PHYSCON 2017, Florence, Italy, 17–19 July, 2017

ROBUST CONTROL DESIGN FOR LINEAR SYSTEMS WITH EXOGENOUS AND SYSTEM DISTURBANCES

Mikhail V. Khlebnikov
Laboratory of Adaptive and Robust Systems
Trapeznikov Institute of Control Sciences
Russia
khlebnik@ipu.ru

Kirill O. Zheleznov
Laboratory of Adaptive and Robust Systems
Trapeznikov Institute of Control Sciences
Russia
tosha594@mail.ru

Abstract
A simple but versatile approach to robust control design for linear systems with system and exogenous disturbances is proposed. The efficacy of the proposed approach is demonstrated on the benchmark example of the F-16 aircraft model. The approach is easily implemented computationally.

Key words
Linear systems, feedback design, exogenous disturbances, linear matrix inequalities, robustness.

1 Introduction

Feedback design for the system with exogenous disturbances is one of the most important problems of the control theory. We note the survey [Petersen, 2014] of the most significant results within robust control theory. Such issues as robust control design, including approaches based on internal model principle, models with structured uncertainty and robust control of singularly perturbed systems, etc. are considered in a lot of papers; for example, the recent paper [Hien, 2014] deals with linear time-varying systems with delay and bounded disturbances.

The present paper is devoted to the robust feedback synthesis in the presence of bounded exogenous disturbances. We use the approach proposed in [Zheleznov, 2016]; it is based on the invariant ellipsoids concept [Nazin, 2007, Khlebnikov, 2011] and linear matrix inequality technique [Boyd, 1994, Polyak, 2014, Skogestad, 2007]. In this way, the obtained tasks can be reduced to semi-definite programming and one-dimensional optimization. Such problems can be easily and effectively solved computationally with the use of software including (but not limited to) freeware package `cvx` [Grant, 2014] and SDPT3 [Tütüncü, 2003] for MATLAB.

The efficacy of the proposed approach is demonstrated via a benchmark problem associated with the F-16 aircraft model [Liao, 2002].

2 Problem Statement and Main Result

Let us consider a continuous-time linear control system

$$\begin{aligned} &\dot{x} = (A + \Delta A)x + Bu + Dv, \quad x(0) = x_0, \\ &z = Cx + Ev, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times l}$, $C \in \mathbb{R}^{m\times n}$, $D \in \mathbb{R}^{n\times p}$, $E \in \mathbb{R}^{m\times p}$ are given constant matrices; the pair (A, B) is controllable, the pair (A, C) is detectable; $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^l$ is the control input, and $v(t) \in \mathbb{R}^p$ is the external disturbance such that

$$\dot{v} = -\delta v + \Delta_v(t),$$

where $\delta > 0$, and $\Delta_v(t) \in \mathbb{R}^p$ is the unknown bounded additive component satisfying the constraint

$$\|\Delta_v(t)\| \le 1 \quad \text{for all} \quad t \ge 0.$$

We assume that the uncertainty ΔA has the structure

$$\Delta A = M\Delta N,$$

where $M \in \mathbb{R}^{n\times d}$, $N \in \mathbb{R}^{d\times n}$ are given constant matrices, and the matrix uncertainty $\Delta \in \mathbb{R}^{d\times d}$ satisfies the constraint $\|\Delta\| \le 1$. Note that all the results presented below can be immediately extended to time-varying matrix uncertainties $\Delta(t)$.

Here and further $\|\cdot\|$ is the Euclidean vector norm and the spectral matrix norm, I is the identity matrix of the appropriate dimension, and the matrix inequalities are understood in the sense of matrix sign-definiteness.

Definition 1. *The ellipsoid with the center at the origin*

$$\mathcal{E}_x = \left\{x \in \mathbb{R}^n: \quad x^T P^{-1} x \le 1\right\}, \quad P \succ 0, \tag{2}$$

is called invariant for the dynamical system

$$\dot{x} = Ax + Dw, \quad \|w(t)\| \leq 1,$$

if from the condition $x(0) \in \mathcal{E}_x$ it follows that $x(t) \in \mathcal{E}_x$ for all $t \geq 0$.

In other words, any trajectory of the system that comes from the point lying in the ellipsoid $\mathcal{E}_x$ belongs to this ellipsoid at any time.

