HIERARCHICAL ORGANIZATION OF SYNCHRONOUS BEHAVIOR IN ADAPTIVE NETWORKS

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Abstract
We study synchronization behavior in the network of phase oscillators with adaptive couplings. We have found hierarchical formation of complex synchronization patterns including multi-cluster and chimera states. This process represents a sequential formation of a new group of synchronized elements, ordered in hierarchical way. The formation of these groups is accompanied by a transformation of the network interaction structure, when the network splits into a number of weakly interacting subnetworks.

Key words
Synchronization, phase clusters, chimera states, coupled oscillators, adaptive networks.

1 Introduction

Large populations of coupled oscillators occur in a variety of different applications, ranging from physics, chemistry, engineering to biology and neuroscience. Synchronization is an universal phenomenon, taking place in such systems. For many years, the study of various aspects of synchronization phenomena in networks of coupled oscillators are successfully carried out within the framework of the Kuramoto model [Kuramoto, 1984]. Since its original formulation, the Kuramoto model has been extensively studied in many different versions (see [Acebrón, Bonilla, Vicente, Ritort and Spigler, 2005; Rodrigues,Peron and Kurths, 2016] and Refs. therein), taking into account the influence of various factors on the collective behavior of coupled oscillator networks. Most of the early works on Kuramoto model have focused on studying of the dynamics of networks with global coupling.

Recently one has observed the emergence of a huge number of works devoted to the study of the effects, caused by the introduction of a complex configuration of connections between the oscillators of network. In the most cases of these works, attention was exclusively concentrated on complex networks with either static structures [Gomes, Moreno and Arenas, 2007; Zhu, Zheng and Yang, 2014; Coutinho, 2013; Ujjwal, Punetha and Ramaswamy, 2016], or networks with randomly evolving topology of connection [Laing, 2012; Kohar, 2014; Singh and Bagarti, 2015; Buscarino, 2015]. However, the formation of structure in some real networks is not random, but represents the result of co-evolution of the coupling connections and the states of the elements. For example, such adaptation mechanisms of the network structure can be found in neural systems, where strength of synaptic connections vary depending on the activity of neurons. This have motivated some recent studies [Ren and Zhao, 2007; Niyogi and English, 2009; Aoki and Aoyagi, 2011] that seek to understand the influence of the dynamic (adaptive) couplings to collective behavior of the networks. However, most of these studies has mainly aimed at determining how different schemes of coupling adaptation affect the onset of synchronization in assumption that the complex topology of the network is not changed through the co-evolving dynamics. At the same time, the properties of synchronization patterns emerging in networks of coupled oscillators depend largely on the underlying topology of interaction. Therefore, the influence of two interrelated factors such as adaptive changing of couplings and dynamically evolving topology of connections on the dynamics of complex networks is also an interesting open question.

In this work, we consider the complex networks of phase oscillators with adaptive couplings and study the properties of synchronization patterns that emerge as result of interplay of the own dynamics of network elements and the process of adaptive reorganization of network interaction structure. For this purpose, we will consider generalization of the Kuramoto model on complex network whose interactions evolve depending on the relative phases between the oscillators.

2 Model

We consider a system of N coupled phase oscillators described by the following equation:

$$\frac{d\phi_i}{dt} = \omega_i + \frac{1}{N_i}\sum_{j=1}^{N} a_{ij}\kappa_{ij}F(\phi_i - \phi_j), \qquad (1)$$

where ϕ_i represents the phase of the ith oscillator $(i = 1, \ldots, N)$, ω_i is its intrinsic frequency. The function $F(\phi)$ characterizes the interaction between the oscillators and κ_{ij} denotes the coupling strength of the connection from the jth to the ith oscillator. The topology of the network is determined by the connectivity matrix $\mathbf{A} = \{a_{ij}\}$ whose elements $a_{ij} = a_{ji} = 1$ when oscillators i and j are connected and $a_{ij} = 0$ otherwise. N_i is the number of elements in the set of oscillators connected to oscillator i, i.e. $N_i = \sum_j a_{ij}$. We consider a simple model $F(\phi) = -\sin(\phi + \alpha)$, where parameter α can be regarded as the phase difference induced by a short delay of the coupling.

