) of $\mathcal{X}(T)$ is shown at Fig. 6.

The tube of trajectories of the system (18) and the dynamics of internal estimates of $\mathcal{X}(T)$ are indicated at Fig. 8.

Figure 9 shows the tube of trajectories of the system (18) and its internal and external estimates.

## REFERENCES


[1] A. B. Kurzhanski, *Control and Observation under Conditions of Uncertainty.* Moscow: Izdatel'stvo Nauka, 1977. (In Russian.)

[2] A. B. Kurzhanski, T. F. Filippova, "On the Theory of Trajectory Tubes — a Mathematical Formalism for Uncertain Dynamics, Viability and Control," *Advances in Nonlinear Dynamics and Control: a Report from Russia, Progress in Systems and Control Theory, (A.B. Kurzhanski, (Ed))*, vol. 17, Birkhauser, Boston, pp. 22-188, 1993.

[3] A. B. Kurzhanski, I. Valyi, *Ellipsoidal Calculus for Estimation and Control*. Boston: Izdatel'stvo Birkhauser, 1997.

[4] F. L. Chernousko, *State Estimation for Dynamic Systems*. Moscow: Izdatel'stvo Nauka, 1988. (In Russian.)

[5] S. T. Zavalischin, A. N. Sesekin, *Impulsive Processes. Models and Applications*. Moscow: Izdatel'stvo Nauka, 1991. (In Russian.)

[6] S. T. Zavalishchin, A. N. Sesekin, *Dynamic Impulse Systems. Theory and Applications,* Dordrecht, Netherlands: Kluwer Academic Publishers, 1997

[7] V. A. Dykhta, O. N. Sumsonuk, *Optimal Impulse Control with Applications*. Moscow: Izdatel'stvo Fizmatgiz, 2000. (In Russian.)

[8] R. B. Vinter, F. L. Pereira, "A maximum principle for optimal processes with discontinuous trajectories," in *SIAM J. Contr. and Optimization*, vol. 26, pp. 155-167, 1988.

[9] F. L. Pereira, G. N. Silva, "Necessary Conditions of Optimality for Vector-Valued Impulsive Control Problems," *Systems and Control Letters*, vol. 40, pp. 205-215, 2000.

[10] B. M. Miller, E. Ya. Rubinovich *Optimization of dynamic systems with impulsivecontrols*. Moscow: Izdatel'stvo Nauka, 2005. (In Russian.)

[11] V. A. Honin, "Guaranteed Estimation of the State of Linear Systems by Means of Ellipsoids," in *Evolutionary Systems in Estimation Problems*, A. B. Kurzhanski, T. F. Filippova, eds., Sverdlovsk, 1985. (In Russian.)

[12] O. G. Vzdornova, T. F. Filippova, "External Ellipsoidal Estimates of the Attainability Sets of Differential Impulse Systems". *J. Computer and Systems Sciences Intern.*, vol.45, **1**, pp. 34-43, 2006. (Transl. from *Teoriya i Sistemi Upravlenia*, no. 1, pp.38-47, 2006.)

[13] O. G. Vzdornova, T. F. Filippova, "State estimation for impulse control systems with ellipsoidal constraints," in *Proc. of The Fourth International Conference on Tools for mathematical modelling: .* St. Petersburg, Russia, 2003, pp.34-41. (in Russian)

[14] P. Lankaster, *The Theory of Matrices*. Moscow: Izdatel'stvo Nauka, 1978. (In Russian.)

[15] R. T. Rockafellar, *Convex Analysis*. Moscow: Izdatel'stvo Nauka, 1988. (In Russian.)


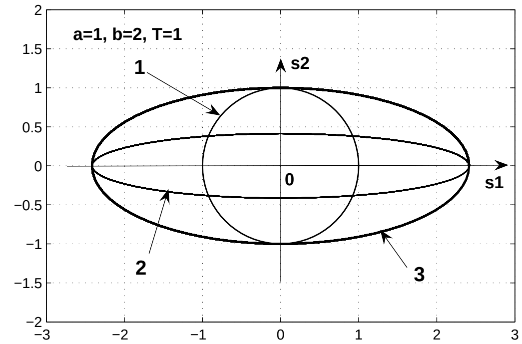


Fig. 1. Auxiliary ellipsoids $\tilde{\mathcal{E}}_0, \tilde{\mathcal{E}}_1, \tilde{\mathcal{E}}^+$ for external estimation.

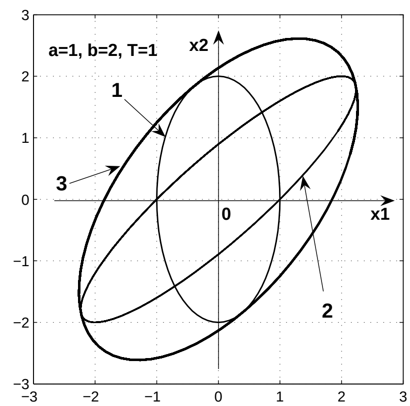


Fig. 2. External estimation $\mathcal{E}^+$ of union of two ellipsoids $\mathcal{E}_0, \mathcal{E}_1$.