Definition 2. *The ellipsoid with the center at the origin*

$$\mathcal{E}_z = \left\{ z \in \mathbb{R}^l : \quad z^T (CPC^T)^{-1} z \leq 1 \right\}$$

is said to be bounding for the dynamical system

$$\begin{aligned} &\dot{x} = Ax + Dw, \quad x(0) = x_0, \\ &z = Cx, \end{aligned}$$

corresponding to the invariant ellipsoid $\mathcal{E}_x$.

Accordingly, the condition $x(0) \in \mathcal{E}_x$ implies that the system output $z(t) \in \mathcal{E}_z$ for all $t \geq 0$.

Our goal is to design the linear static state feedback

$$u = Kx, \tag{3}$$

which robustly stabilizes the system (1) for all admissible disturbances $v(t)$ and uncertainties Δ, and minimizes (via certain criteria) the bounding ellipsoid for the output of the closed-loop system.

As a criterion we adopt the spectral norm of the matrix that specifies the bounding ellipsoid, i.e., the minimal radius of the containing ball.

Lemma 1. *[Nazin, 2007] The ellipsoid* (2) *is invariant for the dynamical system*

$$\dot{x} = Ax + Dw, \quad \|w(t)\| \leq 1,$$

if and only if its matrix P satisfies the LMIs

$$AP + PA^T + \alpha P + \frac{1}{\alpha} DD^T \preceq 0, \quad P \succ 0,$$

for a certain positive scalar α.

The following theorem presents the main result of the paper.

Theorem 1. *Let $\widehat{P}_{11}$, $\widehat{P}_{22}$, and $\widehat{Y}$ be the solution of the minimization problem*

$$\min \|\widetilde{C} P \widetilde{C}^T\| \tag{4}$$

subject to the constraints

$$\begin{pmatrix} AP_{11} + P_{11}A^T + BY + Y^T B^T + \alpha P_{11} + \beta M M^T & DP_{22} & P_{11} N^T \\ * & (\alpha - 2\delta) P_{22} + \frac{1}{\alpha} I & 0 \\ * & * & -\beta I \end{pmatrix} \preceq 0, \qquad P \succ 0, \tag{5}$$

where

$$P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}, \qquad \widetilde{C} = \begin{pmatrix} C & E \end{pmatrix},$$

with respect to the matrix variables $P_{11} = P_{11}^T \in \mathbb{R}^{n \times n}$, $P_{22} = P_{22}^T \in \mathbb{R}^{p \times p}$, $Y \in \mathbb{R}^{l \times n}$, the scalar variable β, and the scalar parameter $\alpha > 0$.

Then the matrix

$$\widetilde{C} \widehat{P} \widetilde{C}^T$$

defines the bounding ellipsoid for the output of system (1) *with zero initial condition, and the state feedback controller with matrix*

$$\widehat{K} = \widehat{Y} \widehat{P}_{11}^{-1}$$

robustly stabilizes the closed-loop system and rejects the effects of admissible disturbances $v(t)$.

Proof. Introducing the composite vector

$$g = \begin{pmatrix} x \\ v \end{pmatrix} \in \mathbb{R}^{n+p},$$

and embracing the system with feedback (3), we obtain the system

$$\begin{aligned} &\dot{g} = \underbrace{\begin{pmatrix} A + BK + M\Delta N & D \\ 0 & -\delta I \end{pmatrix}}_{\widetilde{A}} g + \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}}_{\widetilde{D}} \underbrace{\begin{pmatrix} 0 \\ \Delta_v \end{pmatrix}}_{\widetilde{v}}, \\ &z = \underbrace{\begin{pmatrix} C & E \end{pmatrix}}_{\widetilde{C}} g. \end{aligned} \tag{6}$$

Taking into account $\widetilde{v} = \begin{pmatrix} 0 \\ \Delta_v \end{pmatrix}$, we have

$$\|\widetilde{v}(t)\| = \left\| \begin{pmatrix} 0 \\ \Delta_v \end{pmatrix} \right\| \le 1, \quad \forall t \ge 0.$$

Therefore, Lemma 1 is applicable to system (6); i.e., the matrix $P \succ 0$ of the invariant ellipsoid satisfies the LMI

$$\widetilde{A}P + P\widetilde{A}^T + \alpha P + \frac{1}{\alpha}\widetilde{D}\widetilde{D}^T \preceq 0. \tag{7}$$

Imposing a simplifying assumption, we will seek the matrix $P = P^T \in \mathbb{R}^{(n+p)\times(n+p)}$ in the block-diagonal form

$$P = \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix}, \quad P_{11} = P_{11}^T \in \mathbb{R}^{n\times n}.$$

