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ENUMERATION OF FEEDBACK EQUIVALENCE CLASSES OF LINEAR CONTROL SYSTEMS OVER A COMMUTATIVE RING VS. PARTITIONS OF ELEMENTS OF A MONOID

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Abstract

The problem of finding all feedback equivalence classes of Brunovsky and locally Brunovsky linear systems defined on a commutative ring is related with combinatorial problem of visiting all partitions of elements in a concrete monoid.

Key words

Feedback classification; Projective module; Enumeration; K-theory

1 Introduction

The theory of linear control systems over a commutative ring R goes back to the models of [Morse, 1976] for delay systems. The main example in our study will be the ring of continuous real functions $R = \mathcal{C}(K)$ defined on a compact topological space K (which was introduced in the control theory framework in [Bumby, Sontag, Sussmann and Vasconcelos, 1981] as model for studying parametrized families of systems). Rings of continuous functions also apply to the geometric study of differential deformations of linear systems (see [O'Halloran, 1987] or [Ferrer, García-Planas and Puerta, 1997]).

This paper deals with the feedback classification problem for linear systems over a commutative ring. To be concise, we are interested in the enumeration of all feedback classes of reachable linear systems. For general reading on the subject we refer to [Brunovsky, 1970], [Brewer, Bunce and VanVleck, 1986], [Hermida-Alonso, Pérez and Sánchez-Giralda, 1996], [Carriegos, 2003] and references therein.

Geometric properties of commutative ring R are crucial: In fact we prove that the problem of enumerating all feedback classes of reachable linear systems over R^n is equivalent to the problem of enumerating all direct-sum decompositions of R^n as an element of monoid $\mathbf{P}(R)$ of finitely generated projective R-modules. Moreover, note that in the particular case of $R = \mathcal{C}(K)$ being the ring of real continuous functions defined on a compact topological space K then monoid $(\mathbf{P}(R), \oplus)$ is equivalent to monoid of finite dimensional real vector bundles over K (see [Swan, 1962]).

The description of monoid $\mathbf{P}(R)$ is in general very difficult. To avoid this, we also provide a new *stable* feedback equivalence relation and conjecture that this new relation may be studied by working with the Grothendieck's group completion $K_0(R)$ of monoid $\mathbf{P}(R)$.

Organization of the paper

In section 2 we review some well known facts about linear systems over a commutative ring, feedback equivalence. Main invariants are also revised. Section 3 is devoted to study Brunovsky case: This is the case of constant linear systems over a field, and it is solved by studying classical partitions of an integer. Section 4 deals with the more general case of locally Brunovsky linear systems: This is the case of projective invariants or, if $R = \mathcal{C}(K)$, the case of smooth invariants. We solve the problem by studying direct-sum decompositions of elements of monoid $\mathbf{P}(R)$. The cases of real circle and real sphere are specially focused. Section 5 is devoted to describe a dynamic study of feedback equivalence related to the Grothendieck's group completion $K_0(R)$ of monoid $\mathbf{P}(R)$. Finally we include a conclusion.

2 Preliminaries

Let R be a commutative ring with identity element $1 \neq 0$.

2.1. *A linear system over R is given by a linear rule (or right hand side) on the form $x^+ = Ax + Bu$ where $x \in X$ are states, $u \in U$ are inputs, and x^+ is the time-derivative or time-shift in the sequential case.*

Sets of states X and of inputs U are R-modules while maps A and B are R-linear maps.

$$\Sigma : \boxed{\begin{array}{ll} U & \\ \quad \searrow^{B} & \\ & X \to^{A} X \end{array}} \tag{1}$$

2.2. *Above linear system Σ and linear system*

$$\Sigma' : \boxed{\begin{array}{ll} U' & \\ \quad \searrow^{B'} & \\ & X' \to^{A'} X' \end{array}} \tag{2}$$

are said to be Feedback Equivalent *if one can bring one of them into the another by a finite composition of the following* Basic Feedback Actions*:*

1. *Isomorphisms $Q : U \to U'$ in the input module which transforms $(A, B) \to (A, BQ)$*
2. *Isomorphisms $P : X \to X'$ in the state module which transforms $(A, B) \to (PAP^{-1}, PB)$*
3. *Feedback actions $F : X \to U$ which transforms $(A, B) \to (A + BF, B)$*

Consequently a general feedback action (P, Q, F) brings linear system $\Sigma = (A, B)$ to system

$$(P(A + BF)P^{-1}, PBQ). \tag{3}$$

2.3. *Partial reachability linear map given by*

$$\varphi_i^{\Sigma} = \begin{pmatrix} B & AB & \cdots & A^{i-1}B \end{pmatrix} : U^{\oplus i} \longrightarrow X \tag{4}$$

is a feedback invariant, up to equivalence, associated to Σ (see [Hermida-Alonso, Pérez and Sánchez-Giralda, 1996] and [Carriegos, 2003]).

