AUTORESONANCE IN KLEIN-GORDON CHAINS

Agnessa Kovaleva
Space Research Institute
Russian Academy of Sciences
Russia
agnessa_kovaleva@hotmail.com

Abstract
In this paper we study the emergence and stability of autoresonance (AR) in the nonlinear Klein-Gordon chain consisting of identical linearly coupled Duffing oscillators. The chain is excited by an external periodic force with a slowly varying frequency applied to one of the oscillators. Explicit asymptotic equations describing the averaged amplitudes and phases of oscillations are derived. These equations demonstrate that, in contrast to the chains with linear attachments, the nonlinear chain can be entirely captured into resonance. As shown in this paper, AR in the entire chain gives birth to the asymptotic equipartition of energy amongst the oscillators, which is manifested as the convergence of the amplitudes of oscillations to a monotonically increasing mean value equal for all oscillators. The thresholds of the structural and excitation parameters, which allow the emergence of AR in the entire chain, are determined. The derived analytic results are in good agreement with numerical simulation.

Key words
Nonlinear oscillations, asymptotic methods, autoresonance.

1 Introduction

In this work we investigate the emergence of high-energy autoresonant (AR) oscillations in the Klein-Gordon chain of finite length consisting of n identical linearly coupled Duffing oscillators. The chain is excited by a harmonic force with a slowly varying frequency applied at an edge of the chain.

An idea of AR, or "resonance under the action of a force produced by the system's itself" was first suggested in [Andronov, Vitt and Khaikin, 1966]. Feedback control schemes building on this idea and using self-sustained oscillations with predefined energy as a working process are widely used in engineering, see, e.g., [Astashev and Babitsky, 2007]. Note that feedback does not need an additional source of energy. However, its design and tuning may be expensive and cumbersome in practical situations, especially in multi-dimensional structures. A large class of systems can avoid feedback, still producing the required state with the help of time-variant feed-forward frequency control, which employs an intrinsic property of a nonlinear oscillator to change both its amplitude and natural frequency when the driving frequency changes. The ability of a nonlinear oscillator to remain captured into resonance due to variance of its structural or/and excitation parameters is termed autoresonance (AR). AR leads to persistently growing mean energy of oscillations, and thus, this process may be employed to attain the required energy level.

After first studies for the purposes of particle acceleration [Veksler, 1944; McMillan, 1945; Livingston 1954], a large number of theoretical approaches, experiments and applications of AR and similar effects in different fields of natural science, from plasmas to nonlinear optics and hydrodynamics, have been reported in literature (see, e.g., [Blekhman, 2012; Dauxois at al., 2004; May and Kühn, 2011; Milant'ev, 2013; Vazquez, MacKay, and Zorzano, 2003] and references therein). The analysis, first concentrated on the study of AR in a single nonlinear oscillator, was then extended to oscillator arrays with two- or three-degree-of-freedom. Examples in this category are excitations of phase-locked plasma waves with laser beams [Chapman et. al., 2010], particle transport in a weak external field with slowly changing frequency [Galow et.al., 2013], isotope separation processes [Rax, Robiche, and Fisch, 2007], control of nanoparticles [Klughertz, Hervieux, and Manfredi, 2014], etc. It is important to note that multi-dimensional nonlinear non-stationary systems seldom yield explicit analytical solutions needed for understanding and modelling the transition phenomena, so that the above-mentioned studies have not suggested any general conclusions concerning the occurrence of AR in multidimensional systems.

In most of previous studies, AR in the forced oscillator was considered as an effective tool for exciting high-energy oscillations in the entire array. However, recent results [Kovaleva, 2014a,b; 2016a; Kovaleva and Manevitch, 2016] have shown that this principle is not universal because the behavior of each element in the chain may drastically differ from the dynamics of a single oscillator. This effect was recently analyzed for oscillator arrays, which comprise a chain of time-invariant linear oscillators weakly coupled to a nonlinear actuator [Kovaleva, 2016a; Kovaleva and Manevitch, 2016]. It was shown that an external periodic force with slowly-varying frequency gives rise to AR only in the excited nonlinear oscillator (the actuator), while the response of the linear attachment remains bounded.