Then we rewrite (7) as

$$\begin{pmatrix} A + BK + M\Delta N & D \\ 0 & -\delta I \end{pmatrix} \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} + \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} \begin{pmatrix} (A+BK)^T + N^T\Delta^T M^T & 0 \\ D^T & -\delta I \end{pmatrix} + \alpha \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22} \end{pmatrix} + \frac{1}{\alpha}\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \preceq 0$$

or equivalently,

$$\begin{pmatrix} \begin{matrix} AP_{11} + P_{11}A^T + \\ BKP_{11} + P_{11}K^TB^T + \\ \alpha P_{11} + \\ M\Delta N P_{11} + \\ P_{11}N^T\Delta^T M^T \end{matrix} & DP_{22} \\ * & (\alpha - 2\delta)P_{22} + \frac{1}{\alpha}I \end{pmatrix} \preceq 0. \tag{8}$$

Condition (8) can be reformulated as

$$\begin{pmatrix} \begin{matrix} AP_{11} + P_{11}A^T + \\ BKP_{11} + P_{11}K^TB^T + \\ \alpha P_{11} \end{matrix} & DP_{22} \\ * & (\alpha - 2\delta)P_{22} + \frac{1}{\alpha}I \end{pmatrix} + \begin{pmatrix} M \\ 0 \end{pmatrix} \Delta \begin{pmatrix} NP_{11} & 0 \end{pmatrix} + \begin{pmatrix} P_{11}N^T \\ 0 \end{pmatrix} \Delta^T \begin{pmatrix} M^T & 0 \end{pmatrix} \preceq 0. \tag{9}$$

By Petersen's lemma [Petersen, 1987], condition (9) is fulfilled for all admissible uncertainties Δ if there exists β such that

$$\begin{pmatrix} \begin{matrix} AP_{11} + P_{11}A^T + \\ BKP_{11} + \\ P_{11}K^TB^T + \\ \alpha P_{11} + \beta MM^T \end{matrix} & DP_{22} & P_{11}N^T \\ * & (\alpha - 2\delta)P_{22} + \frac{1}{\alpha}I & 0 \\ * & * & -\beta I \end{pmatrix} \preceq 0.$$

Introducing the auxiliary variable $Y = KP_{11}$, we arrive at (5).

As the invariant ellipsoid for the state g of system (6) is specified by the matrix P, the bounding ellipsoid for the output z is specified by the matrix $\widetilde{C}P\widetilde{C}^T$ due to Definition 2. The proof is complete.

Remark 1. *The problem* (4)–(5) *is a semi-definite programming with 1D optimization over the scalar parameter α. Such a procedure can be easily accomplished in* MATLAB, *e.g., by using the toolbox* cvx.

Remark 2. *Theorem 1 establishes a sufficient condition due to the simplifying assumption of the block-diagonal form of the matrix P. Obtaining an analogue of Theorem 1 for general-form matrices P remains an open problem.*

It is natural to require the following constraint on the control input:

$$\|u(t)\| \le \mu \quad \forall t \ge 0 \tag{10}$$

along the trajectory for a given level μ. The following lemma states a sufficient condition for the control bound (10). It is formulated as a linear matrix inequality in the matrix variables P_{11} and Y in Theorem 1.

Lemma 2. *Let the matrices $P_{11} \succ 0$ and Y satisfy inequalities* (5)*. Then the LMI*

$$\begin{pmatrix} P_{11} & Y^T \\ Y & \mu^2 I \end{pmatrix} \succeq 0 \tag{11}$$

guarantees the satisfaction of constraint (10) *along the trajectory of the system* (6) *with controller* (3).

Proof. By (3), constraint (10) can be represented in the form

$$g^T \begin{pmatrix} K^TK & 0 \\ 0 & 0 \end{pmatrix} g \le \mu^2. \tag{12}$$

Consider the ellipsoid

$$\mathcal{E}_g = \left\{ g \in \mathbb{R}^{n+p}\colon \quad g^T P^{-1} g \le 1 \right\}, \quad P \succ 0,$$

specified by the matrix P which satisfies condition (5), and impose the following condition:

$$g^T \begin{pmatrix} K^TK & 0 \\ 0 & 0 \end{pmatrix} g \le \mu^2 \quad \forall g\colon \quad g^T P^{-1} g \le 1. \tag{13}$$

