CYBERNETICS AND PHYSICS, VOL. 1, NO. 1, 2012 , 51–54

ANALYTICAL APPROACH FOR ATTRACTOR LOCALIZATION IN DYNAMICAL MODEL OF SOLAR WIND-MAGNETOSPHERE-IONOSPHERE

Gennady Leonov
Applied cybernetics Dept.
St. Petersburg State University
Russia
leonov@math.spbu.ru

Nikolay Kuznetsov
Applied cybernetics department
St. Petersburg State University (Russia)
Mathematical Information Technology Dept.
University of Jyväskylä (Finland)
nkuznetsov@gmail.com

Svetlana Seledzhi
Applied cybernetics Dept.
St. Petersburg State University
Russia
ssm@ss1563.spb.edu

Konstantin Starkov
CITEDI-IPN
Mexico
konstarkov@hotmail.com

Abstract

The problem of localization of attractor of "solar wind-magnetosphere-ionosphere" (WINDMI) three-dimensional model has stimulated further development of method of conical nets. In the paper the development of this method is carried out and analytical localization of the attractor of WINDMI model is performed.

Key words

frequency method of positively invariant cone grids, attractor localization, WINDMI model, solar wind-magnetosphere-ionosphere.

1 Introduction

In the present paper we consider the localization problem of attractor of 3-dimensional simplified model [Spencer et al., 2006] of 6-dimensional system [Horton et al., 2001], which is used for analysis of geomagnetic storms and substorms and modeling the energy flow through the solar wind-magnetosphere-ionosphere system (WINDMI) system. The study of attractors of such systems is important task because it allows to understand the possible magnetospheric plasma states [Smith et al., 2000].

The problem of attractor localization has stimulated the development of method of positively invariant cone grids (which is often used for study of control systems) [Leonov et al., 1996[1]; Yakubovich et al., 2004; Leonov et al., 1996[2]; Leonov, 2006; Leonov et al., 2009] for its study. Further development of this method is carried out and analytical estimates of the attractor of the model are obtained.

2 Further development of the method of positively invariant cone grids

Consider a system

$$\frac{dx}{dt} = Px + q\varphi(r^*x), \quad x \in \mathbb{R}^n. \tag{1}$$

Here P is a constant degenerate $(n \times n)$-matrix, q and r are n-dimensional vectors, $*$ is an operation of transposition, and $\varphi(\sigma)$ is a differentiable scalar function satisfying the following sector conditions

$$\varphi(\sigma) < \mu(\sigma - \alpha), \quad \forall\sigma \geq \alpha, \tag{2}$$

$$\mu(\sigma - \beta) < \varphi(\sigma), \quad \forall\sigma \leq \beta, \tag{3}$$

where μ is a certain positive number, $\alpha < \beta$.

Let the pair (P, q) be totally controllable, the pair (P, r) be totally observable, and system (1) have a unique equilibrium.

Theorem. *Suppose, $r^*q \leq 0$, there exists a number $\lambda > 0$ such that the matrix $P + \lambda T$ has $(n-1)$ eigenvalues with negative real parts, and the following inequality*

$$\mathrm{Re}\, W(i\omega - \lambda) + \mu|W(i\omega - \lambda)|^2 \leq 0, \quad \forall\omega \in \mathbb{R}^1 \tag{4}$$

is satisfied. Then for any solution $x(t)$ of system (1) there exists a number T such that

$$r^*x(t) \in (\alpha, \beta), \quad \forall t > T. \tag{5}$$

Here $W(p) = r^*(P - pI)^{-1}q$ is a transfer function of system (1), I is a unit $(n \times n)$-matrix.

We give a scheme of the proof of Theorem 1. Condition (4) implies the existence of symmetric $n \times n$-matrix H such that for it there are valid the following conditions (the detailed proof of this fact can be found in [Leonov et al., 1996[1]]):

1) the matrix H has one negative and $(n-1)$ positive eigenvalues,

2) for all $z \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^1$ the inequality

$$2z^*H[(P + \lambda I)z + q\xi] + r^*z(r^*z - \mu^{-1}\xi) \leq 0 \quad (6)$$

is satisfied. Note that relation (6) yields the relation

$$2Hq = \mu^{-1}r.$$

Then from Theorem 1 it follows that $r^*H^{-1}r = 2\mu r^*q \leq 0$ and, therefore, [Leonov et al., 1996[1]] we have

$$z^*Hz \geq 0, \quad \forall z \in \{z^*r = 0\}. \quad (7)$$

Let be $d \in \mathbb{R}^n$ such that $d \neq 0$ and $Pd = 0$, $r^*d = 1$. Then relation (6) implies that $d^*Hd < 0$.