This work demonstrates that, in contrast to the abovementioned examples, all oscillators in the nonlinear chain can be captured into resonance. The difference in the dynamics of the two types of systems is closely connected with their resonance properties. High-energy resonant oscillations in a linear time-invariant oscillator can be generated by an external force whose frequency is equal or close to the natural frequency of the oscillator; deviations of the forcing frequency result in escape from resonance. On the contrary, the natural frequency of a nonlinear oscillator changes as its amplitude changes, and the oscillator may remain in resonance with its drive if the driving frequency varies slowly in time to be consistent with the slowly changing frequency of the oscillator.

It was shown in recent papers that the emergence and stability of AR in a single Duffing oscillator [Kovaleva, Manevitch, 2013a,b] as well as in two coupled Duffing oscillators [Kovaleva, 2014b] directly depends on the forcing and coupling parameters. It was demonstrated both theoretically and numerically that AR can occur only if the considered parameters exceed a certain threshold. Our consideration of a multi-particle chain is also focused on the study of the threshold phenomena, with the purpose to identify a set of parameters allowing stable AR in the entire chain.

The analysis of AR in this paper is motivated by the methods and the results, earlier developed for the study of resonance in a Klein-Gordon chain subjected to periodic forcing with constant frequency [Kovaleva, 2016b]. Section 2 introduces the equations of the chain dynamics. The small parameter of the system is defined as the dimensionless coefficient of linear coupling between the oscillators. Averaging of the dimensionless equations gives birth to the equations for the slow envelopes and the phases of resonant oscillations. In Section 3 we analytically calculate the steady-states values of the amplitudes and the phases of AR oscillations. Both analytical results and numerical simulations demonstrate that AR entails the asymptotic equipartition of energy with the amplitudes converging to the slowly growing quasi-steady mean values equal for all oscillators. The critical thresholds for the structural and excitation parameters are derived in Section 4. Numerical examples are discussed in Section 5.

2 The Model

The dynamics of the chain is governed by the nonlinear Klein-Gordon equations:

$$\begin{aligned}\frac{d^2u_r}{dt^2} + \omega^2 u_r + \gamma u^3 + \kappa[\eta_{r,r+1}(u_r - u_{r+1}) + \\ \eta_{r,r-1}(u_r - u_{r-1})] = A_r \sin\theta, \\ \frac{d\theta}{dt} = \omega + \zeta(t), \zeta(t) = k_1 + k_2 t. \quad (1)\end{aligned}$$

Here and below u_r denotes the displacement of the r-th oscillator; the frequency $\omega = (c/m)^{1/2}$, with m and c being the mass and the coefficient of linear stiffness of each oscillator, respectively; $\gamma > 0$ is the cubic stiffness coefficient; the parameter κ denotes the stiffness of linear coupling between the oscillators. The first oscillator is subjected to periodic forcing with amplitude A and time-dependent frequency; the parameters $k_1 > 0$ and $k_2 > 0$ denote the initial constant detuning and the detuning rate, respectively. The coefficients $\eta_{r,l}$ $= \{1, l \in [1, n]; 0, l = 0, l = n + 1\}$ depict unilateral coupling of the edge oscillators ($r = 1, r = n$) with the adjacent elements. Since the attachment is not directly driven by an external force, we let $A_r = 0$ if $r \geq 2$.

Note that Eqs. (1) depict a general oscillator model, which corresponds to a wide variety of physical objects. The physical interpretation of the coefficients in Eqs. (1) depends on the model under consideration. For examples, intensively studied micro-electromechanical systems (MEMS) are modelled as micro-cantilever arrays composed of the nonlinear elastic cantilever beams. Approximate models of MEMS are depicted by the nonlinear Klein–Gordon equations with coefficients expressed through the parameters of the cantilevers, see, e.g., [Balachandran, Perkins, and Fitzgerald, 2015].

For the study of the transient motion, it is convenient to separate the dimensionless equations of the slow dynamics, which have a simpler structure and involve a lesser number of independent parameters than the original equations (1). First, we introduce the dimensionless small parameter as $\varepsilon = k_1/\omega << 1$. Then, we define the recalled parameters and the new dimensionless time variables as follows:

$$\begin{aligned}&\gamma = 8\varepsilon\alpha\omega^{\bar{2}}, A_r = 2\varepsilon F_r \omega^{\bar{2}}, \kappa = 2\varepsilon k \omega^{\bar{2}}, \\ &k_2 = 2\varepsilon^2 \beta \omega^{\bar{2}}; \qquad (2)\\ &\tau_0 = \omega t, \tau = \varepsilon \tau_0,\end{aligned}$$

where $F_1 = F > 0$ but $F_r = 0$ at $r \geq 2$. Inserting (2) into (1), we then have

$$\begin{aligned}\frac{d^2u_r}{d\tau_0^2} + u_r + 8\varepsilon\alpha u^3 + 2\varepsilon k[\eta_{r,r+1}(u_r - u_{r+1}) + \\ \eta_{r,r-1}(u_r - u_{r-1})] = 2\varepsilon F_r \sin\theta, \\ \frac{d\theta}{d\tau_0} = 1 + \zeta_0(\tau); \zeta_0(\tau) = 1 + \beta\tau. (3)\end{aligned}$$