Consequently we obtain our main set of feedback invariants:

2.4. *Quotient modules*

$$\begin{gathered} N_{i+1}^{\Sigma}/N_i^{\Sigma} = \\ \mathit{Im}(B, AB, ..., A^iB)/\mathit{Im}(B, AB, ..., A^{i-1}B) \end{gathered} \tag{5}$$

are feedback invariants, up to isomorphism, associated to system Σ.

3 n-dimensional Brunovsky systems and partitions of integer n

3.1. *A linear system $\Sigma = (A, B)$ is a Brunovsky system if it is equivalent to a Brunovsky canonical form.*

In the case of $R = \mathbb{K}$ being a field, a Brunovsky linear system is just a reachable linear system. Then one has the following result.

3.2. *The problem of enumerating all feedback classes of reachable linear systems over $\mathbb{K}^n$ is equivalent to the classical problem of enumerating all partitions of integer n.*

Consequently, classical enumeration algorithms [Knuth, 2004] may be directly translated from partitions of a given integer n to reachable linear systems over $\mathbb{K}^n$.

The problem can also be attacked in the case of linear systems such that all its invariants are free defined over commutative rings R such that finitely generated projective R-modules are free:

3.3. *Assume that commutative ring R is projectively trivial (i.e. all projective R-modules are free) then linear system $\Sigma = (A, B)$ over R^n is equivalent to a Brunovsky canonical form if and only if all invariant R-modules $N_{i+1}^{\Sigma}/N_i^{\Sigma}$ are free.*

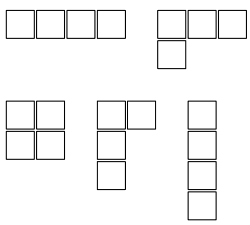
Thus the problem of enumerating all feedback classes of reachable linear systems with free invariants over a projectively trivial ring is actually equivalent to the problem of enumerating all Brunovsky canonical forms and thus equivalent to the problem of enumerating all partitions of the integer n.

3.4. *The key is that, in the case of reachable linear systems over a field or, in the more general framework of projective-free rings, if all the R-modules $N_{i+1}^{\Sigma}/N_i^{\Sigma}$ are free then they are really a complete set of invariants verifying that*

$$X = N_1^{\Sigma} \oplus (N_2^{\Sigma}/N_1^{\Sigma}) \oplus \cdots \oplus (N_s^{\Sigma}/N_{s-1}^{\Sigma}) \tag{6}$$

are in one-to-one correspondence with the set of partitions of integer n in decreasing sequences, or, equivalently or by all the Ferrers diagrams of integer n.

For example, if we set $n = 4$ we have the following Ferrers diagrams visited following the reverse lexicographical order:



or, equivalently the following partitions of $4 \in \mathbb{N}$:

$$\begin{gathered} 4 = 4, \quad 4 = 3 + 1, \quad 4 = 2 + 2, \\ 4 = 2 + 1 + 1, \quad 4 = 1 + 1 + 1 + 1. \end{gathered} \tag{7}$$

Consequently, the description of all types of Brunovsky linear systems over a commutative ring R does not depend on the ring R but on the dimension n

of free state module X. In fact, there are exactly $p(n)$ Brunovsky linear systems over a free module $X \cong R^n$, where $p(n)$ is the number of partitions of integer n.

4 Locally Brunovsky systems and partitions of an element of a monoid

Let Σ be a reachable linear system over R^n.