Consider now the Lyapunov-type function

$$V_1(x) = V(x - \alpha d) = (x - \alpha d)^*H(x - \alpha d),$$
$$V_2(x) = V(x - \beta d) = (x - \beta d)^*H(x - \beta d).$$

From relations (2), (3), (6) it follows that

$$\dot{V}_1(x(t)) + 2\lambda V_1(x(t)) < 0 \quad \text{for } r^*x(t) > \alpha,$$
$$\dot{V}_2(x(t)) + 2\lambda V_2(x(t)) < 0 \quad \text{for } r^*x(t) < \beta.$$

These inequalities can be rewritten in the following form

$$\begin{array}{l} V_1(x(t)) \leq V_1(x(0))e^{-2\lambda t} \forall t \geq 0 : \ r^*x(t) > \alpha, \\ V_2(x(t)) \leq V_2(x(0))e^{-2\lambda t} \forall t \geq 0 : \ r^*x(t) < \beta. \end{array} \quad (8)$$

In this case relation (7) implies positive invariance of the sets [Leonov et al., 1996[1]]

$$\Omega_1(\alpha) = \{(x - \alpha d)^*H(x - \alpha d) < 0, \quad r^*x \geq \alpha\},$$
$$\Omega_2(\beta) = \{(x - \beta d)^*H(x - \beta d) < 0, \quad r^*x \leq \beta\}.$$

Then it is easily seen that the closures $\overline{\Omega}_1(\alpha)$, $\overline{\Omega}_2(\beta)$ are also positively invariant.

In this case from (8) we have that the boundaries $\partial\Omega_1(\alpha)$ and $\partial\Omega_2(\beta)$ do not involve whole trajectories and they are almost everywhere transverse to vector field of system (1). In the phase space of system (1) these boundaries make up a continuum set of surfaces (conical net), which is shown in Fig. 1.

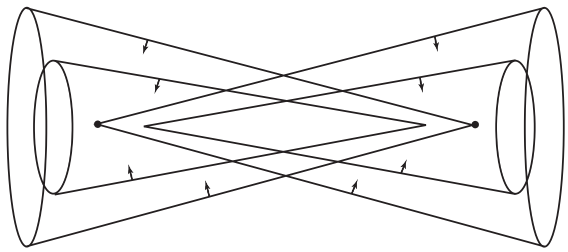

Figure 1. Cone grids.

Let us fix $x(0)$ and determine α_0 and β_0 such that

$$x(0) \in \Omega_1(\alpha_0) \cap \Omega_2(\beta_0).$$

Then consider sets $\Omega_1(\nu)$ and $\Omega_2(\mu)$ where $\nu \in (\alpha, \alpha_0)$ and $\mu \in (\beta, \beta_0)$. Boundaries $\partial\Omega_1(\nu)$ and $\partial\Omega_2(\mu)$ form continual contactless surfaces in phase space (Fig. 1). Hence there is time $T > 0$ such that for $t \geq T$ for solution $x(t)$ the following inclusion

$$x(t) \in \Omega_1(\alpha) \cap \Omega_2(\beta).$$

is valid. This inclusion proofs relation (5).

In increasing time t the structure constructed "huddles" any solution in the set $\Omega_1(\alpha) \cap \Omega_2(\beta)$. The latter proves the assertion of theorem.

Note that the estimate obtained cannot be improved in the considered class of nonlinearities since if for all $\sigma \in (\alpha, \beta)$ $\varphi(\sigma) = 0$, then for $\nu \in (\alpha, \beta)$ $x = \nu d$ is a stationary solution of system.

3 WINDMI system

Consider a system

$$\begin{array}{l} \dddot{x} + b\ddot{x} + c_1\dot{x} + \varphi(x) = 0, \\ \varphi(x) = (c_2 + c_3\tanh(x)), \end{array} \quad (9)$$

where

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

System (9) is 3-dimensional simplified model of 6-dimensional system (obtained in [Horton et al., 2001]) used for analysis of geomagnetic storms and substorms [Spencer et al., 2006] and modeling the energy flow through the solar wind-magnetosphere-ionosphere system.

System (9) is called a simplified WINDMI model. As in [Horton et al., 2001], the parameters of (9), by assumption, are the following

$$b > 0, c_1 > 0, c_3 > c_2 > 0.$$

These parameters are given in the dimensionless form and computed by the formulas from [Horton et al., 2001]. Some dynamical features of model (9) (such as the graphs of the largest Lyapunov exponent; Lyapunov dimension versus the solar wind dynamo voltage; the bifurcation diagram) are shown in [Horton et al., 2001].
Our aim is to study a location of invariant sets of simplified WINDMI model.
Let us apply the theory, developed above, to investigation of equation (9). In this case the maximal coefficient μ can be computed [Yakubovich et al., 2004] by the formula

$$\mu = \left[\begin{array}{ll} \frac{b}{3}\left(c_1 - \frac{2}{9}b^2\right), & b^2 \le 3c_1 \\ \frac{b}{3}\left(c_1 - \frac{2}{9}b^2\right) + 2\left(\frac{b^2}{9} - \frac{c_1}{3}\right)^{3/2}, & b^2 \ge 3c_1 \end{array}\right.$$

Determine a point $x_0 > 0$ such that $\varphi'(x_0) = \mu$,

$$x_0 = \operatorname{arccosh}\sqrt{\frac{c_3}{\mu}} \qquad \text{for } \frac{c_3}{\mu} > 1.$$