The array is assumed to be initially at rest, i.e. $\theta = 0, u_r = 0, \frac{du_r}{d\tau_0} = 0$ at $\tau_0 = 0$. We recall that zero initial conditions identify the so-called Limiting Phase Trajectory (LPT) corresponding to maximum possible energy transfer from a source of energy to a receiver [Manevitch, Kovaleva, and Shepelev, 2011].

Next, we reduce Eqs. (3) to the so-called standard form [Sanders, Verhulst and Murdock, 2007]. To this end, we introduce the dimensionless complex-conjugate vector envelopes Ψ and Ψ^* with components Ψ_r and Ψ_r^* $(r = 1, ..., n)$, respectively, and related dimensionless parameters by the following formulas:

$$\begin{aligned} &\Psi_r = \Lambda^{-1}(v_r + iu_r)e^{-i\theta}, \\ &\Psi_r^* = \Lambda^{-1}(v_r - iu_r)e^{i\theta}, \\ &\Lambda = (1/3\alpha)^{1/2}, f_r = F_r/\Lambda, \mu = k/\Lambda. \end{aligned} \tag{4}$$

It follows from (4) that the real-valued dimensionless amplitudes and the phases of oscillations are defined as $\widetilde{a}_r = |\Psi_r|$ and $\widetilde{\Delta}_r = \arg \Psi_r$, respectively. Substituting (4) into (3), we obtain the following (still exact) equations for the envelopes Ψ_r:

$$\begin{aligned} \frac{d\Psi_r}{d\tau_0} = -\varepsilon i[(\zeta_0(\tau) - |\Psi_r|^2)\Psi_r - \\ \mu\eta_{r,r+1}(\Psi_r - \Psi_{r+1}) - \\ \mu\eta_{r,r-1}(\Psi_r - \Psi_{r-1}) + f_r + G_r \end{aligned} \tag{5}$$

with initial conditions $\Psi_r(0) = 0$ $(r = 1, \ldots, n)$. The coefficients G_r include fast harmonics with coefficients depending on Ψ and Ψ^* but explicit expressions of these coefficients are insignificant for further analysis.

Finally, the multiple scales expansion is introduced:

$$\Psi_r(\tau_0, \varepsilon) = \psi_r(\tau) + \varepsilon\psi_r^{(1)}(\tau_0, \tau) + O(\varepsilon^2), \tag{6}$$

where the leading-order term $\psi_r(\tau)$ depict the slow component of the complex envelope but additional terms define small fast deviations near $\psi_r(\tau)$. The equations for the slow components $\psi_r(\tau)$ can be obtained by straightforward averaging of Eqs. (5) [Sanders, Verhulst, and Murdock, 2007]. Using the standard averaging procedure, we obtain the following equations for the slow components of the complex envelopes:

$$\begin{aligned} \frac{d\psi_r}{d\tau} = -i[(\zeta_0(\tau) - \\ |\psi_r|^2)\psi_r - \mu\eta_{r,r+1}(\psi_r - \psi_{r+1}) - \\ -\mu\eta_{r,r-1}(\psi_r - \psi_{r-1}) + f_r] \end{aligned} \tag{7}$$

with initial conditions $\psi_r(0) = 0$. Note that the averaged equations include three independent coefficients instead of six parameters in Eqs. (1).