4.1. *If invariant modules $N_{i+1}^{\Sigma}/N_{i+1}^{\Sigma}$ are projective then system Σ is locally Burnovsky (see [Carriegos, 2003]).*

4.2. *The feedback classification problem for locally Brunovsky linear systems over R^n (i.e. the case of projective invariants) is equivalent to the problem of characterization of all possible decompositions of finitely generated R-modules U and X on the form*

$$\begin{array}{rl} U = & Q \oplus P_1 \\ X = & P_1 \oplus P_2 \oplus \cdots \oplus P_s \end{array} \tag{8}$$

With the only restriction to solve the system of equations is that P_{i+1} must be a direct summand of P_i for all i.

On the other hand if we are not worried about he generators of input space or, in other words, we allow ancillary blank inputs, the following result applies:

Theorem 4.3. *Enumerating all feedback equivalence classes of reachable linear systems over X allowing ancillary inputs is equivalent to enumerating all decompositions*

$$X = P_1 \oplus P_2 \oplus \cdots \oplus P_s \tag{9}$$

where P_{i+1} is a direct summand of P_i.

4.4. *Let's denote by $p_R(n)$ the number of non-isomorphic decompositions of R^n while $\tilde{p}_R(n)$ denotes the number of non-isomorphic decompositions $R^n \cong P_1 \oplus \cdots \oplus P_s$ where P_{i+1} is a direct summand of P_i.*

Note that if R is projectively trivial then $\tilde{p}_R(n) = p_R(n) = p(n)$ is the number of partitions of integer n but in general $\tilde{p}_R(n) \leq p_R(n)$.

Anyway, one needs to in order to know exactly the monoid $(\mathbf{P}(R), \oplus)$ of isomorphism classes of finitely generated R-modules in order to give the complete classification of locally Brunovsky linear systems.

The full description of the monoid $(\mathbf{P}(R), \oplus)$ is a great task. Of course if finitely generated projectives are free then $(\mathbf{P}(R), \oplus)$ is isomorphic to $(\mathbb{N} \cup \{0\}, +)$ but in general this is not the case.

If $R = \mathcal{C}(K)$ is the ring of continuous functions defined on a compact topological space K then $(\mathbf{P}(R), \oplus)$ depends on the topology of K. For instance if $K = \mathbb{S}^1$ is the real unit circumference then $(\mathbf{P}(R = \mathcal{C}(\mathbb{S}^1)), \oplus)$ is the commutative monoid generated by the symbols R (trivial vector bundle) and P (Möbius Strip) modulo the relation $P \oplus P = R \oplus R = R^2$ (see [Rosenberg, 1994]), in other words, if $K = \mathbb{S}^1$ is the above example of real unit circumference then $(\mathbf{P}(\mathcal{C}(\mathbb{S}^1)), \oplus)$ is the commutative monoid given, in terms of generators and relations, by $\langle R, P : P^2 = R^2 \rangle$.

Thus a feedback class of a n-dimensional locally Brunovsky linear system over the real unit circumference is determined by a partition of element $R^n \in \langle R, P : P^2 = R^2 \rangle$. Allowed partitions for $n = 3$ (in the sense of Theorem 4.3 are the following:

$$\begin{array}{rl} R^3 \cong & R^3 \\ R^3 \cong & R^2 \oplus R \\ R^3 \cong & (R \oplus P) \oplus P \\ R^3 \cong & R \oplus R \oplus R \end{array} \tag{10}$$

and hence $\tilde{p}_{\mathbb{S}^1}(n) = 4$ (though $p_{\mathbb{S}^1}(n) = 5$).

Example 4.5. *We can compute how many feedback classes are there in terms of R and n. As matter of example we next compute the cases (R a projectively trivial ring, R being the ring of continuous functions defined over the real unit circle and R being the ring of real continuous functions defined over the real unit sphere:*

n	$p_{\mathbb{N}}(n)$	$\tilde{p}_{\mathbb{S}^1}(n)$	$\tilde{p}_{\mathbb{S}^2}(n)$
1	1	1	1
2	2	3	2
3	3	4	3
4	5	9	∞
5	7	11	∞
6	11	24	∞
$\vdots$	$\vdots$	$\vdots$	$\vdots$

(11)

Above computations have been performed as follows: For the case of partitions of integer $p_{\mathbb{N}}(n)$ we have the usual Euler theory (see [Knuth, 2004]).