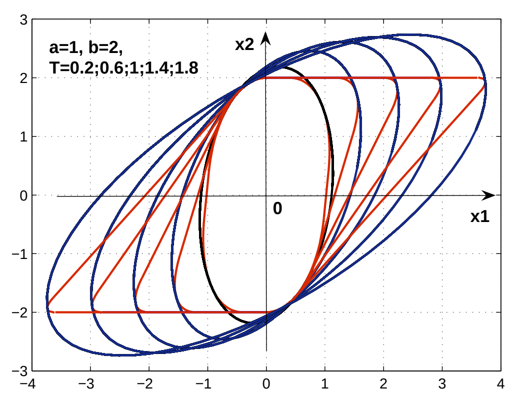


Fig. 3. The dynamics of external estimates of $\mathcal{X}(T)$.

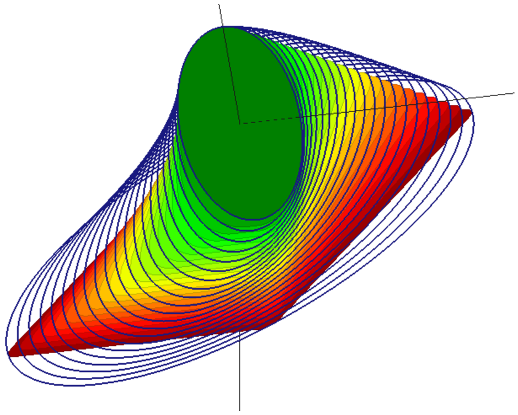

Fig. 4. The dynamics of external estimates of $\mathcal{X}(T)$ and the tube of trajectories of the system (18).

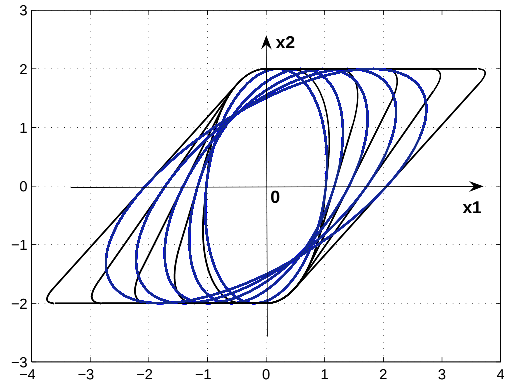


Fig. 7. The dynamics of internal estimates of $\mathcal{X}(T)$.

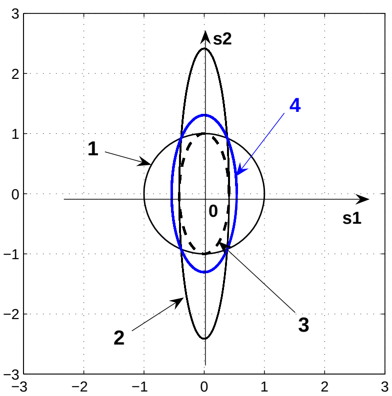


Fig. 5. Auxiliary ellipsoids $\tilde{\mathcal{E}}_0^*, \tilde{\mathcal{E}}_1^*, \tilde{\mathcal{E}}_2^*, (\tilde{\mathcal{E}}^-)^*$ for internal estimation.

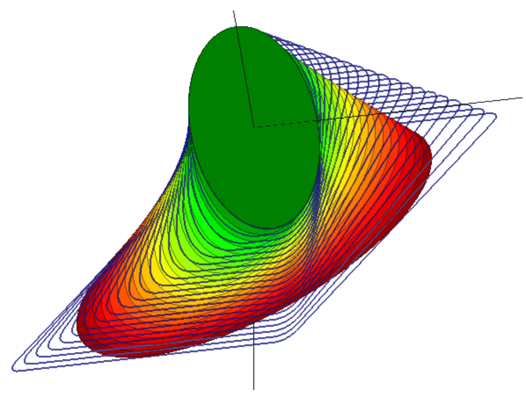

Fig. 8. The dynamics of internal estimates of $\mathcal{X}(T)$ and the tube of trajectories of the system (18).

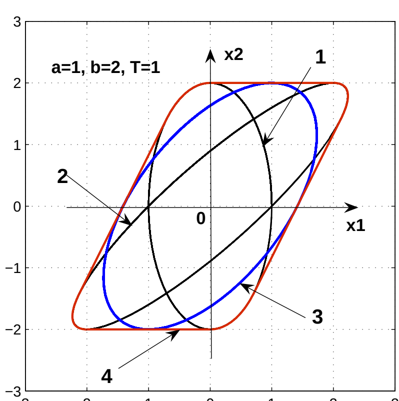


Fig. 6. Internal estimate $\mathcal{E}^-$ of union of two ellipsoids $\mathcal{E}_0, \mathcal{E}_1$.

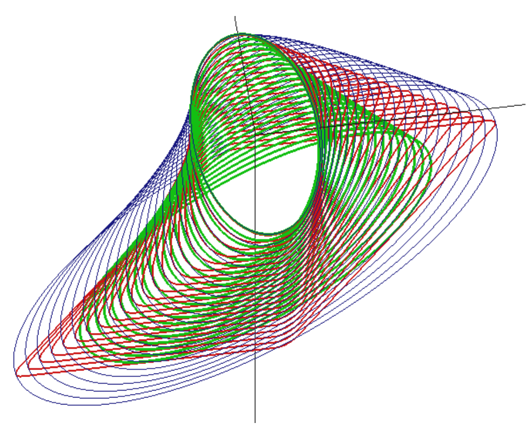

Fig. 9. The dynamics of external and internal estimates of $\mathcal{X}(T)$ and the tube of trajectories of the system (18).