The change of variables $\psi_r = a_r e^{i\Delta_r}$ yields the following equations for the real-valued dimensionless amplitudes a_r and the phases Δ_r:

$$\begin{aligned} \frac{da_r}{d\tau} &= \mu[\eta_{r,r+1}a_{r+1}\sin(\Delta_{r+1} - \Delta_r) + \\ &+\eta_{r,r-1}a_{r-1}\sin(\Delta_{r-1} - \Delta_r)] - f_r \sin\Delta_r, \\ a_r\frac{d\Delta_r}{d\tau} &= \mu[\eta_{r,r+1}(a_r - a_{r+1}\cos(\Delta_{r+1} - \Delta_r)) + \\ &+\eta_{r,r-1}(a_r - a_{r-1}\cos(\Delta_{r-1} - \Delta_r))] - \\ &(\zeta_0(\tau) - a_r^2)a_r - f_r\cos\Delta_r \end{aligned} \tag{8}$$

with initial amplitudes $a_r(0) = 0$ and indefinite initial phases $\Delta_r(0), r = 1, \ldots, n$. These initial conditions make the system singular at $\tau = 0$. To overcome these difficulties, one needs to solve the non-singular complex-valued Eqs. (7) and then find the real-valued amplitudes from Eqs. (8). The accuracy of asymptotic approximations for systems with slowly varying parameters has been discussed, e.g., in [Arnold, Kozlov, and Neishtadt, 2006]. Recall that the errors of approximation $|\widetilde{a}_r(\tau, \varepsilon) - a_r(\tau)| \to 0$ as $\varepsilon \to 0$ in the time interval of interest, which is, at least, of order $O(1/\beta)$.

3 Quasi-Steady States

Given that $\beta << 1$, the quasi-steady solutions can be calculated by the following formulas:

$$P_r = \frac{da_r}{d\tau} = 0; Q_r = \frac{d\Delta_r}{d\tau} = 0, \ r = 1, \ldots, n. \tag{9}$$

The last condition $P_n = 0$ implies $\sin(\bar{\Delta}_n - \bar{\Delta}_{n-1}) = 0$. Inserting this equality into the condition $P_{n-1} = 0$, we obtain $\sin(\bar{\Delta}_{n-1} - \bar{\Delta}_{n-2}) = 0$. Repeating this procedure for each equation $P_r = 0$, we find that $\sin(\bar{\Delta}_r - \bar{\Delta}_{r-1}) = 0, \ r > 1$. Finally, the equation $P_1 = 0$ yields $\sin\bar{\Delta}_1 = 0$. This means that either $\bar{\Delta}_r = 0 \pmod{2\pi}$ or $\bar{\Delta}_r = \pi \pmod{2\pi}$, $r = 1, \ldots, n$. As in [Kovaleva and Manevitch, 2013a; Kovaleva, 2014b], one can find that the phases $\bar{\Delta}_r = 0$ correspond to stable oscillations but the phases $\bar{\Delta}_r = \pi$ are unstable for all $r \geq 1$. Under these conditions, the second group of Eqs. (9) takes the form

$$\begin{aligned} \mu(1 - a_2/a_1) - \zeta_0(\tau) + a_1^2 - f = 0, \\ \mu[2 - (a_{r-1} + a_{r+1})/a_r] - -\zeta_0(\tau) + a_r^2 = 0, \\ r \in [2, n], \\ \mu(1 - a_{n-1}/a_n) - \zeta_0(\tau) + a_n^2 = 0. \end{aligned} \tag{10}$$

The following solutions of Eqs. (10)

$$\begin{aligned} \bar{a}_1(\tau) \approx \sqrt{\zeta_0(\tau)} + [f/2\zeta_0(\tau)] + O(\mu f/\zeta_0(\tau)), \\ \bar{a}_r(\tau) \approx \sqrt{\zeta_0(\tau)} + O(\mu^{r-1}f/\zeta_0^r(\tau), r \geq 2 \end{aligned} \tag{11}$$

describe the maximal quasi-steady amplitudes corresponding to AR in the entire chain. Note that the slowly increasing functions $\bar{a}_r(\tau)$ $(r = 1, \ldots, n)$ can be interpreted as the backbone curves.

It follows from (11) that the higher-order corrections may be ignored; furthermore,

$$\bar{a}_r(\tau) \to \bar{a}(\tau) = \sqrt{\zeta_0(\tau)} \text{as } \tau \to \infty \qquad (12)$$

for all $r \geq 1$.. Formulas (11), (12) clearly indicate that the energy initially placed in the first oscillator tends to equipartition amongst all oscillators with amplitudes of oscillations $\bar{a}_r(\tau) \to \bar{a}(\tau)$ as $\tau \to \infty$, $r = 1, \ldots, n$. In Section 5, this conclusion is illustrated by the results of numerical simulations for the chains with different number of particles.