If $R = \mathbb{S}^1_{\mathbb{R}} = \mathbb{R}[\sin\theta, \cos\theta]$ then: $\mathbf{P}(R) = \langle a, b \rangle / \{ab = ba, a^2 = b^2\}$, the calculation of $p_{\mathbb{S}^1}(n)$ can be performed by using "colored" Ferrers' Diagrams [Carriegos, 2007]. In particular, it is not hard to prove that $p_{\mathbb{S}^1}(n) < \infty$, but it may be interesting to evaluate the asymptotic behavior $\mathcal{O}(p_{\mathbb{S}^1}(n))$ in terms of the dimension n of state-space.

Finally if $R = \mathbb{S}^2_{\mathbb{R}} = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ then $\mathbf{P}(R)$ is the following monoid [Rosenberg, 1994]:

$$\begin{array}{c} (\mathbf{P}(R), \oplus) = \\ (\{(0,0), (1,0), (2,\alpha), (n,\beta) : \alpha \in \mathbb{Z}, \beta \in \mathbb{Z}_2\}, ✠) \end{array} \tag{12}$$

where ✠ operates as follows: If $a + c \leq 2$ then

$$(a, b) ✠ (c, d) = (a + c, b + d) \tag{13}$$

while if $a+c \geq 3$ one has

$$(a,b)\maltese(c,d) = (a+c, b+d(\mathrm{mod}\ 2)) \tag{14}$$

Thus $p_{\mathbb{S}^2}(n)$ can be directly computed for $n \leq 3$. To check that $p_{\mathbb{S}^2}(n) = \infty$ for $n \geq 4$ only note that for $n \geq 2$ and all j we have

$$(2n,0) = (2,2j)\maltese \cdots \maltese(2,2j) \tag{15}$$

and

$$(2n+1,0) = (3,0)\maltese(2,2j)\maltese \cdots \maltese(2,2j) \tag{16}$$

4.6. *Note that in the general case a recursive procedure calculating $P_M(n)$ and in particular $\tilde{P}_M(n)$ is needed.*

5 Future work: Dynamic study and $K_0(R)$

The usual dynamic study of a system (A,B) (e.g. dynamic stabilization, see [Brewer, Bunce and VanVleck, 1986], [Hermida-Alonso, López-Cabeceira and Trobajo, 2005], [Hermida-Alonso and López-Cabeceira, 2006] or [Hermida-Alonso and Trobajo, 2003]) allows to introduce ancillary variables an the augmented system

$$(\widehat{A},\widehat{B}) = (\mathbf{0},\mathbf{1}) \oplus (A,B) = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix}, \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \tag{17}$$

A generalization of dynamic study is the following:

Definition 5.1. *We say that system (A,B) is stably equivalent to system (A',B') if there exists a Brunovsky system (I,J) such that*

$$(I,J) \oplus (A,B) = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix} \begin{pmatrix} J & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} \tag{18}$$

are equivalent

The following result can be obtained by direct calculations:

Theorem 5.2. *Feedback invariants of augmented system splits:*

$$\frac{N_{i+1}^{(I,J)\oplus(A,B)}}{N_i^{(I,J)\oplus(A,B)}} \cong \frac{N_{i+1}^{(I,J)}}{N_i^{(I,J)}} \oplus \frac{N_{i+1}^{(A,B)}}{N_i^{(A,B)}} \tag{19}$$

5.3. *As consequence of above result we conjecture that two locally Brunovsky linear systems are stably equivalent if and only if their feedback invariants lie in the same class in the group $K_0(R)$.*

Therefore feedback equivalence would be related to the study of some kind of partitions in $\mathbf{P}(R)$ while stable feedback equivalence deals with the study of some kind of partitions in $K_0(R)$.

Groups $K_0(R)$ are described for the case of spheres of any dimension both in the real, complex and quaternion cases (see [Rosenberg, 1994], [Weibel, 2009]). Hence in order to describe all the stable feedback classes for the case of systems over the ring of continuous functions over real or complex spheres we only need to compute partitions in the following K_0-groups depending on the dimension n of the sphere:

Real case $R = \mathcal{C}(\mathbb{S}^n_{\mathbb{R}})$:

$n(\mathrm{mod}\ 8)$	$K_0(R)$
1	$\mathbb{Z} \oplus \mathbb{Z}_2$
2	$\mathbb{Z} \oplus \mathbb{Z}_2$
3	$\mathbb{Z}$
4	$\mathbb{Z} \oplus \mathbb{Z}$
5	$\mathbb{Z}$
6	$\mathbb{Z}$
7	$\mathbb{Z}$
8	$\mathbb{Z} \oplus \mathbb{Z}$