Clearly, the fulfillment of (13) is equivalent to

$$\frac{1}{\mu^2}\begin{pmatrix} K^TK & 0 \\ 0 & 0 \end{pmatrix} \preceq P^{-1} = \begin{pmatrix} P_{11}^{-1} & 0 \\ 0 & P_{22}^{-1} \end{pmatrix}$$

or

$$\frac{1}{\mu^2}K^TK \preceq P_{11}^{-1}.$$

Since $K = YP_{11}^{-1}$, this matrix inequality takes the form

$$\frac{1}{\mu^2}P_{11}^{-1}Y^TYP_{11}^{-1} \preceq P_{11}^{-1}.$$

Pre- and post-multiplying this inequality by P_{11} and using the Schur lemma, we complete the proof.

Finally, Theorem 1 is formulated for zero initial condition. Otherwise, we should require

$$g_0^TP^{-1}g_0 \le 1$$

or by the Schur lemma

$$\begin{pmatrix} 1 & g_0^T \\ g_0 & P \end{pmatrix} \succeq 0. \tag{14}$$

Therefore, Theorem 1 modifies as follows. Linear matrix inequalities (11) and (14) are to be attached to inequalities (5).

3 Example

We demonstrate the efficacy of proposed approach via the well-known benchmark of the F-16 aircraft model, see [17], with scalar uncertainty and the matrices A, B, C, D, E, M, N (see below).

Applying the Theorem 1 for

$$\delta = 0.03, \qquad \mu = 10,$$

we obtain the matrix $\widetilde{C}\widehat{P}\widetilde{C}^T$ of the bounding ellipsoid, and the gain matrix $\widehat{K}$:

$$A = \begin{pmatrix} -0.0153 & 0.0481 & -5.9420 & 0.0021 & 0 & 0 \\ -0.0910 & -0.9568 & 138.3608 & 0.0163 & 0 & 0 \\ 0.0002 & 0.0046 & -1.0220 & -0.0005 & 0 & -0.0029 \\ 0 & 0 & 0 & -0.2804 & 6.2667 & -151.1435 \\ 0 & 0 & 0.0003 & -0.1821 & -3.4192 & 0.6401 \\ 0 & 0 & 0.0025 & 0.0454 & -0.0304 & -0.4535 \end{pmatrix}, \qquad M = N^T = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$B = \begin{pmatrix} 0.0239 & 0.0239 & 0.0250 & 0.0250 & 0 \\ -0.1722 & -0.1722 & -0.1799 & -0.1799 & 0 \\ -0.0873 & -0.0873 & -0.0076 & -0.0076 & 0 \\ -0.3149 & 0.3149 & 0.0233 & -0.0233 & 0.1205 \\ -0.1892 & 0.1892 & -0.3464 & 0.3464 & 0.1237 \\ -0.1678 & 0.1678 & -0.0147 & 0.0147 & -0.0587 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.0481 \\ -0.9568 \\ 0.0046 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 57.2958 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 57.2468 & 2.3696 \\ 0 & 0 & 0 & 0 & 2.3696 & 57.2468 \\ -0.0155 & 0.3756 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.03760 & 0 & 0 \end{pmatrix},$$

$$\widetilde{C}\widehat{P}\widetilde{C}^T = \begin{pmatrix} 1195.7274 & 2.5029 & -4.4952 & -2574.4520 & 2.0365 \\ * & 6439.0260 & -957.4741 & 2.7459 & -233.6640 \\ * & * & 597.9295 & 12.2821 & -8.4607 \\ * & * & * & 6072.4507 & -7.5914 \\ * & * & * & * & 2540.4079 \end{pmatrix},$$

$$\widehat{K} = \begin{pmatrix} 3.4781 & 0.2734 & 38.7728 & -0.1303 & -1.3647 & 14.3270 \\ -3.6517 & -0.0999 & 35.3838 & 0.1309 & 1.3704 & -14.4680 \\ -17.9018 & -0.8690 & 5.2321 & 0.4505 & 4.2587 & 13.0617 \\ 17.8935 & 1.0104 & 22.7458 & -0.4674 & -4.2859 & -13.1402 \\ -17.3187 & -0.9159 & -10.1840 & -0.3547 & 3.7430 & 3.7302 \end{pmatrix}.$$

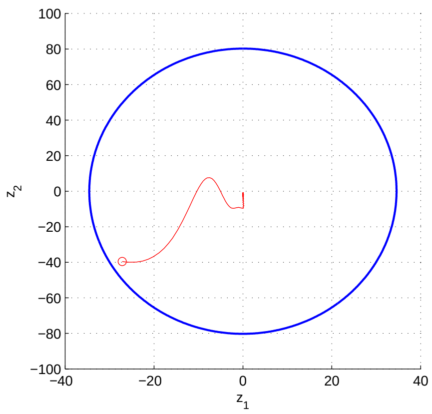

Figure 1. The 2D projections of the bounding ellipsoid and the output trajectory on the plane (z_1, z_2).