4 Parametric Thresholds

Note that solutions (11) formally exist for arbitrary values of structural and excitation parameters. However, AR in the entire chain is physically realizable when the amplitude of excitation is large enough to generate AR in the actuator and the coupling strength is sufficient to maintain large oscillations of all attached oscillators. In this section we establish the parametric thresholds, which allow the emergence of AR in the entire chain. We demonstrate that, in contrast to the array with a linear attachment [Kovaleva, 2016a], a proper choice of the structural and excitation parameters guarantees the emergence of AR in the nonlinear chain. In order to simplify an analytical framework and elucidate the physical interpretation of the results, we assume that $f \sim o(1), \mu \sim o(1)$ but $\beta << 1$. These assumptions agree with the earlier obtained results for a pair of coupled Duffing oscillators [Kovaleva, 2014b] as well as with the results of numerical simulations for multi-particle chains presented in Sec. 5 below.

As remarked above, AR in the forced oscillator represents the necessary condition for the emergence of resonance in the passive attachment. This means that the first parametric boundary can be found assuming small oscillations of the attachment. Under this assumption, the equations of the excited oscillator are reduced to the form

$$\begin{aligned} \frac{da_1}{d\tau} &= -f \sin \Delta, \qquad (13)\\ a_1 \frac{d\Delta_1}{d\tau} &= -(\zeta_0(\tau) - \mu)a_1 + a_1^3 - f \cos \Delta_1. \end{aligned}$$

It was shown [Kovaleva and Manevich, 2013a] that in the adiabatic system $(\beta << 1)$ the amplitude of the forced oscillator within the first cycle of oscillations is close to the amplitude of an identical oscillator excited by the same force with constant frequency $(\beta = 0)$. Thus, the first step towards analyzing AR is the study of the transition from small to large oscillations in the time-invariant analog of (13) described by the following equations:

$$\begin{aligned} \frac{da_1}{d\tau} &= -f \sin \Delta, \qquad (14)\\ a_1 \frac{d\Delta_1}{d\tau} &= -(1 - \mu)a_1 + a_1^3 - f \cos \Delta_1 \end{aligned}$$

with initial conditions $a_1(0) = 0, \Delta_1(0) = -\pi/2$ corresponding to the LPTs of the oscillator (14). It was proved [Manevitch, Kovaleva, and Shepelev, 2011] that the parametric thresholds that determine the boundaries between small and large oscillations are expressed as

$$(a) f_{1\mu} = f_1 \sqrt{1 - \mu^3}, (b) f_{2\mu} = f_2 \sqrt{1 - \mu^3}, \quad (15)$$

where $f_1 = \sqrt{2/27}, f_2 = 2/\sqrt{27}$. At $f < f_{1\mu}$, the LPT represents an outer boundary for a set of closed trajectories encircling the stable center on the axis $\Delta_1 = -\pi$, while at $f > f_{1\mu}$ the LPT depicts an outer boundary for the trajectories encircling the stable center on the axis $\Delta_1 = 0$. It was proved [Manevitch, Kovaleva, and Shepelev, 2011] that the transition from small (non-resonant) to large (resonant) oscillations for the oscillator being initially at rest occurs due to the loss of stability of the LPT of small oscillations at $f = f_{1\mu}$. At $f = f_{2\mu}$, the stable center at $\Delta_1 = -\pi$ vanishes, and only a single stable center corresponding to nonlinear resonance remains on the axis $\Delta_1 = 0$.

We note that, if the actuator is captured into resonance, resonant oscillations in the attachment can occur if the coupling stiffness is strong enough to transfer the required amount of energy. This implies that the coupling parameter μ cannot be negligibly small. It was shown that in the time-invariant analogue of (8) the admissible parameter $\mu = 0.189$ for all attached oscillators from $r = 2$ to $r = n - 1$ but $\mu = 0.25$ for $r = 1$ and $r = n$ [Kovaleva, 2016b]. This implies that in the adiabatic case an admissible parametric domain for a multi-particle chain is determined by the same conditions as for a pair of oscillators [Kovaleva, 2014b], namely,

$$f > f_{1\mu} = \sqrt{2(1 - \mu)^3/27}; \mu > \mu_{cr} = 0.25. \quad (16)$$

Parametric thresholds (16) are shown in Fig. 1.

