STABILITY ANALYSIS OF THE LINEAR TIME DELAY SYSTEMS WITH LINEARLY INCREASING DELAY

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Abstract
The linear differential-difference systems with constant coefficients and linearly increasing time delay are considered. A double stage approach is applied for stability analysis of these systems. As a main result the sufficient conditions of the asymptotic stability and instability are obtained. The results are illustrated on the example of multi-agency systems.

Key words
Differential-difference system, linearly increasing delay, stability, multi-agency system.

1 Introduction

The differential-difference systems with proportional time-delay are worse studied than the same systems with constant delay. However the time-delay is not always constant. For example a mathematical model of the dynamic of the information server [Zhabko and Chizhova, 2015b] takes account the time required for information processing or the technological time-delay. This delay is proportional to the volume of information being processed. The same time-delay is in the model considered in [Zhabko and Chizhova, 2015a]. An increasing transport time-delay occurs in the mathematical model of the information exchange between objects receding from each other. It should be noted that this time-delay is unbounded and well known approaches are not applicable for stability analysis such systems.

The method of the power series was suggested for constructing of the system solutions [Valeev, 1964; Valeev, 1967]. Some sufficient conditions of the asymptotic stability for the high order equations with linearly increasing time-delay were obtained there. Some types of such equations and systems were investigated in [Grebenshchikov, 1983; Grebenshchikov, 1986; Grebenshchikov and Novikov, 2010]. The estimates of solutions were derived and sufficient conditions of the asymptotic stability and instability were obtained. The asymptotical behavior of the solutions of the system with several linearly increasing time-delays was studied in [Laktionov and Zhabko, 1998; Zhabko and Laktionov, 1997]. Some algebraic conditions of asymptotic stability and instability of the systems was also given there. Based on the linear approximation stability criteria are true [Krasovskij, 1955; Bellman and Kuk, 1967; Zubov, 1973] for differential-difference systems with bounded time-delay. But considered in the paper systems are not time-invariant so Laplas transformations are not applicable in this case.

The main goal of this paper is the application of the new approach to the study of stability of the systems of linear equations with constant coefficients and linearly increasing time-delay. The analysis of stability such system in the coefficient space was given in [Valeev, 1964; Valeev, 1967; Laktionov and Zhabko, 1998; Zhabko and Laktionov, 1997]. However auxiliary transformation is used in this paper. The application of a double-stage approach allows reducing the problem of stability analysis by Lyapunov to the serial approach Razumihin for the auxiliary system of differential-difference equations and the stability analysis of difference systems. This approach allows improving the previous results and obtaining the estimates of the asymptotical behavior of the solutions.

The proposed method can be effective in the following cases

- the analysis of stability the systems of linear equations with constant coefficients and linearly increasing time-delay by using of Lyapunov functions [Razumikhin, 1956].
- the application of modifications of the Lyapunov-Krasovsrii approach [Kharitonov, 2005a; Kharitonov, 2005b] to the linear systems with undefined coefficients and linearly increasing time-delay.
- the application of Razumihin approach to systems of homogeneous differential-difference equations with linearly increasing delay [Alexandrov and

Zhabko, 2012; Alexandrov and Zhabko, 2013].
some multidimensional of the mathematical physic equations [Provotorov, 2014] should be considered as dynamical systems with unbounded delays.

2 A Dynamic Model of Traffic on the Ring Road

Let us consider the RR (ring road) which divided into the sections between the two nearest descents. Let the sections are numbered from 1 to N so that the first section and section number N are nearest descents. Let $x_i(t)$ and $v_i(t)$ be average traffic density and average speed on the section number i at the moment t. Let $r_i(t)$ denote average traffic density at the entrance in the beginning of the section number i and let $d_i(t)$ be average traffic density at the exit at the end of the section number $i-1$. Let the traffic on RR occurs with the maximum admissible speed $\tilde{v}_0$ if possible. So we define the following function

$$v_i = v_i(x_i) = \left.\begin{array}{l} \tilde{v}_0, \text{ if } x_i \leq \tilde{x}_0 \\ \tilde{v}(x_i), \text{ if } x_i > \tilde{x}_0. \end{array}\right\} \quad (1)$$

Here $\tilde{x}_0$ is limit value of the traffic density when the admissible speed $\tilde{v}_0$ would be reached. If traffic density is increasing then admissible speed is decreasing.