The dynamics of the coupling strength k_{ij} is described by the following equation:

$$\frac{d\kappa_{ij}}{dt} = \varepsilon(\Lambda(\phi_i - \phi_j) - \kappa_{ij}), \qquad (2)$$

where $\Lambda(\phi)$ is the adaptation function, which determines the character of variation of the coupling strength as a function of the states of interacting oscillators, namely, their relative phase differences. We choose the adaptation function in the form $\Lambda(\phi) = -\sin(\phi + \beta)$, where parameter β controls the properties of adaptation function. In simulation, we choose the parameter $\varepsilon = 0.01$, assuming that the evolution of couplings strength is slower than the evolution of the oscillators. Additionally, if oscillator j does not interact with oscillator i, then coupling between them is always equal to zero, i.e. $\kappa_{ij} \equiv 0$ if $a_{ij} = 0$.

Thus, the adaptive networks considered here is described by

$$\frac{d\phi_i}{dt} = 1 - \frac{1}{N}\sum_{j=1}^{N} \kappa_{ij}\sin(\phi_i - \phi_j + \alpha), \qquad (3)$$
$$\frac{d\kappa_{ij}}{dt} = -\varepsilon(\sin(\phi_i - \phi_j + \beta) + \kappa_{ij}).$$

We have analyzed the self-organization processes in the networks consisting of $N = 500$ identical oscillators $(\omega_i = 1)$ with different topology of connections. The state of the network was studied as a function of control parameters α and β. Initial conditions were chosen randomly, with uniform distributions of the phase ϕ_i in the interval $[0, 2\pi]$ and coupling strengths κ_{ij} in $[-1, 1]$.

3 Dynamical states of a network

Figure 1 represents the diagrams illustrating the distribution of domains with different dynamical behavior

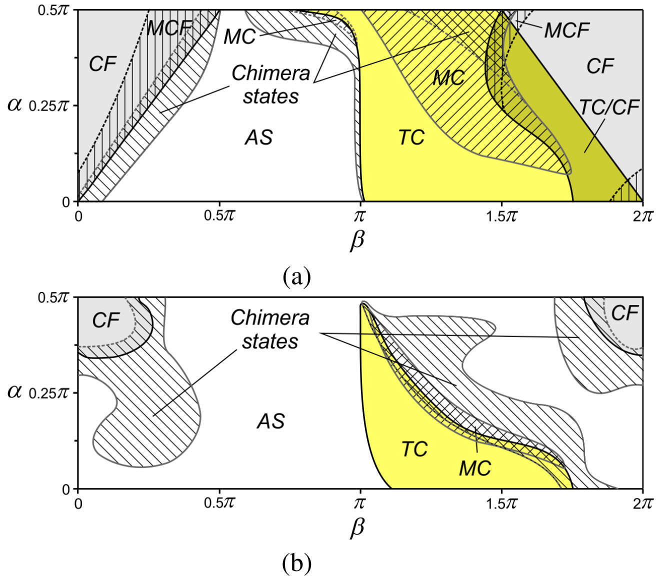


Figure 1. Diagrams of of dynamical states of network with all-to-all connections (a) and random network with average node degree $k = 10$ (b). The dynamical states of the network: (TC) - two-cluster state, (MC) - multi-cluster state, the states characterized by the formation of one (CF) or several (MCF) coherent groups with a fixed phase relationship between oscillators within the groups, (AC) - asynchronous state.

of the networks with all-to-all and random connection topology. These diagrams was obtained as a result of averaging 20 numerical trials carried out for different sets of initial conditions. During each trial, we constructed two-parametric diagrams of a number of characteristics, such as time-averaged order parameters

$$\langle R_k \rangle = \frac{1}{\Delta t}\int_T^{T+\Delta t} R_k dt, \qquad (4)$$

where

$$R_k = \frac{1}{N}\left|\sum_{j=1}^{N} e^{-ik\phi_j}\right|, k = 1, 2. \qquad (5)$$

The calculation of these characteristics was carried out over a large time interval $\Delta t = 10^3$, assuming that transient processes had ended to the moment of time $T = 10^5$. Parameter R_1 characterizes the degree of complete synchronization of the network and can take a value in the interval $[0, 1]$. The second order parameter R_2 can also take values in the interval $[0, 1]$ and indicates the formation of two synchronized groups with anti-phase relationship.