It follows from (16) that if the parameters f, μ lie within the domain D_0, then the entire chain is captured into resonance; oscillators with the parameters from the domain D_1 below $f_{1\mu}$ execute small quasi-linear oscillations; if the parameters belong to the domain D, then the dynamics of the chain should be investigated separately (an example is discussed in Sec. 5). It is important to note that these conditions have been obtained approximately for the time-invariant system. However, numerical examples in Section 5 prove that slow variations of the forcing frequency weakly affect the conditions of the emergence of AR in a multi-particle chain.

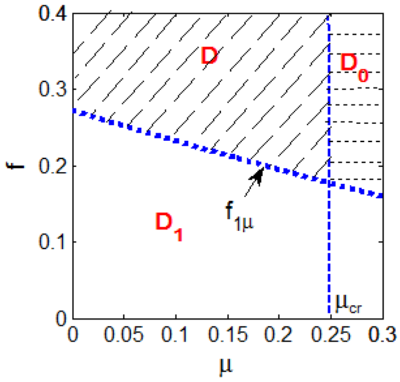

Figure 1. Parametric thresholds (16).

It is important to note that the emergence of AR also depends on the critical detuning rate β^*, at which the transition from bounded to unbounded oscillations occurs. It was shown [Kovaleva and Manevitch 2013a] that AR can occur if $\beta < \beta^* << 1$. Note that the analytical approximation of the critical rate β^* is a rather cumbersome task [Kovaleva and Manevitch 2013a]. In this paper the derivation of the threshold parameter β^* is omitted but the numerical results presented in Sec. 5 have been obtained for the oscillators subjected to forcing with adiabatically increasing frequency satisfying the required condition.

5 Numerical Results

In this section we illustrate the effect of structural and excitation parameters on the formation of stable AR in the chain. It was shown [Kovaleva, 2016b] that a decrease of the coupling stiffness μ results in escape from resonance for the chain excited by an external force with constant frequency. Figure 2 confirms a similar effect for the 4-particle chain excited by an external force with slowly-varying frequency. The excitations with amplitude $f = 0.4$ and detuning rate $\beta = 0.01$ is considered. Figures 2(a) and 2(b) show that in the chain with the stiffness parameter $\mu = 0.2 < \mu_{cr}$ the energy is localized in the excited oscillator but the attachment exhibits small non-resonant oscillations. Further increase of the coefficient μ enhances the coupling response thus resulting in the occurrence of AR in the entire chain (Fig. 2(c)). Note that both sets of the parameters ($f = 0.4$, $\mu = 0.2$ and $f = 0.4$, $\mu = 0.23$) belong to the domain D (Fig. 1) and the behavior of the chain cannot be predicted beforehand but an approach of the parameter $\mu < \mu_{cr}$ to μ_{cr} increases the probability of the emergence of AR.

Figure 3 illustrates the dynamics of the 4-particle chain with parameters $\mu = 0.25, \beta = 0.005$ but with different amplitudes of excitation.

It follows from (16) that $f_{1\mu} = 0.176$ at $\mu = 0.25$. Figure 3(a) depicts small oscillations of the chain at $f = 0.17 < f_{1\mu}$. An increase of the forcing amplitude entails energy localization in the excited oscillator

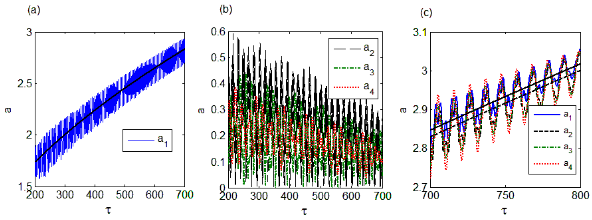

Figure 2. Amplitudes of oscillations in the 4-particle chain: (a) AR in the actuator and (b) small oscillations of the attached oscillators at $\mu = 0.2$; (b) AR in all oscillators at $\mu = 0.23$. The solid and dashed bold lines depict backbones (11) for the actuator and the attached oscillators, respectively.