Then the function $\tilde{v}(x)$ is strictly decreasing. We denote as l_i the length of the section number i. Let the functions $x_i(t)$, $v_i(t)$, $r_i(t)$ and $d_i(t)$ are continuously differentiable. Then the balance equation for section number i may be written in the form

$$\begin{gathered} x_i(t+\Delta)l_i = x_i(t)l_i - x_i(t)v_i(x_i(t))\Delta + \\ +x_{i-1}(t)v_{i-1}(x_{i-1}(t))\Delta - d_i(t)v_{i-1}(x_{i-1}(t))\Delta + \\ +r_i(t)v_i(x_i(t))\Delta + O(\Delta^2). \end{gathered}$$

Since $x_i(t+\Delta) = x_i(t) + \Delta\dot{x}_i(t) + O(\Delta^2)$ we can divide by Δ last equality and consider the limit for $\Delta \to 0$. Then the next system of equations may be obtained

$$\begin{gathered} l_i\dot{x}_i = -x_iv_i(x_i) + x_{i-1}v_{i-1}(x_{i-1}) - \\ -d_iv_{i-1}(x_{i-1}) + r_iv_i(x_i(t)) \\ i = 1, 2, \ldots, N, \end{gathered} \quad (2)$$

where $x_0(t) \equiv x_N(t)$ and $v_0(x) \equiv v_N(x)$. The system (2) is multi-agency system [Amelina et al., (2015)] described as a system of ordinary differential equations.

Let us formulate a problem of the RR traffic control. The entrance traffic density $r_i(t)$ $(i = 1, \ldots, N)$ could be control law and the RR traffic parameters would be used as observations.

Let us consider the control function $u(t) = (r_1(t), r_2(t), \ldots, r_N(t))$ and admissible set $U = R_n^+$ which is positive subset in the space R_n. Let the performance criterion is of the form

$$V\left(u(\cdot)\right) = \min_{u(t)\in U, t\geq 0} v_i(x_i(t)).$$

Then the simple problem formulated as finding an admissible control $u(t)$ for $t \geq 0$ such that $V\left(u(\cdot)\right) \geq \tilde{v}_{zad}$.

In accordance with the physical essence of the model we can say that $\tilde{v}_{zad} \leq \tilde{v}_0$ (sec.(1)). Then the problem will be solved if $v_i(x_i(t)) \geq \tilde{v}_{zad}$ for $i = 1, 2, \ldots, N$ and $t \geq 0$. Without loss of generality we can assume that $\tilde{v}_{zad} = \tilde{v}_0$.

Let we know the values $x_i(t)$ for $i = 1, 2, \ldots, N$. We will look for a control $u(t)$ in the form

$$u_i(t) = \sum_{\nu=0}^{N-1} \alpha_{i\nu}x_{i+\nu}(t - h_{i\nu}), i = 1, 2, \ldots, N \quad (3)$$

Here $x_{i+\nu} = x_{i+\nu-N}$ if $i+\nu > N$. It should be noted that $\alpha_{i\nu} \geq 0$, $\alpha_{i\nu} \neq \alpha_{\nu i}$, $h_{i\nu} \neq h_{\nu i}$, for $i \neq \nu$ and $0 = h_{i0} < h_{i1} < \ldots < h_{iN-1}$.

The system (2), (3) is the multi-agency controllable system of differential equations with delay. The delay $h_{i\nu}$ is the function of the system state. This value shows that RR section number ν affects the section number i after the time for which a car moves from section number ν to section number i.

Let the function $\tilde{v}(x)$ is defined as

$$\tilde{v}(x) = \frac{\tilde{v}_0\tilde{x}_0}{\tilde{x}_0 + a(x - \tilde{x}_0)}$$

for $x > \tilde{x}_0$ and $\dot{\bar{x}}_{i\nu}$ is average speed of traffic density increasing between the section number ν and the section number i at the moment $\bar{t}$.

Then the value $h_{i\nu}$ may be defined as

$$h_{i\nu} = l_{i\nu}\frac{x_0 + a\dot{\bar{x}}_{i\nu}(t - \bar{t})}{\tilde{v}_0\tilde{x}_0} = h_{i\nu}^0 + \gamma_{i\nu}(t - \bar{t}), \quad (4)$$

where $l_{i\nu}$ is the distance between the section number ν and the section number i, $a > 0$.

Therefore if the average traffic density is exceeded in some sections RR then traffic dynamic is described by the multi-agency system with constantly and linear increasing delay.