To obtain complete information on the possibility of formation of complex synchronous states, including partial synchronization modes, we additionally introduced a parameter that determines the fraction of synchronized pairs of connected oscillators in the network.

For this purpose, analogous to [Gomes, Moreno and Arenas, 2007], we calculate the degree of mutual synchronization for each pair of oscillators i and j

$$R_{ij} = \left| \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_T^{T+\Delta t} e^{i(\phi_i(t) - \phi_j(t))} dt \right|. \quad (6)$$

The value of each element in the resulting matrix $\mathbf{R}$ is bounded in the interval $[0, 1]$, being $R_{ij} = 1$ when oscillators i and j are synchronized, i.e. phase difference $\phi_i(t) - \phi_j(t) = const$. Once the degree of synchronization between the oscillators is measured, we construct a new matrix $\widetilde{\mathbf{R}}$ whose elements take a value $\widetilde{R}_{ij} = 1$, when oscillators i and j are synchronized or $\widetilde{R}_{ij} = 0$ otherwise. Thus, the fraction of synchronized pairs of connected oscillators is determined as follows

$$R_{link} = \frac{1}{N_L} \sum_{i,j=1}^{N} a_{ij} \widetilde{R}_{ij}, \quad N_L = \sum_{i,j=1}^{N} \frac{a_{ij}}{2}, \quad (7)$$

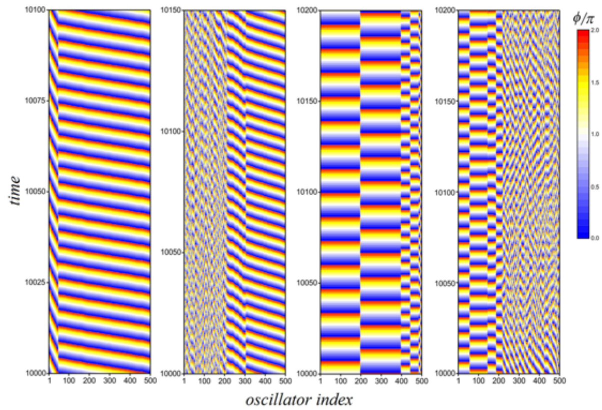
where N_L - is the total number of connections between oscillators in the network. In order to distinguish states when the network splits into several coherent groups from chimera states, we analyzed the parameter P_S defined as

$$P_S = \frac{1}{N} \sum_{i=1}^{N} \max_j \{\widetilde{R}_{ij}\}. \quad (8)$$

The fulfillment of the condition $P_S < 1$ indicates the formation of a chimera state in the network. A joint analysis of the characteristics (5)-(8) allowed us to obtain several different types of synchronous behavior observed in the system (3).

In the region around $\beta = 0$ one or several synchronized groups are organized in the network. The oscillators within each group demonstrate coherent behavior, maintaining a fixed phase relationship between the oscillators (Fig.2a). The oscillators within each group have the same frequency and a fixed phase relationship. Frequencies of oscillators belonging to different groups are different, and the relative phase differences between them continuously change in time. Another type of synchronous behavior is observed in the region $\beta \in (\pi, 2\pi)$. In this case the network may demonstrate two-cluster or multi-cluster states, in which one or several pairs of anti-phase clusters are formed. The frequencies of the oscillators belonging to different pairs of anti-phase clusters differ from each other. The example of such synchronous behavior is presented in Fig.2c.

We find that the network may also demonstrate chimera states, in which some part of the network forms a group of synchronized oscillators, while the other oscillators remain desynchronized. The synchronous part of the chimera states can represent several pairs of anti-phase clusters (Fig.2d), one or more groups of elements exhibiting coherent behavior with a fixed phase relationship within the groups(Fig.2b), and a combination of these states. These complex synchronization patterns are observed both in networks with all-to-all initial connection topology and random networks. In this case, a decrease of the number of connections in the network leads to an increase of the parameters regions where the system demonstrates chimera states (Fig.1b).