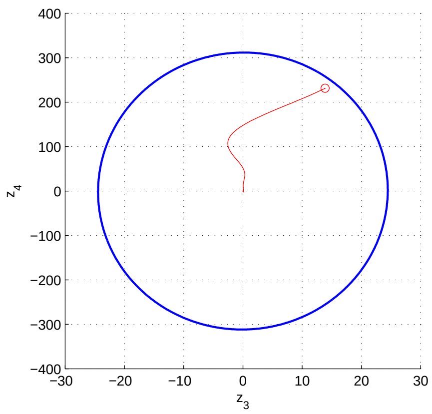

Figure 2. The 2D projections of the bounding ellipsoid and the output trajectory on the plane (z_3, z_4).

Fig. 1 depicts the projections of the bounding ellipsoid and the output trajectory on the plane defined by axes z_1 and z_2 with

$$\Delta_v(t) = \operatorname{sign} \sin t, \qquad \Delta(t) \equiv 1,$$

and the initial point

$$x_0 = \begin{pmatrix} -7.8948 \\ 153.6350 \\ -0.4741 \\ 2.2227 \\ -0.6997 \\ 0.2135 \end{pmatrix}.$$

Fig. 2 depicts the projections of the bounding ellipsoid and the output trajectory on the plane defined by the axes z_3 and z_4.

4 Conclusion

A simple but versatile approach to robust control design for linear systems with system and exogenous disturbances is proposed. The efficacy of the proposed approach is demonstrated via the benchmark example of the F-16 aircraft model. The approach is easily implemented computationally.

The authors plan to adopt the approach to discrete-time systems and to tracking systems.

References

Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory.* SIAM. Philadelphia.

Grant, M. and Boyd, S. (2014) *CVX: Matlab Software for Disciplined Convex Programming, version 2.1.* `http://cvxr.com/cvx`.

Hien, L. V., and Trinh, H. M. (2014). A new approach to state bounding for linear time-varying systems with delay and bounded disturbances. *Automatica*, **50**(6), pp. 1735–1738.

Khlebnikov, M. V., Polyak, B. T., and Kuntsevich, V. M. (2011). Optimization of linear systems subject to bounded exogenous disturbances: The invariant ellipsoid technique. *Automation and Remote Control*, **72**(11), pp. 2227–2275.

Liao, F., Wang, J. L., and Yang, G. H. (2002). Reliable robust flight tracking control: An LMI approach. *IEEE Transactions on Control Systems Technology*, **10**(1), pp. 76–89.

Nazin, S. A., Polyak, B. T., and Topunov, M. V. (2007). Rejection of bounded exogenous disturbances by the method of invariant ellipsoids. *Autom. Remote Control*, **68**(3), pp. 467–486.

Petersen, I. (1987). A stabilization algorithm for a class of uncertain linear systems. *Systems & Control Letters*, **8**(4), pp. 351–357.

Petersen, I. R., and Tempo, R. (2014). Robust control of uncertain systems: Classical results and recent developments. *Automatica*, **50**(5), pp. 1315–1335.

Polyak, B. T., Khlebnikov, M. V., and Scherbakov, P. S. (2014). *Control of Linear Systems Subjected to Exogenous Distrubances: An LMI Approach* [in Russian]. LENAND. Moscow.

Skogestad, S., and Postlethwaite, I. (2007). *Multivariable Feedback Control: Analysis and Design.* Wiley. New York.

Tütüncü, R. H., Toh, K. C., and Todd, M. J. (2003). Solving semidefinite-quadratic-linear programs using SDPT3. *Mathematical programming*, **95**(2), pp. 189–217.

Zheleznov, K. O., and Khlebnikov, M. V. (2016). Tracking problem for dynamical systems with exogenous and system disturbances. In *20th International Conference on System Theory, Control and Computing.* Sinaia, Romania, October 13–15, 2016, pp. 125–128.