The problem of stability and instability of systems with limited delay was studied in the previously quoted papers . The same problem for system

$$\dot{x}(t) = Ax(t) + Bx(\gamma t), 0 < \gamma < 1$$

was solved in [Zhabko and Chizhova, 2015a].

We will show that the problem of maintaining a specified speed on RR is reduced to stability analysis of the system with linear increasing delay. Indeed, system (2) has a set of equilibrium solutions

$$\begin{gathered} x_i(t) = \overline{x}_i, v_i(x_i(t)) = \tilde{v}(\overline{x}_i), \\ r_i(t) = d_i(t) = \overline{d}_i, i = 1, 2, \ldots, N. \end{gathered}$$

These solutions can be found from the system of algebraic equations

$$\left.\begin{array}{l}(\overline{x}_i - \overline{d}_i)\cdot\tilde{v}(\overline{x}_i) = (\overline{x}_{i-1} - \overline{d}_i)\cdot\tilde{v}(\overline{x}_{i-1})\\ i = 1, 2, \ldots, N\end{array}\right\}$$

In particular, if

$$\begin{array}{c} x_i = \overline{x}_i > \tilde{x}_0, v_i = \overline{v}, r_i = d_i = \overline{d}_i,\\ \dot{\overline{x}}_{i\nu} = \dot{\overline{x}} \text{ for } i = 1, 2, \ldots, N \end{array} \quad (5)$$

then

$$h_{i\nu} = l_{i\nu}\frac{\tilde{x}_0 + a(\overline{x} - \tilde{x}_0 + \dot{\overline{x}}t)}{\tilde{v}_0\tilde{x}_0} = l_{i\nu}(h + \gamma t). \quad (6)$$

Now we introduce the new values

$$x_i = \overline{x} + y_i, r_i = \overline{d}_i + z_i, d_i = \overline{d}_i + \Delta_i, i = 1, 2, \ldots, N.$$

Then the system of linear approximation for system (2), (3), (4) in a neighborhood of the equilibrium solution (5) has a form

$$\begin{array}{c}\dot{y}_i = \frac{\overline{v}}{l_i}\left((1 - c\overline{x} + cd_i)(y_{i-1} - y_i) + z_i + \Delta_i\right),\\ i = 1, 2, \ldots, N,\end{array} \quad (7)$$

where

$$c = \frac{a}{\tilde{x}_0 + a(\overline{x} - x_0)}; \; z_i(t) = \sum_{\nu=0}^{N-1}\alpha_{i\nu}y_{i+\nu}(t - h_{i\nu}).$$

System (6), (7) is linear retarded type system with constant and linearly increasing delay.

Then the approach for analysis asymptotic stability of a system with distributed delay will be given.

Consider the linear retarded type system with linearly increasing delay

$$\dot{y}(t) = Ay(t) + \int_p^q d_\gamma G(\gamma)y(\gamma t), \quad (8)$$

where $y(t)$ is n – dimensional state vector, A is real constant $(n\times n)$ matrix, $G(\gamma)$ is real bounded variation matrix; $0 < p < q < 1$[1].

3 Main Symbols and Definitions

Let us consider system (8). We introduce continuous initial functions

$$\varphi : [pt_0; t_0] \to R^n; y(t_0) = y_0; t_0 > 0. \quad (9)$$

Obviously, zero solution of system (8) corresponds to zero initial functions $\varphi(t) \equiv 0$. It is well known [Kharitonov, 2013] that the initial data problem (8), (9) has uniqueness solution for any $\varphi : [pt_0; t_0] \to R^n$ and $t_0 > 0$, moreover any solution is defined for any $t \geq t_0 > 0$.

Introduce the vector functions $y^k(t)$ for $k = 0, 1, \ldots$ by the equalities

$$y^{k+1}(t) = Ay^k(t) + \int_p^q \gamma^k d_\gamma G(\gamma)y^k(\gamma t), \quad (10)$$

where $y^0(t) = y(t)$. These functions are defined and continuous for $t \geq p^{1-k}t_0$ any and continuously differentiable for any $t \geq p^{-k}t_0$. Moreover, for any $t \geq p^{-k}t_0$ the following equalities hold

$$\dot{y}^k(t) = Ay^k(t) + \int_p^q \gamma^k d_\gamma G(\gamma)y^k(\gamma t).$$

Remark 1. *If the vector function $y(t, t_0, \varphi)$ is the solution of the initial data problem (8), (9) then following equalities hold*

$$y^k(t) = \frac{d^k y(t, t_0, \varphi)}{dt^k} \quad (11)$$

for any $t \geq p^{1-k}t_0$.