(20)

Complex case $R = \mathcal{C}(\mathbb{S}^n_{\mathbb{C}})$:

$n(\mathrm{mod}\ 2)$	$K_0(R)$
1	$\mathbb{Z}$
2	$\mathbb{Z} \oplus \mathbb{Z}$

(21)

6 Conclusion

The problem of enumerating all feedback classes of reachable linear systems over R^n where R is a commutative ring is related to the problem of enumerating all possible direct-sum decompositions of R^n into $\mathbf{P}(R)$. Thus we need to study partitions of elements in a commutative monoid.

It is conjectured that the problem may be simplified by introducing a new *stable* feedback equivalence which carries the problem to studying partitions in the K-theory group $K_0(R)$.

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References

Brewer, J. W., Bunce, J. W., VanVleck, F. S. (1986) *Linear systems over commutative rings*. Marcel Dekker. New York.

Brunovsky, P. A. (1970) A classification of linear controllable systems. *Kybernetika*, **3**, pp. 173–187.

Bumby, R.,Sontag, E. D., Sussmann, H. J., Vasconcelos, W. V. (1981) Remarks on the pole-shifting problem over rings. *Journal of Pure and Applied Algebra*, **20**, pp. 113–127.

Carriegos, M. V. (2003) On the local-global decomposition of linear control systems. *Communications in Nonlinear Science and Numerical Simmulation* , **9**, pp. 149–156.

Carriegos, M. V. (2007) A study of control systems from the geometry of ring of scalars. *IPACS Electronic Library*, http://lib.physcon.ru

Carriegos, M. V., Sánchez-Giralda, T. (2001). Canonical forms for linear dynamical systems over commutative rings: The local case. In *Ring Theory and Algebraic Geometry.* Marcel Deker. New York. pp. 113–131.

Ferrer, J., García-Planas,Mª . I., Puerta, F. (1994) Differentiable families of subspaces. *Linear Algebra and Its Applications*, **199**, pp. 229–252.

Ferrer, J., García-Planas,Mª .I., Puerta, F. (1997) Brunowsky local form of a holomorphic family of pairs of matrices. *Linear Algebra and Its Applications*, **253**, pp. 175–198.

Hermida-Alonso, J. A., López-Cabeceira M. M., Trobajo M. T. (2005) When are dynamic and static feedback equivalent?. *Linear Algebra and its Applications*, **405**, pp. 74–82.

Hermida-Alonso, J. A., López-Cabeceira M. M. (2006) Dynamic feedback over principal ideal domains and quotient rings. *Linear Algebra and its Applications*, **413**, pp. 235–244.

Hermida-Alonso, J. A., Trobajo M. T. (2003) The dynamic feedback equivalence over principal ideal domains. *Linear Algebra and its Applications*, **368**, pp. 197–208.

Brunovsky's canonical form for linear dynamical systems over commutative rings. *Linear Algebra and Its Applications*, **233**, pp. 131–147.

Husemoller, D. (1994) *Fibre Bundles* Springer-Verlag. New York.

Knuth, D. E. (2004) *The art of computer programming. Prefascicle 3B: Sections 7.2.1.4-5 "Genetrating all partitions"*. Addison-Wesley.

McDonald, B. R. (1984) *Linear Algebra over Commutative Rings*. Marcel Dekker.

Milnor, J. (1971) *Introduction to algebraic K-theory*. Princeton University Press.

Morse, A. S. (1976) Ring models for delay-differential systems. *Automatica*, **12**, pp. 529–531.

O'Halloran, J. (1987) Feedback equivalence of constant linear systems. *Systems & Control Letters*, **8**, pp. 241–246.

Rosenberg, J. (1994) *Algebraic K-theory and its applications*. Springer.

Rosenthal, J., York, E. V. (1999) BCH convolutional codes. *IEEE Transactions on Information Theory* **45(6.1)**, pp. 1833–1844.

Swan, R. G. (1962) Vector Bundles and Projective Modules *Transactions of The Amererican Mathematical Society*, **105(2)**, pp. 264–277.

Weibel, C. A. *An Introduction to algebraic K-Theory*. e-book in progress. http://www.math.rutgers.edu/weibel