(a) (b) (c) (d)

Figure 2. Dynamical states of the network with all-to-all connections. Parameter values: $\varepsilon = 0.01, \alpha = 0.3\pi, \beta = 0.23\pi$ (a), $\beta = 0.3\pi$ (b), $\beta = 1.47\pi$ (c); $\alpha = 0.45\pi, \beta = 1.4\pi$ (d).

4 Scenarios of organization of synchronous behavior

The process of formation of the network states characterized by the presence of several group of synchronized oscillators is accompanied by transformation of interaction structure of the network. As a result of coevolving dynamics of the phases of oscillators and coupling strength the network splits into several weakly coupled subnetworks. We have found hierarchical formation of synchronization patterns which represents a sequential formation of new groups of synchronized oscillators. Besides that these synchronous groups arise in different time scales, the size of the groups decreases at each subsequent stage of the network evolution.

To illustrate the organization of hierarchical synchronization, Fig. 3 displays a series of snapshots of coupling matrix calculated at different stages of the formation of synchronization pattern. At the initial stage of the evolution of random initial state (Fig. 3a), there appears a group of elements which are relatively quickly synchronized. Formed group includes most of the network elements and is characterized by strong inter-element couplings within the group (Fig. 3b) against the background of relatively weak interaction between the rest of the network elements. Further, in the remaining incoherent part of the network, there forms the second synchronous group (Fig. 3c), which is also

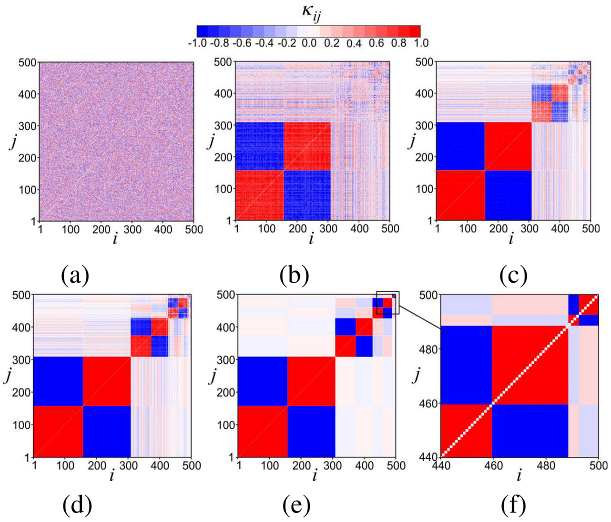


Figure 3. Sequential formation of multi-cluster state in the network with all-to-all connections. (a)-(f) The matrix of couplings at the different stage of formation of synchronization pattern: (a) $t = 0$, (b) $t = 200$, (c) $t = 350$,(d) $t = 500$, (e)-(f) $t = 10000$. Parameter values: $\varepsilon = 0.01, \alpha = 0.3\pi, \beta = -0.53\pi$.

characterized by a strong interaction between the elements within the group. At the same time, this process is accompanied by the suppression of couplings between the elements of synchronous groups formed at different stages of the evolution of the network. Next similar processes continue in the incoherent part of the network until the network reaches the final clustered state.

Depending on the parameters the process of formation of synchronous groups can terminate at some stage of network evolution. As a result the system demonstrates chimera state, in which some part of the network forms one or several groups of synchronized oscillators, while the other oscillators remain nonsynchronized.

5 Conclusion

We have investigated the organization of synchronous behavior in network of phase oscillators with adaptive couplings. We have found hierarchical formation of complex synchronization patterns including multi-cluster and chimera states. This process represents a sequential formation of a new group of synchronized oscillators, ordered in hierarchical way. Depending on the properties of the adaptation function the organized groups may exhibit different local properties of synchronous behavior. In final state, the network can consist of a finite number of groups of synchronized oscillators (multi-cluster states), split into several coherent groups with a fixed phase relationship within the groups, and also demonstrate a great number of chimera states. The formation of these states is accompanied by a transformation of the network interaction structure, when the network splits into a number of weakly interacting subnetworks.

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