Let us consider the system of equations

$$\dot{y}^k(t) = Ay^k(t) + \int_p^q \gamma^k d_\gamma G(\gamma)y^k(\gamma t), \quad (12)$$

for $t \geq t_0$.

Lemma. *Let vector function $y^k(t)$ be a continuous solution of system (12) for $t \geq t_0$ with continuous initial function $\varphi^k(t)$ for $t \in [pt_0; t_0]$ and*

$$\det\left(A + \int_p^q \gamma^s d_\gamma G(\gamma)\right) \neq 0$$

for $s = 0, 1, \ldots, k-1$. Then there exists a continuous function $\varphi(t)$ for $t \in [pt_0; t_0]$ such that equality (11) holds for any $t \geq p^{1-k}t_0$.

Proof. The proof of the lemma is similar to that of lemma in [Zhabko and Chizhova, 2015a].

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4 Stability Analysis

In this paragraph we will investigate Lyapunov stability of zero solution of system (8) under well-known definitions [Zubov, 1973; Kharitonov, 2013] of stability, asymptotic stability and instability of solutions of the differential-difference systems.

Remark 2. *As system (8) is linear so all solutions of the system have the same stability properties.*

Therefore we can say that system (8) as a whole is stable, asymptotically stable or instable.

Remark 3. *In general, system (8) is not uniformly asymptotically stable* [Zhabko and Chizhova, 2015a].

Theorem 1. *If all roots of equations* $\det(\lambda E - A) = 0$ *and* $\det\left(\lambda E + A^{-1}\int_p^q e^{\lambda ln\gamma} d\gamma G(\gamma)\right) = 0$ *have negative real parts then system (8) is asymptotically stable.*

Proof. We consider a positive definite symmetric matrix V as a solution of Lyapunov matrix equation

$$VA + A^T V = -E.$$

Then we choose an integer k such that inequality

$$-\|y\|^2 + 2q^k \cdot g \cdot \|y\| \cdot \|z\| \leq -\beta y^T V y$$

holds on the set $S = \{z : z^T V z \leq 2y^T V y\}$ for some $\beta > 0$. Here $g = \|V\| \cdot \int\limits_p^q \|d_\gamma G(\gamma)\|$. We introduce a notation $v(y^k) = (y^k)^T V y^k$. Then the inequality

$$\left.\frac{dv(y^k(t))}{dt}\right|_{(12)} \leq -\beta v(y^k(t))$$

holds for $y^k(\gamma t) = z \in S$. In accordance with [Razumikhin, 1956] system (12) is asymptotically stable, therefore $y^k(t) \to 0$ for $t \to +\infty$.

Let the vector function $y(t, t_0, \varphi)$ be a solution of system (8). In accordance with equality (11) we have

$$\frac{d^k y(t, t_0, \varphi)}{dt^k} = y^k(t) \to 0 \text{ for } t \to +\infty.$$

Now we will show that

$$\frac{d^{k-1} y(t, t_0, \varphi)}{dt^{k-1}} = y^{k-1}(t) \to 0 \text{ for } t \to +\infty. \quad (13)$$

Using equality (10) we have

$$y^{k-1}(t) + \int_p^q \gamma^{k-1} A^{-1} d_\gamma G(\gamma) y^{k-1}(\gamma t) = = A^{-1} y^k(t). \quad (14)$$

Let $t \in [p^{1-m}t_0; p^{-m}t_0]$ and $m \geq k$. Then the function $y^{k-1}(t)$ can be written as a sum of solution of the linear homogeneous system

$$\hat{y}^{k-1}(t) + \int\limits_p^q \gamma^{k-1} A^{-1} d_\gamma G(\gamma) \hat{y}^{k-1}(\gamma t) = 0 \quad (15)$$

with initial conditions $\hat{y}^{k-1}(t) = y^{k-1}(t)$ for $t \in [p^{2-k}t_0; p^{1-k}t_0]$ and the solution of inhomogeneous system (14) with zero initial conditions.

System (15) is asymptotically stable in accordance with the condition of the theorem and $y^k(t) \to 0$ for $t \to +\infty$. Using Cauchy formula [Kharitonov, 2013] we obtain that (13) is true. Applying this method k times we obtain that $y(t, t_0, \varphi) \to 0$ for $t \to +\infty$.