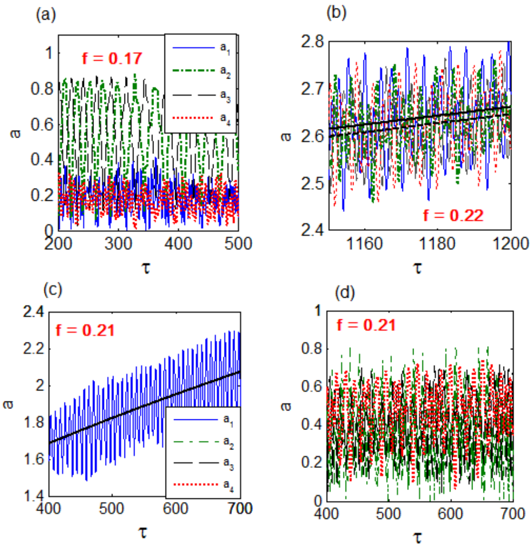

Figure 3. Amplitudes of oscillations in the 4-particle chain: (a) small oscillations at $f = 0.17 < f_{1\mu}$; (b) AR at $f = 0.22 > f_{1\mu}$; (c) energy localization in the excited oscillator and (d) small oscillations of the attachment at $f = 0.21 > f_{1\mu}$. The solid and dashed bold lines depict backbones (11) for the actuator and all attached oscillators, respectively.

against small oscillations of the attachment at $f = 0.21$ (Figs. 3(c), 3(d)) and capture into resonance of the entire chain at $f = 0.22$ (Fig. 3(b)). The initial segments of chaotic motion and transitions to regular oscillations are excluded from consideration

The results of numerical simulations for the eight-particle chain subjected to forcing with parameters $f = 0.25$, $\beta = 0.001$ are shown in Figs. 4, 5. For brevity, only the dynamics of the actuator and the last (eight) oscillator is illustrated.

Figure 4 indicates that an increase of the coupling strength μ leads to the transformations of small oscillations (Fig. 4(a)) into large oscillations of the actuator (Fig. 4(b)) and then into AR in the entire chain (Fig. 4(c))

Figure 5 illustrates the transitions from small oscillations to AR with an increase of the forcing amplitude f

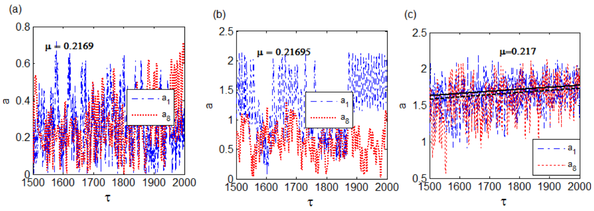

Figure 4. Amplitudes $a_1(\tau)$ and $a_8(\tau)$ in the 8-particle chain subjected to forcing with parameters $f = 0.25$, $\beta = 0.001$: (a) small oscillations of the entire chain at $\mu = 0.2169$; (b) energy localization in the excited oscillator against small oscillations of the attachment at $\mu = 0.21695$; (c) AR in the entire chain at $\mu = 0.217$. The solid and dashed bold lines in plot (c) depict backbones (11) for the actuator and the attached oscillator, respectively.

in the chain with parameters $\mu = 0.25$, $\beta = 0.001$.

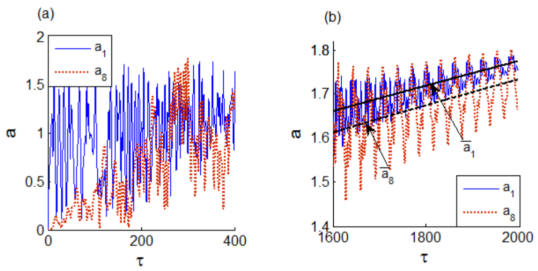

Figure 5. The dependence of the response amplitudes $a_1(\tau)$ and $a_8(\tau)$ from the forcing amplitude f in the 8-particle chain with parameters $\mu = 0.25$, $\beta = 0.001$: (a) small oscillations of the entire chain at $f = 0.12$; (b) AR in the entire chain at $f = 0.18$. The solid and dashed bold lines depict backbones (11) for the actuator and the attached oscillators, respectively.

6 Conclusions

In this work, the emergence of autoresonance (AR) in the Klein-Gordon chain of finite length has been studied. The chain is excited by periodic forcing with a slowly varying frequency applied at one edge of the chain. This work has shown that, in contrast to the earlier investigated arrays with a linear attachment, the nonlinear chain can be entirely captured into resonance. Furthermore, in the main approximations the amplitudes of AR for all oscillators converge to the equal monotonically growing quasi-steady amplitudes. The threshold values of the structural and excitation parameters, which allow the emergence of AR in the entire chain, are determined. Close agreement between the analytical and numerical results has been demonstrated.

Acknowledgements

Partial financial support for this work received from the Russian Foundation for Basic Research (grants 14-01-00284, 16-02-00400) is acknowledged with thanks.