Taking into account the relation $\varphi'(x_0) = \varphi'(-x_0) = \mu$, we obtain restrictions (2) and (3) on the nonlinearity $\varphi(x)$ (Fig. 2). Denote

$$\alpha_0 = -\frac{\varphi(x_0)}{\mu} + x_0, \quad -\frac{\varphi(-x_0)}{\mu} - x_0 = \beta_0.$$

Then by (2), (3), and (5) we obtain

$$\alpha_0 \le \lim_{t\to+\infty} \inf x(t), \quad \lim_{t\to+\infty} \sup x(t) \le \beta_0. \qquad (10)$$

For the estimation of $\dot{x}$ and $\ddot{x}$ we remark that by the conditions of positiveness of coefficients the characteristic polynomial of linear part of equation (9) has roots to the left of imaginary axis and the nonlinearity $\varphi(x)$ is bounded:

$$|\varphi(x)| \le c_2 + c_3.$$

In this case, using Cauchy formula, we can obtain [Cesari, 1959; Leonov, 2001; Leonov and Kuznetsov, 2007] the following estimates for $|\dot{x}(t)|$:

$$y_{att} = \left[\begin{array}{l} \lim_{t\to+\infty} \sup |\dot{x}(t)| \le \frac{c_2 + c_3}{c_1}, b^2 \ge 4c_1, \\ \lim_{t\to+\infty} \sup |\dot{x}(t)| \le \frac{2(c_2 + c_3)}{b\sqrt{c_1 - \frac{b^2}{4}}}, b^2 < 4c_1, \end{array}\right.$$

and for $|\ddot{x}(t)|$:

$$\lim_{t\to+\infty} \sup |\ddot{x}(t)| \le \frac{(c_1 y_{att} + c_2 + c_3)}{b}.$$

These estimates together with estimate (10) for $|x(t)|$ localize an attractor of equation (9).

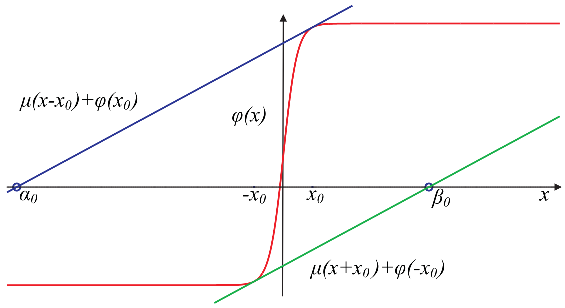

Figure 2. Estimation of nonlinearity.

Conclusion
Application of frequency methods allows us to further develop this area. In particular, for further revision of the criteria one can consider Lyapunov functions of the quadratic form plus integral of the nonlinearity. In this case, instead of analogues of the circular criterion, it is possible to obtain analogues of Popov criterion [Leonov et al., 1996[1]; Leonov et al., 1996[2]].

Acknowledgement
This work was partly supported by Academy of Finland, Ministry of Education and Science of the Russian Federation (Federal target program “Scientific and Scientific-Pedagogical Cadres for Innovative Russia” for the 2009–2013 years), Russian Foundation for Basic Research and Saint-Petersburg State University.

References
Horton, W., Weigel, R.S., and Sprott, J.C. (2011) Chaos and the limits of predictability for the solar-wind-driven magnetosphere-ionosphere system. *Physics of Plasmas*, **6**(8), pp. 2946–2952.
Spencer, E., Horton, W., and Doxas, I. (2006) The dynamics of storms and substorms with the WINDMI model. *J. Adv. Space Res.*, **38**(8), pp. 1657–1668.
Smith, J.P., Thiffeault, J.-L., and Horton, W. (2000) Dynamical range of the WINDMI model: An exploration of possible magnetospheric plasma states. *Journal of geophysical research*, **105**(A6), 12, 98312, 996.
Leonov, G.A., Burkin, I.M., Shepeljavyi A.I. (1996) *Frequency methods in oscillation theory*. Kluwer Academic Publishers.

Leonov, G.A., Ponomarenko, D.V., and Smirnova, V.B. (1996) *Frequency-Domain Methods for Nonlinear Analysis. Theory and Applications*. World Scientific.

Leonov, G.A. (2006) Phase Synchronization: Theory and Applications. *Automation and Remote Control*, **67**(10), pp. 1573–1609.

Leonov, G.A., Kuznetsov, N.V., and Seledzhi, S.M. (2009) Nonlinear Analysis and Design of Phase-Locked Loops. in the book *Automation and Control – Theory And Practice*, In-Tech, pp. 89–114.

Yakubovich, V.A., Leonov, G.A., and Gelig, A.Kh. (2004) *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities.* World Scientific.

Cesari, L. (1959) *Asymptotic Behavior and Stability Problems in Ordinary Differential Equations*. Springer.

Leonov, G.A. (2001) *Mathematical Problems of Control Theory. An Introduction.* World Scientific.

Leonov, G.A., Kuznetsov, N.V. (2007) Time-Varying Linearization and the Perron effects. *International Journal of Bifurcation and Chaos*, **17**(4), pp. 1079–1107.