Then we will give some instability conditions of system (8), where the second term has a form

$$\int\limits_p^q d_\gamma G(\gamma) y(\gamma t) = \sum_{i=1}^m A_i y(\gamma_i t) + \int\limits_p^q \hat{G}(\gamma) y(\gamma t) d\gamma.$$

Let us consider the system

$$\dot{y}(t) = Ay(t) + \sum_{i=1}^m A_i y(\gamma_i t) + \int\limits_p^q \hat{G}(\gamma) y(\gamma t) d\gamma, \quad (16)$$

where $0 < p \leq \gamma_1 < \gamma_2 < \cdots < \gamma_m \leq q < 1$, and matrix $\hat{G}(\gamma)$ is piecewise continuous for $t \in [p, q]$.

Theorem 2. *If matrix A has eigenvalues with the positive real parts then system (16) is instable.*

Proof. At first let matrix A has not eigenvalues with zero real parts and

$$\det\left(A + \sum_{i=1}^m \gamma_i^s A_i + \int\limits_p^q \gamma^s \hat{G}(\gamma) d\gamma\right) \neq 0 \text{ for } s = 0, 1, \ldots.$$

Let us make the coordinates transformation $y = Sz = S\binom{\hat{z}}{\tilde{z}}$, $S^{-1}AS = diag(A_+; A_-)$ in order to obtain the matrix A_+ with all the eigenvalues in the open right half-plane and the matrix A_- with all the eigenvalues in the open left half-plane of the complex plane. Then system (16) may be written in the form

$$\dot{z}(t) = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} z(t) + + \sum_{i=1}^m \tilde{A} z(\gamma_i t) + \int\limits_p^q \hat{G}(\gamma) z(\gamma t) d\gamma.$$

Let the matrixes $\hat{V}$ and $\tilde{V}$ are solutions of Lyapunov matrix equations $\hat{V}A_+ + (A_+)^T\hat{V} = E$ and $\tilde{V}A_- + (A_-)^T\tilde{V} = -E$. Note that the quadratic forms $v_1 = \hat{z}^T\hat{V}\hat{z}$ and $v_2 = \tilde{z}^T\tilde{V}\tilde{z}$ are positive definite.

Now we introduce the auxiliary system

$$\dot{z}^k(t) = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix} z^k(t) + \\ + \sum_{i=1}^{m} \gamma_i^k \tilde{A}_i z^k(\gamma_i t) + \int_p^q \gamma^k \hat{G}(\gamma) z^k(\gamma t) d\gamma \quad (17)$$

and the functional

$$v(z^k) = v_1(\hat{z}^k(t)) - v_2(\tilde{z}^k(t)) - d \cdot \sum_{i=1}^{m} \int_{\gamma_i t}^{t} \left\| z^k(\tau) \right\|^2 d\tau - \\ -d \cdot \int_p^q \int_{\gamma t}^{t} \left\| z^k(\tau) \right\|^2 d\tau d\gamma.$$

The time derivative of this functional along the solutions of system (17) is

$$\left.\frac{dv(z^k)}{dt}\right|_{17} = \|z^k(t)\|^2 + 2(z^k(t))^T \cdot \begin{pmatrix} \hat{V} & 0 \\ 0 & -\tilde{V} \end{pmatrix} \times \\ \times \left[\sum_{i=1}^{m} \gamma_i^k \tilde{A}_i z^k(\gamma_i t) + \int_p^q \gamma^k \tilde{G}(\gamma) z^k(\gamma t) d\gamma \right] + \\ +d \cdot \sum_{i=1}^{m} (\gamma_i \|z^k(\gamma_i t)\|^2 - \|z^k(t)\|^2) + \\ +d \cdot \left(\int_p^q \gamma \|z^k(\gamma t)\|^2 d\gamma - (q-p)\|z^k(t)\|^2 \right).$$

Define the following value

$$r = (\|\hat{V}\| + \|\tilde{V}\|) \cdot \max \left\{ \max_{i=1,\dots,m} \|\tilde{A}_i\|; \sup_{\gamma \in [p,q]} \|\tilde{G}(\gamma)\| \right\}.$$