References

Andronov, A.A., Vitt, A.A., and Khaikin, S.E. (1966). Theory of Oscillators. Pergamon Press, Oxford (In Russian: 1st ed. - 1937, 2nd ed. - 1959).

Arnold, V.I., Kozlov, V.V., and Neishtadt, A.I. (2006). Mathematical Aspects of Classical and Celestial Mechanics. Springer, Berlin.

Astashev, V.K. and Babitsky, V.I. (2007). Ultrasonic Processes and Machines. Springer, Berlin.

Balachandran, B., Perkins, E., and Fitzgerald, T. (2015). Response localization in micro-scale oscillator arrays: influence of cubic coupling nonlinearities. Int. J. Dynam. Control 3(2), 183–188.

Blekhman, I. (2012). Oscillatory strobodynamics — a new area in nonlinear oscillations theory, nonlinear dynamics and cybernetical physics. Cybernetics and Physics 1(1), 5-10.

Chapman, T., Hüller, S., Masson-Laborde, P. E., Rozmus, W., and Pesme, D. (2010). Spatially autoresonant stimulated Raman scattering in inhomogeneous plasmas in the kinetic regime. Phys. Plasmas 17(12), 122317 (1-8).

Dauxois, T., Litvak-Hinenzon, A., MacKay, R., and Spanoudaki, A. (2004). Energy Localization and Transfer, World Scientific, Singapore.

Galow, B.J., Li, J.X., Salamin, Y.I., Harman, Z., and Keitel, C.H. (2013). High-quality multi-GeV electron bunches via cyclotron autoresonance. Phys. Rev. STAB, 16(8), 081302(1–6).

Klughertz, G., Hervieux, P.-A., and Manfredi, G. (2014). Autoresonant control of the magnetization switching in single-domain nanoparticles. J. Phys. D: Appl. Phys. 47, 345004 (1-9).

Kovaleva, A. (2014a). Autoresonance in a pair of coupled oscillators. Cybernetics and Physics 3(4), 166–173.

Kovaleva, A. (2014b). Capture into resonance of coupled Duffing oscillators. Phys. Rev. E 92, 022909 (1-7).

Kovaleva, A. (2016a). Response enhancement in an oscillator chain, Commun. Nonlinear Sci. Numer. Simulat. 30, 373–386.

Kovaleva, A. (2016b). Energy localization in weakly dissipative resonant chains. Phys. Rev. E 94, 022208 (1-9).

Kovaleva, A., and Manevitch, L.I. (2013a). Limiting phase trajectories and emergence of autoresonance in nonlinear oscillators. Phys. Rev. E, 88(2), 02490-1 - 02490-6.

Kovaleva, A., and Manevitch, L.I. (2013b). Emergence and stability of autoresonance in nonlinear oscillators. Cybernetics and Physics, 2(1), 25–30.

Kovaleva, A., and Manevitch, L.I. (2016). Autoresonance versus localization in weakly coupled oscillators, Physica D 320, 1-8.

Livingston, M.S. (1954). High-Energy Particle Accelerators, Interscience, New York.

Manevitch, L.I, Kovaleva, A., and Shepelev, D. (2011). Non-smooth approximations of the limiting phase trajectories for the Duffing oscillator near 1:1 resonance. Physica D 240 (1), 1-12.

May, V., and Kühn, O. (2011). Charge and Energy Transfer Dynamics in Molecular Systems (2011) Wiley-VCH, Weinheim.

McMillan, E.M. (1945). The synchrotron—a proposed high energy particle accelerator. Phys. Rev., 68(5-6), 143–144.

Milant'ev, V. P. (2013). Cyclotron autoresonance — 50 years since discovery. Phys. Usp. 56(8), 823–832.

Rax, J.-M. , Robiche, J., and Fisch, N.J. (2007). Autoresonant ion cyclotron isotope separation, Phys. Plasmas 14, 043102.

Sanders, J.A., Verhulst, F., and Murdock, J. (2007). Averaging Methods in Nonlinear Dynamical Systems, Springer, New York.

Vazquez, L., MacKay, R., and Zorzano, M.P. (2003). Localization and Energy Transfer in Nonlinear Systems. World Scientific, Singapore.

Veksler, V.I. (1944). Some new methods of acceleration of relativistic particles. Comptes Rendus (Dokaldy) de l'Academie Sciences de l'URSS, 43(8), 329-331.