It is easy to see that if

$$d = \frac{q^k r}{p} \text{ and } h = 1 - q^k \left(1 + \frac{r(m+q-p)}{p} \right)$$

then

$$\left.\frac{dv(z^k)}{dt}\right|_{(17)} \geq h \cdot \|z^k(t)\|^2,$$

Integrating the last inequality from t_0 to t we obtain that

$$v_1(\hat{z}^k(t)) \geq v_2(\tilde{z}^k(t)) + d \cdot \sum_{i=1}^{m} \int_{\gamma_i t}^{t} \|z^k(\tau)\|^2 d\tau + \\ +d \cdot \int_p^q \int_{\gamma t}^{t} \|z^k(\tau)\|^2 d\tau d\gamma + h \int_{t_0}^{t} \|z^k(\tau)\|^2 d\tau + \\ +v(\varphi^k) \geq v(\varphi^k) + h \int_{t_0}^{t} \|z^k(\tau)\|^2 d\tau$$

Now we choose k such that $h > 0$. Using the last inequality and Gronuoll lemma we obtain the next inequality

$$\|z_k(t)\|^2 \geq \frac{v(\varphi^k)}{a} \cdot e^{\frac{h}{a}(t-t_0)}, \ t \geq t_0,$$

where a is the largest eigenvalue of the matrix $\hat{V}$.

Thus the solution of system (17) is unbounded under the initial function φ^k such that $v(\varphi^k) > 0$.

In according to Lemma the corresponding solution of system (16) is unbounded too.

Let matrix A has eigenvalue with zero real part or $\det \left(A + \sum_{i=1}^{m} \gamma_i^s \cdot A_i + \int_p^q \gamma^s \cdot \hat{G}(\gamma) d\gamma \right) = 0$ for some s. Using the variable transformation $y = e^{\varepsilon t} \cdot z$, where ε is sufficiently small we obtain the previous case.

5 Example

Let us consider system (7), where $l_i = l$, $\bar{d}_i = d$, $\alpha_{i0} = c_1$, $\alpha_{i1} = c_2$, $\alpha_{i\nu} = 0$ for $\nu = 2, 3, \dots, N-1$ and $i = 1, 2, \dots, N$. An addition let $h = 0$ so $h_{i1} = e \cdot \gamma \cdot t$.

Then system (7) has the form

$$\frac{dy}{dt} = (pE + qJ)y(t) + rJy(\beta t) + \Delta, \quad (18)$$

where E is identity matrix,

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}, J = \begin{pmatrix} 0\ 1\ 0 \dots 0 \\ 0\ 0\ 1 \dots 0 \\ \dots\dots\dots\dots \\ 0\ 0\ 0 \dots 1 \\ 1\ 0\ 0 \dots 0 \end{pmatrix}, \Delta = \frac{\bar{v}}{l} \begin{pmatrix} \Delta_1 \\ \vdots \\ \Delta_N \end{pmatrix},$$

$p = \frac{\bar{v}}{l}\left(-1 + c_1 + c(\bar{x} - \bar{d})\right)$, $q = \frac{\bar{v}}{l}\left(1 + c(\bar{x} - \bar{d})\right)$, $r = \frac{\bar{v}}{l} c_2$, $\beta = 1 - l\gamma > 0$.

In accordance to Theorem 1 if $pE + qJ$ is Hurwitz matrix and all roots of equation $\det(pE + qJ + r\beta^\lambda J) = 0$ are in the open left half - plane of the complex plane then system (18) is asymptotically stable.

Now we calculate $\det(\lambda E - pE - qJ) = (\lambda - p)^N - q^N$. In accordance with the physical essence of the model we can say that $1 + c\bar{d} - c\bar{x} > 0$. It follows that $c_1 < 0$. The second equation has the form $(q + r\beta^\lambda)^N = (-p)^N$. By considering that $q > 0, -p = q - c_1 > 0, r > 0$ we have

$$\beta^\lambda = \frac{-q + (q - c_1)\sqrt[N]{1}}{r}.$$

Since $0 < \beta < 1$ we obtain the next inequality

$$|\beta^\lambda| = \frac{|-q + (q - c_1)\sqrt[N]{1}|}{r} \geq \frac{-c_1}{r} > 1.$$

Therefore system (18) is asymptotically stable, if $c_1 < 0$ and $0 \leq c_2 < -\frac{c_1 l}{v}$.

6 Conclusion

In the paper a double-stage approach for stability analysis is propagated to the linear differential-difference systems with linear increasing and distributed delay. Sufficient stability and instability conditions are obtained and applied for the stability analysis of the multi – agency system. We hope these results could be basis to construct Lyapunov-Krasovskii functionals for these systems.

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